Economic Growth

Master Course WS 2022/23

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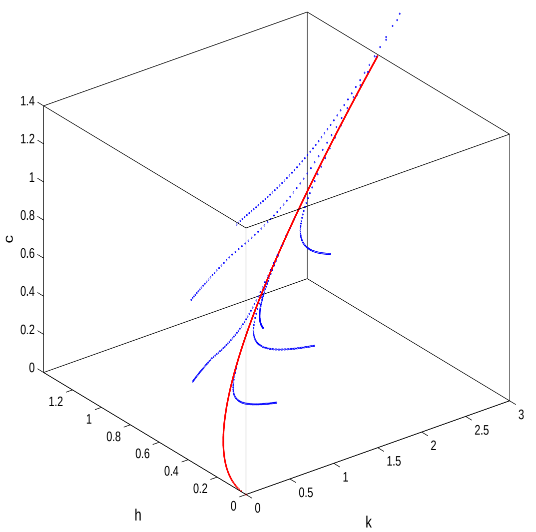
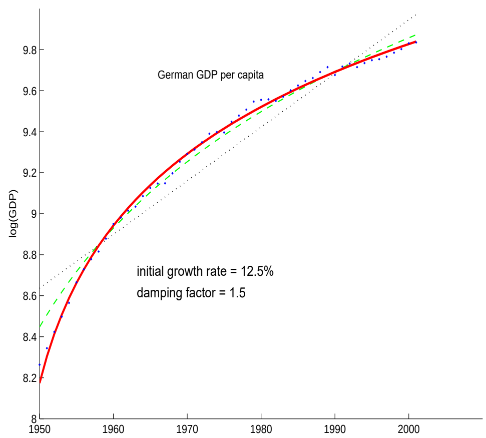
$$\max \int_0^\infty \left(\frac{c^{1-\theta} - 1}{1 - \theta} \right) \cdot e^{-\rho t} \, dt$$





These notes are more or less a one two one print version of the slides of the course. The lecture provides additional information and discussions beyond the content of the slides.

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2 Theories of Economic Growth

2.1 Capital Accumulation and Maximization of Intertemporal Utility

Optimal growth with exogenous technical progress

Solow (1956), Swan (1956), Ramsey (1928) - Cass (1965) - Koopmans (1965)

- Closed economy
- Neoclassical production function
- Intertemporal utility maximization
- Saving equals investment
- Exogenous technical progress

There are three fundamental dynamic effects assumed in this kind of model Factor supply is changed over time by

- exogenous population growth,
- accumulation of physical capital, and
- exogenous human capital growth.

The three dynamic elements interact and jointly change GDP per capita.

There are several issues addressed by this kind of model

- Accumulation of physical capital changes the endogenous relation between capital and labor.
- Exogenous human capital growth affects the capital intensity as well.
- How do savings affect GDP per capita?
- What will determine the propensity to save?

Beyond that we can discuss balance, stability, and convergence.

The macroeconomic production function

$$Y = F(K, L) \qquad \text{e.g. } Y = K^{\alpha} L^{1-\alpha}$$

The marginal product of each factor is positive and decreasing

$$\begin{aligned} \partial F/\partial K > 0 \quad &, \quad \partial^2 F/\partial K^2 < 0 \\ \partial F/\partial L > 0 \quad &, \quad \partial^2 F/\partial L^2 < 0 \end{aligned}$$

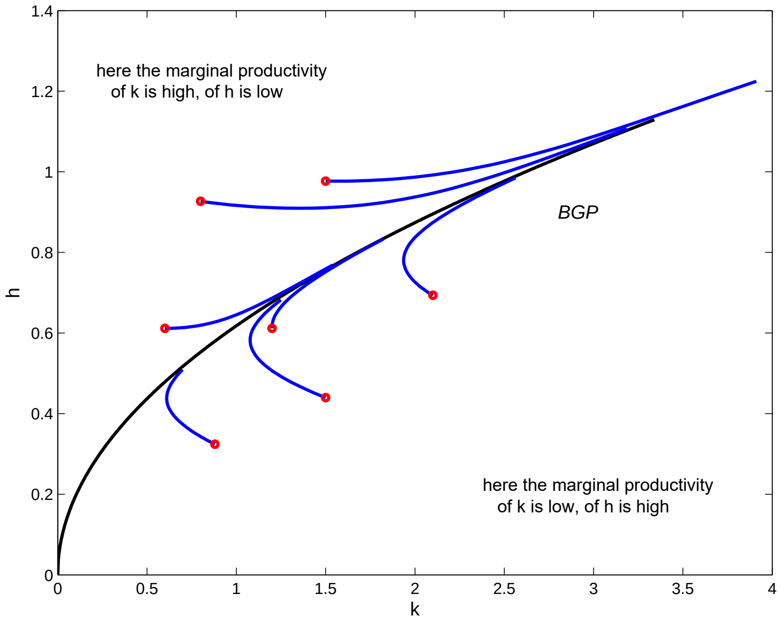
F is linear homogeneous (constant returns to scale)

$$F(\lambda K, \lambda L) = \lambda \cdot F(K, L)$$

and satisfies the Inada conditions

$$\begin{aligned} \lim_{K\to 0}(F_K) &= \lim_{L\to 0}(F_L) = \infty \\ \lim_{K\to \infty}(F_K) &= \lim_{L\to \infty}(F_L) = 0 \end{aligned}$$

The Inada conditions



The model in terms of intensities
The capital intensity k is the ratio of capital over labor

$$k = K/L$$

Constant returns to scale give rise to expressing output as a function of capital intensities

$$Y = F(K, L) = L \cdot F(K/L, 1) = L \cdot F(k, 1) = L \cdot f(k)$$

Hence, production per capita is given by

$$y = Y/L = f(k)$$

Example: Cobb-Douglas production

$$\begin{aligned} Y &= F(K, L) = K^\alpha L^{1-\alpha} \\ \text{yields} & \\ Y/L &= K^\alpha L^{-\alpha} = (K/L)^\alpha \\ & \hookrightarrow \quad y = f(k) = k^\alpha \end{aligned}$$

Marginal productivities

$$\frac{\partial F(K,L)}{\partial K} = \frac{d\, L \cdot f(K/L)}{d\, K} = L \cdot f'(k) \frac{1}{L} = f'(k)$$

and

$$\frac{\partial F(K,L)}{\partial L} = f(k) - L \cdot f'(k) \frac{K}{L^2} = f(k) - k \cdot f'(k)$$

Hence the following identity holds

$$k \cdot F_K(K,L) + F_L(K,L) = f(k)$$

Example:

$$\begin{aligned} f(k) = k^\alpha \quad & \hookrightarrow \quad F_K(K,L) = f'(k) = \alpha k^{\alpha-1} = \alpha f(k)/k \\ & \hookrightarrow \quad F_L(K,L) = f(k) - \alpha f(k) = (1-\alpha) f(k) \end{aligned}$$

Profit maximization
Wage rate w, interest rate r; commodity price index $p = 1$

$$\max_{K,L} \Pi = F(K,L) - (r+\delta) \cdot K - w \cdot L$$

is equivalent to

$$\max_{k,L} \Pi = L\,(f(k) - (r+\delta) \cdot k - w)$$

The solution in terms of L is not determined, but in terms of k

$$\hookrightarrow \qquad f'(k) = (r+\delta)$$

Hence r determines the capital intensity k.

For a market equilibrium $\Pi = 0$ must hold, otherwise L equals 0 or ∞

$$\hookrightarrow \qquad w = f(k) - (r+\delta)k = f(k) - kf'(k)$$

Maximizing utility
Let $c = C/L$ denote per capita consumption.

The objective function of consumers

$$\int_0^\infty u(c)\, e^{-(\rho-n)t}\, dt$$

For technical reasons we consider the special case of a utility function of the $CRRA$-type[1]

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$$

Assume: $\theta > 0$.

Note that $u(c) \to \ln(c)$ as $\theta \to 1$.

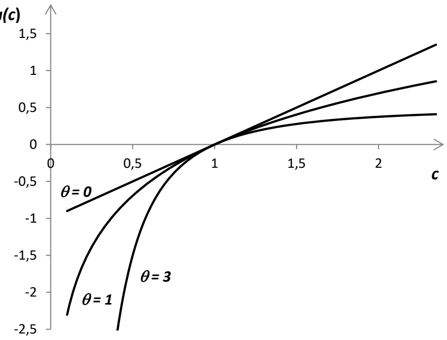
Intertemporal elasticity of substitution
$\theta = -u''(c) \cdot c/u'(c)$ is called the *relative risk aversion* in the context of decisions under uncertainty.

θ large $\longrightarrow$ strong aversion against variation of consumption over time.

$\sigma = 1/\theta$ is the **intertemporal elasticity of substitution**

[1] $CRRA$ stands for constant relative risk aversion in expected utility theory. The concept goes back to Arrow (1965) and Pratt(1964)

Intertemporal elasticity of substitution and smooth consumption
The utility function we consider is strictly concave for $\theta > 0$ and linear in the limit for $\theta \to 0$.



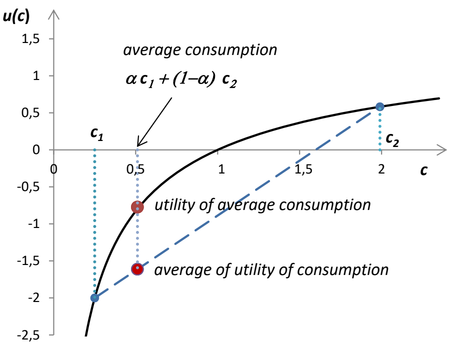
In case of only two periods instead of a time continuum it is obvious what $theta$ implies for the average and dispersion of consumption.

For $\theta > 0$ we see

$$c_1 \neq c_2 \quad \Longrightarrow \quad \alpha u(c_1) + (1-\alpha)u(c_2) < u(\alpha c_1 + (1-\alpha)c_2)$$

Average utility of consumption is smaller than utility of average consumption if there is dispersion. In other words, consumers dislike variation of consumption over time.

and smooth consumption



Accumulation of wealth through savings
Let A be the total amount of assets held by households at some time, w the wage rate, r the interest rate, and n the rate of population growth.

The change of assets holdings is equal to total net savings:

$$\dot{A} = wL + rA - C \qquad \text{together with } a = A/L \text{ yields}$$

$$\dot{a} = w + ra - c - na$$

Indeed, from

$$\hat{A} = wL/A + r - C/A = w/a + r - c/a$$

together with $\hat{a} = \hat{A} - n$ we get the desired result.

Intertemporal utility maximization

$$\max_{c(t)} \quad \int_0^\infty \frac{c^{1-\theta}-1}{1-\theta}\, e^{-(\rho-n)t}\, dt$$

subject to the dynamic constraint for a

i.e. $\dot{a} = w + ra - c - na$

and initial condition $a(0) = a_0$

- Notice that we assume $L_0 = 1$ without loss of generality.
- $\rho > 0$ is the *rate of time preference.*

Solution technique: The Maximum Principle of Pontryagin
Define the Hamiltonian function (in current value form)

$$\mathcal{H} = u(c)e^{nt} \; + \; \lambda(w + (r-n)a - c)$$

$\mathcal{H}$ is a function

- of the state variable a,
- the control variable (co-state variable) c, and
- the shadow price λ.

The Maximum Principle yields first order conditions and a transversality condition.

1. $\mathcal{H}_c = 0$ maximum property
2. $\mathcal{H}_a = -\dot{\lambda} + \lambda\rho$ Euler equation
3. $\lim_{t\to\infty} e^{-\rho t}\lambda\, a = 0$ transversality condition

- We know the differential equation for a and initial condition a_0.
- The evolution of a depends on c
- We look for the differential equation for c and the initial condition c_0.
- It is not necessary to determine the evolution of the shadow price λ.

yields

1. $\mathcal{H}_c = c^{-\theta}e^{nt} \; - \; \lambda = 0$ maximum property
2. $\mathcal{H}_a = \lambda\,(r-n) = -\dot{\lambda} + \lambda\rho$ Euler equation
3. $\lim_{t\to\infty} e^{-\rho t}\lambda\, a = 0$ transversality condition

Differentiate (1) with respect to t:

4. $-\theta c^{-(1+\theta)}\,\dot{c}\, e^{nt} \; + \; c^{-\theta} n\, e^{nt} - \dot{\lambda} = 0$

Substitute $-\dot{\lambda}$ from (2) :

5. $-\theta c^{-(1+\theta)}\,\dot{c}\, e^{nt} \; + \; c^{-\theta} n\, e^{nt} \; + \; \lambda(r - n - \rho) = 0$

Substitute $c^{-\theta}\, e^{nt}$ from (1) :

6. $\big(-\theta c^{-1}\,\dot{c} \; + \; (r-\rho)\big)\,\lambda = 0$

The Keynes-Ramsey-Rule
As λ is positive the last line simplifies to

$$\hat{c} = \frac{1}{\theta}(r-\rho) \qquad \text{the Keynes-Ramsey-Rule}$$

Equilibrium
There is only one asset households can use to invest their savings in: $a = k$ at any point of time.

$$\dot{a} = w + ra - c - na$$

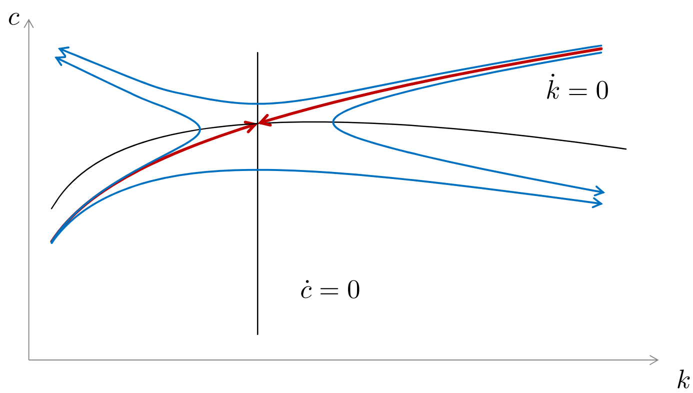
Together with $w + rk = f(k) - \delta k$ this turns into

(1) $\quad \dot{k} = f(k) - c - (n+\delta)k$

The Keynes-Ramsey-rule appears to be

(2) $\quad \dot{c} = \frac{1}{\theta}(f'(k) - \delta - \rho)\, c$

Phase diagram



Technical side notes

Current value and present value form of the Hamiltonian function

The present value form of the Hamiltonian is linked to the current value form by a transformation of coordinates.

$$\mathcal{H}^{present} = e^{-\rho t}\,\mathcal{H}^{current}$$

The shadow price in present values ν turns into the current value shadow price λ by $\lambda = e^{\rho t}\nu$. The derivatives with respect to time are

$$\begin{aligned} \dot{\lambda} &= \rho e^{\rho t}\,\nu + e^{\rho t}\dot{\nu} \\ &= \rho\lambda + e^{\rho t}\dot{\nu} \\ \dot{\lambda} - \rho\lambda &= e^{\rho t}\dot{\nu} \end{aligned}$$

From

$$\frac{\partial \mathcal{H}^{current}}{\partial x} = \dot{\lambda} - \rho\lambda = e^{\rho t}\dot{\nu} \ \text{ together with } \ \frac{\partial \mathcal{H}^{present}}{\partial x} = e^{-\rho t}\,\frac{\partial \mathcal{H}^{current}}{\partial x}$$

we get the first order condition for a control variable x from the present value Hamiltonian

$$\frac{\partial \mathcal{H}^{present}}{\partial x} = -\dot{\nu}$$

Transversality in the Ramsey-Cass-Koopmans Model

In terms of the present value shadow price $\nu = e^{-\rho t}\lambda$ the transversality condition is given by

$$\lim_{t\to\infty} a(t)\nu(t) = 0$$

The Euler equation induces a change of ν of form

$$\dot{\nu} = -(r(t) - n)\nu$$

Integration of the Euler equation yields

$$\nu(t) = \nu(0)e^{-\int_0^\infty (r(\tau)-n)d\tau}$$

$\nu(0)$ is equal to $c(0)^{-\theta}$ due to the maximum property. So it is a positive constant and irrelevant for the validity of the transversality condition.

Using the average interest rate

$$\bar{r}(t) = \frac{1}{t}\int_0^\infty r(\tau)d\tau$$

the transversality condition finally reduces to

$$\lim_{t\to\infty} a(t)e^{-(\bar{r}(t)-n)\,t} = 0$$

I.e. in the long run per capita wealth has to grow with a rate smaller than $\bar{r} - n$. We may evaluate the transversality condition explicitly

$$\begin{aligned} \dot{\nu} &= -\nu\,(r-n) \\ &= -\nu\,(f'(k) - \delta - n) \\ \text{and hence} & \\ \hat{\nu} &= -(f'(k) - \delta - n) \end{aligned}$$

In the capital accumulation equation we use the maximum property $c^{-\theta}\,e^{-(\rho-n)t} - \nu = 0$ to eliminate c. The time scaled shadow price $\mu = e^{(\rho-n)t}\nu$ with property $\hat{\mu} = (\rho - n) + \hat{\nu}$ makes the dynamics even more transparent.

$$c^{-\theta} = e^{(\rho-n)t}\,\nu = \mu$$

$$\begin{aligned} \dot{k} &= f(k) - c - (n+\delta)k \\ &= f(k) - \mu^{-1/\theta} - (n+\delta)k \end{aligned}$$

We have

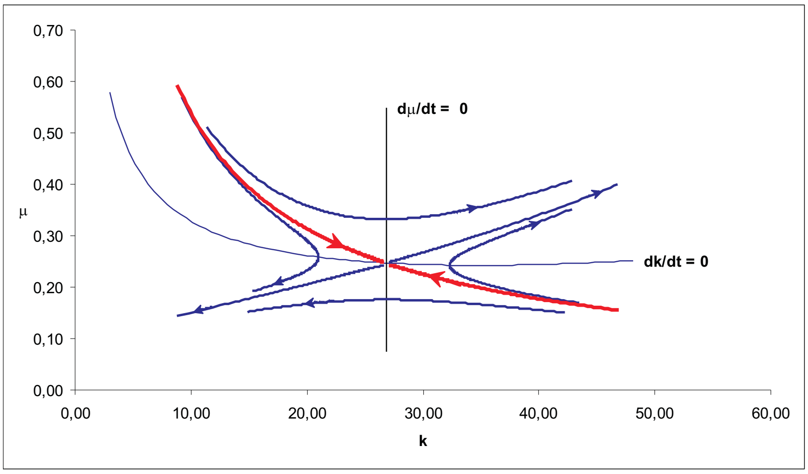
$$\dot{\mu} = (\rho - n)\mu - (f'(k) - \delta - n)\mu = -\mu(f'(k) - \rho - \delta)$$

and hence the system

$$\begin{aligned} \dot{\mu} &= -\mu(f'(k) - \rho - \delta) \\ \dot{k} &= f(k) - (n+\delta)k - \mu^{-1/\theta} \end{aligned}$$

with transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu k = 0$$



Consumption smoothing

$$\max_{c(t)} \int_0^\infty \frac{c^{1-\theta}-1}{1-\theta}\, e^{(n-\rho)t}\, dt$$

- A large θ means a strong aversion against intertemporal variation of consumption

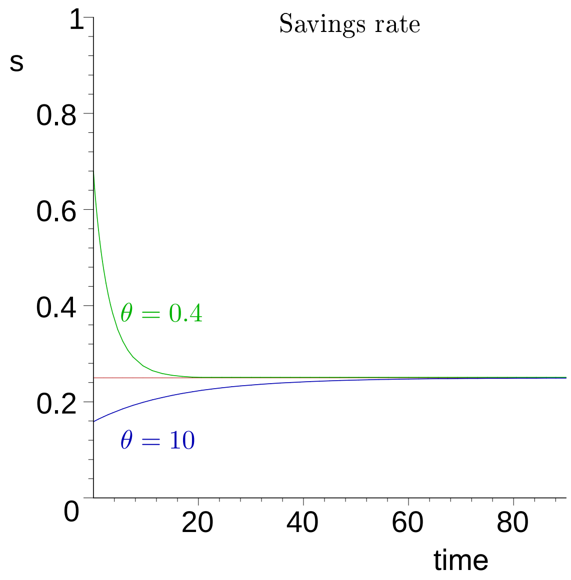
Numerical simulation of the model with the following parameters

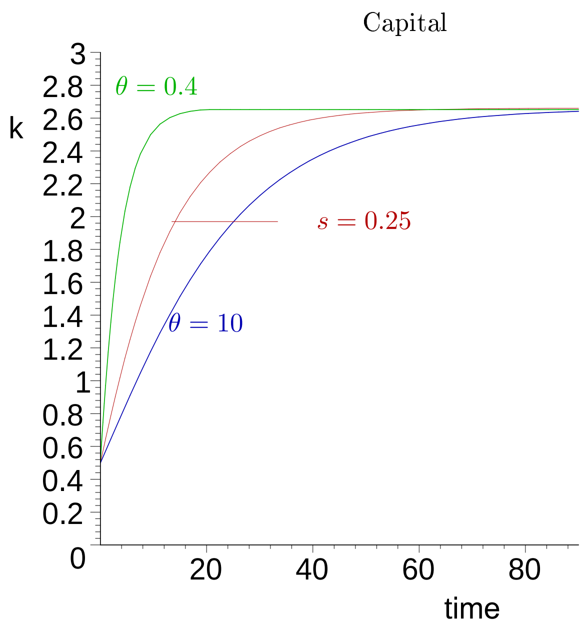
$$\alpha = 0.25\,, \;\; n = 0.01\,, \;\; \delta = 0.1\,, \;\; \rho = 0.02\,, \;\; \theta = 0.4\,, \theta = 10 \text{ resp.}$$

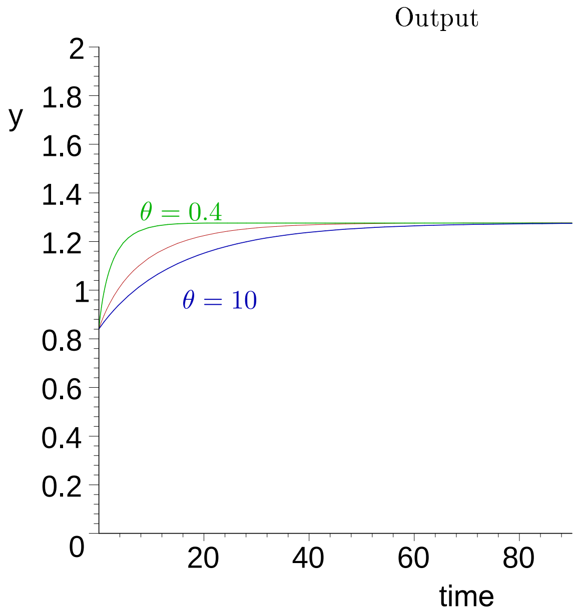
Below we demonstrate the role of the intertemporal elasticity of substitution by comparing simulations for the two values of θ and a constant savings ratio.

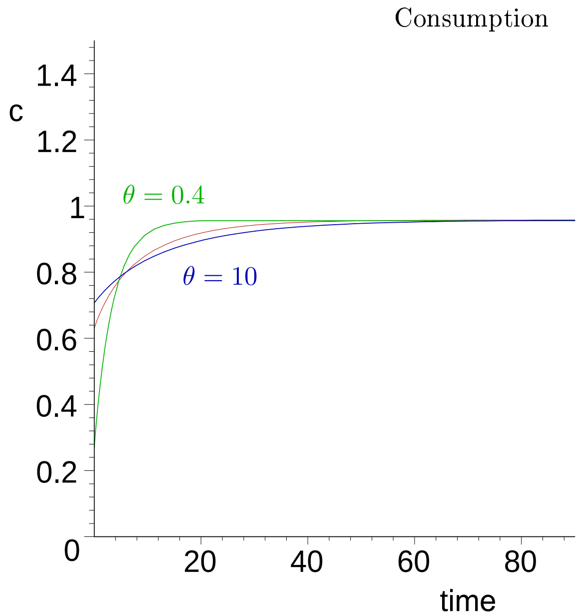
The constant savings ratio is calibrated such that in the long run the same capital intensity k^* is reached with and without intertemporal optimization.

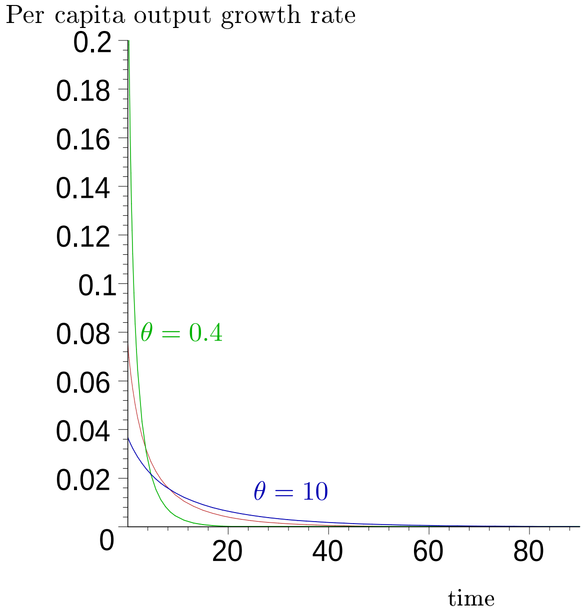
$$\begin{array}{lll} \text{from } \hat{c} = 0 & \text{we get} & (k^*)^{\alpha-1} = (\delta+\rho)/\alpha \\ \text{from } \hat{k} = 0 & \text{we get} & s \cdot (k^*)^{\alpha-1} = n + \delta \\ \text{and hence} & & s = \dfrac{\alpha(\delta+n)}{\delta+\rho} \end{array}$$


Savings rate
s
1
0.8
0.6
0.4
0.2
0
$\theta = 0.4$
$\theta = 10$
20
40
60
80
time


Capital
k
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$\theta = 0.4$
$s = 0.25$
$\theta = 10$
20
40
60
80
time


Output
y
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$\theta = 0.4$
$\theta = 10$
20
40
60
80
time


Consumption
c
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$\theta = 0.4$
$\theta = 10$
20
40
60
80
time



Exogenous technical progress
Without technical progress we had

$$\begin{aligned} F(K,L) &= K^{\alpha}L^{1-\alpha} \\ k &= K/L \\ f(k) &= F(k,1) \end{aligned}$$

In order to augment the model by labor augmenting technical progress we define human capital by $H = E \cdot L = E_0 e^{xt} \cdot L$.

With exogenous technical progress we get

$$\begin{aligned} \tilde{k} &= K/H \quad k = E_0 e^{xt}\tilde{k} \\ F(K,H) &= K^{\alpha}H^{1-\alpha} \\ f(\tilde{k}) &= F(\tilde{k},1) \end{aligned}$$

Exogenous technical progress: The Cobb-Douglas case
Without progress we had

$$F(K,L) = K^{\alpha}L^{1-\alpha}$$

Labor in efficiency units: $H = E \cdot L$

$$F(K,H) = K^{\alpha}H^{1-\alpha}$$

The Cobb-Douglas case in relative terms

$$f(k) = k^{\alpha}$$

or with $k = E\,\tilde{k}$ per efficiency unit of labor

$$f(\tilde{k}) = \tilde{k}^{\alpha}$$

Exogenous technical progress: Details

$$\dot{K} = Y - C - \delta K$$

Use $\tilde{k} = K/EL$ and get

$$\begin{aligned}\frac{\dot{K}}{EL} &= \frac{Y - C - \delta K}{EL} \\ &= \tilde{y} - \tilde{c} - \delta\tilde{k}\end{aligned}$$

Use the time derivative of $\tilde{k}$

$$\begin{aligned}\dot{\tilde{k}} &= \frac{\dot{K}EL - K\dot{E}L - KE\dot{L}}{(EL)^2} \\ &= \frac{\dot{K}}{EL} - \tilde{k}x - \tilde{k}n\end{aligned}$$

Substitute to get the accumulation equation for $\tilde{k}$

$$\dot{\tilde{k}} = \tilde{y} - \tilde{c} - (x + n + \delta)\tilde{k}$$

Set up the Hamiltonian to develop the Keynes -Ramsey rule

$$\mathcal{H} = \frac{\tilde{c}^{1-\theta}E^{1-\theta} - 1}{1-\theta}e^{nt} + \lambda\left(\tilde{y} - \tilde{c} - (x + n + \delta)\tilde{k}\right)$$

Compute and evaluate the FOC's

$$\begin{aligned}\mathcal{H}_{\tilde{c}} = 0 \quad &:: \quad \tilde{c}^{-\theta}E^{1-\theta}e^{nt} = \lambda \\ & \quad\;\; \Rightarrow \hat{\lambda} = -\theta\hat{\tilde{c}} + (1-\theta)x + n \\ \mathcal{H}_{\tilde{k}} = -\dot{\lambda} + \lambda\rho \quad &:: \quad \lambda\left(\alpha\tilde{k}^{\alpha-1} - (x + n + \delta)\right) = -\dot{\lambda} + \lambda\rho \\ & \quad\;\; \Rightarrow \hat{\lambda} = -\alpha\tilde{k}^{\alpha-1} + (x + n + \rho + \delta)\end{aligned}$$

Eliminate $\hat{\lambda}$ and solve for the growth rate of $\tilde{c}$:

$$\begin{aligned}-\theta\hat{\tilde{c}} + (1-\theta)x &= -\alpha\tilde{k}^{\alpha-1} + (x + \rho + \delta) \\ \hat{\tilde{c}} &= \frac{1}{\theta}\left(\alpha\tilde{k}^{\alpha-1} - (\delta + \theta x) - \rho\right)\end{aligned}$$

Dynamic implications of exogenous technical progress

The link between the dynamics of the model with stationary equilibrium and the model with exogenous technical progress is now established. In terms of growth rates it can be demonstrated through the example of k and c

$$\begin{aligned}\tilde{c} = c\,E_0^{-1}e^{-xt},\ \tilde{k} = k\,E_0^{-1}e^{-xt} \quad &\Rightarrow \quad \hat{\tilde{k}} = \hat{k} - x\,,\ \hat{\tilde{c}} = \hat{c} - x \\ \text{or the other way around} & \\ c = \tilde{c}\,E_0e^{xt},\ k = \tilde{k}\,E_0e^{xt} \quad &\Rightarrow \quad \hat{k} = \hat{\tilde{k}} + x\,,\ \hat{c} = \hat{\tilde{c}} + x\end{aligned}$$

Differential equations

$$\begin{aligned}\dot{\tilde{k}} &= \tilde{k}^{\alpha} - \tilde{c} - (n+\delta+x)\tilde{k} \\ \dot{\tilde{c}} &= \frac{1}{\theta}\left(\alpha\tilde{k}^{\alpha-1} - (\delta+\rho+\theta x)\right)\tilde{c}\end{aligned}$$

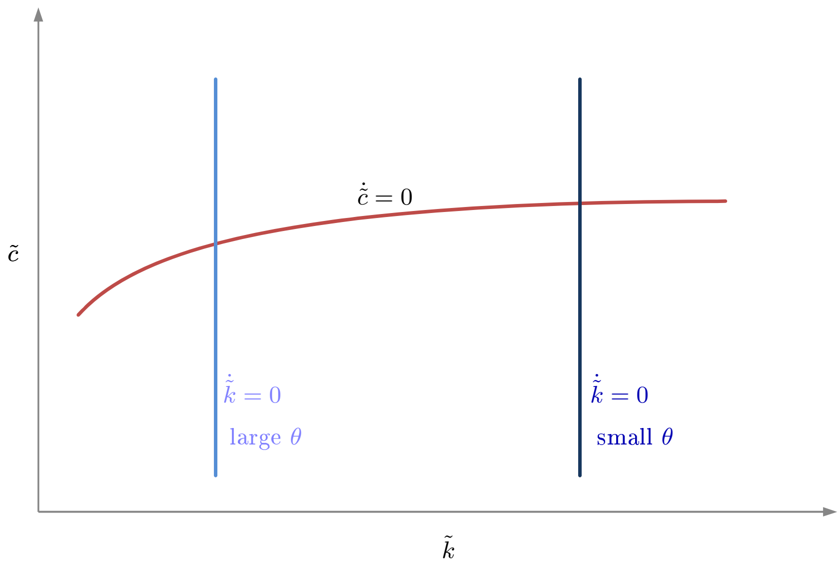
Stationary equilibria and the intertemporal elasticity of substitution

- The stationary value of $\tilde{k}$ depends on θ as $\dot{\tilde{c}}$ depends on θ
- Through $\tilde{k}^*$ the stationary value of $\tilde{c}$ depends on θ too.[2]

Stationary equilibria

$$\begin{aligned}\tilde{k}^* &= \left(\frac{\delta+\rho+\theta x}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ \tilde{c}^* &= \left(\tilde{k}^*\right)^{\alpha} - (n+\delta+x)\tilde{k}^* \\ &= \left(\left(\tilde{k}^*\right)^{\alpha-1} - (n+\delta+x)\right)\tilde{k}^* \\ &= \left[\frac{\delta+\rho+\theta x}{\alpha} - (n+\delta+x)\right]\tilde{k}^*\end{aligned}$$

Different values of θ result in different saddle points. The respective stable manifolds are not shown in the picture below.

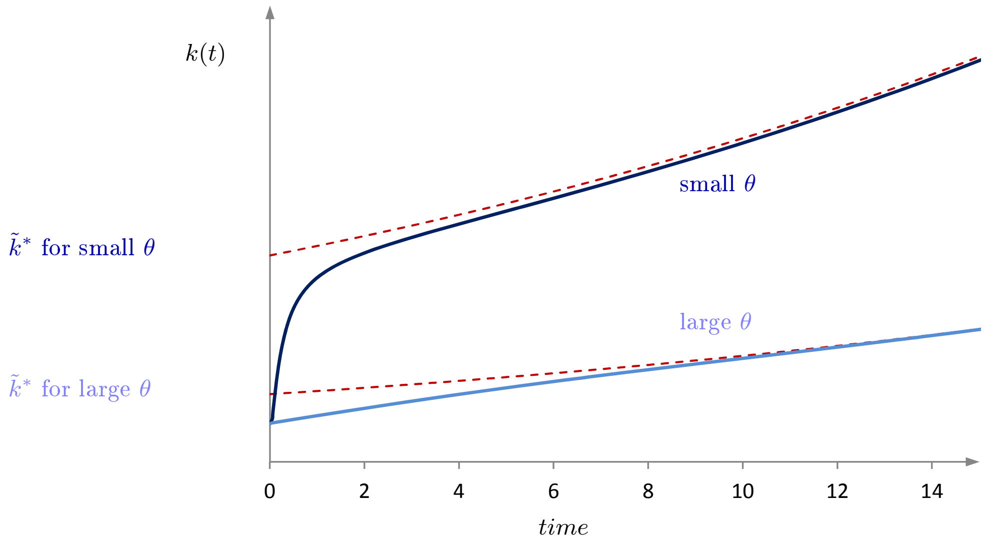


[2]In order that an equilibrium exists ρ must be large enough: $\rho > n + (1-\theta)x$.

Going back to true per capita variables
There is a persistent level effect of growth with different intertemporal rates of substitution due to (exogenous) technical progress. It can be demonstrated by the following inspection of capital intensities k.

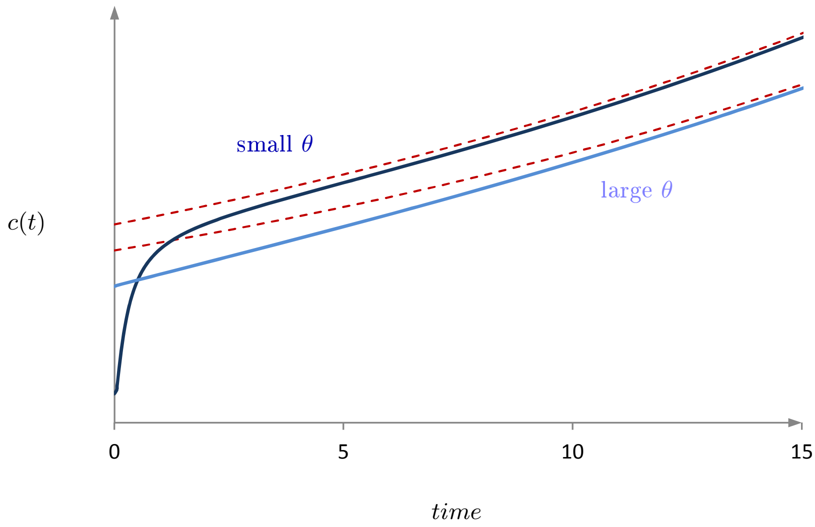
- Assume $E_0 = 1$. A different value of E_0 would only rescale all results.
- At $t = 0$ all variables in efficency units and in per capita units coincide as $E_0 e^{x0} = 1$.
- In particular this holds for $\tilde{k}^*$ and the corresponding $k(t)$ in balanced growth. Recall that $\tilde{k}^*$ is larger if θ is smaller. Now, let time advance continuously. $\tilde{k}(t)$ will stay at the equilibrium level whereas $k(t)$ will start to grow with rate x.

Capital intensity with exogenous progress



The broken lines in red color depict exponential growth with rate x starting from the respective levels $\tilde{k}^*$ for different θ.

Consumption with exogenous progress



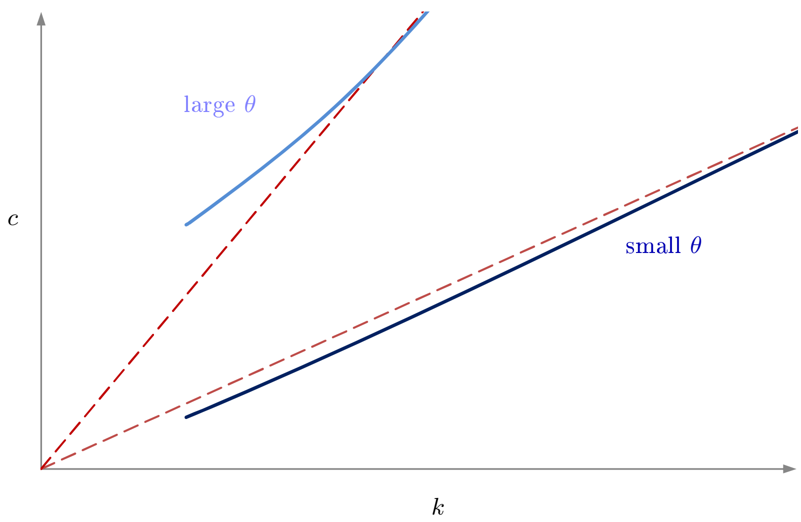
Capital and consumption with exogenous progress
Recall the relation between $\tilde{k}^*$ and $\tilde{c}^*$

$$\tilde{c}^* = \left[\frac{\delta + \rho + \theta x}{\alpha} - (n + \delta + x)\right] \tilde{k}^*$$

Multiplication of both sides with $E_0 e^{xt}$ turns the saddlepoint condition for $(\tilde{k}, \tilde{c})$ into a balanced growth condition for $(k(t), c(t))$.

Notice that we can omit t at the (stationary) saddlepoint, but keep it in the relation for balanced growth (with positive growth rate x)

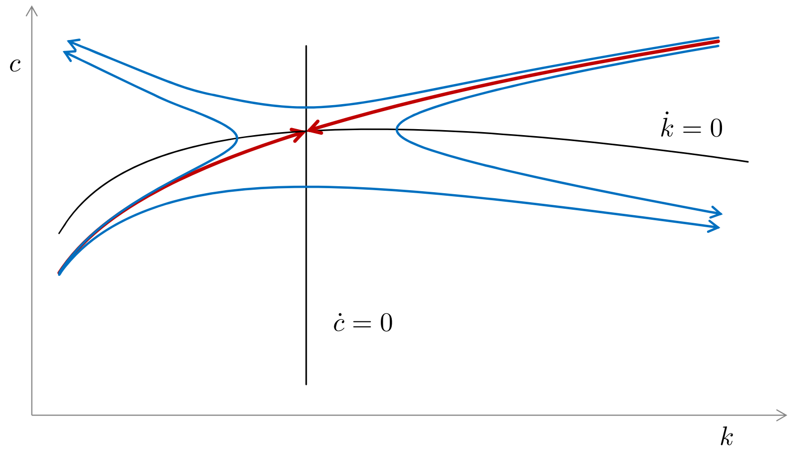
$$c(t) = \left[\frac{\delta + \rho + \theta x}{\alpha} - (n + \delta + x)\right] k(t)$$



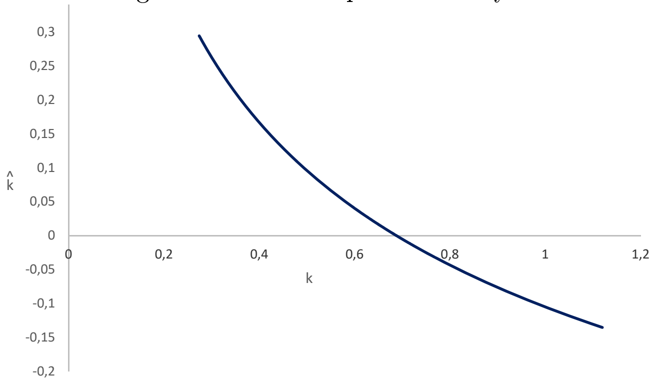
2.2 Returns to Scale and Sustained Growth: The AK-Model

Once more we consider:
Optimal growth without exogenous technical progress Recall the Solow, Swan, Ramsey-Cass-Koopmans model

$$\begin{array}{lrcl}(1) & \dot{k} & = & f(k) - c - (n+\delta)k \\ (2) & \dot{c} & = & \frac{1}{\theta}(f'(k) - \delta - \rho)\, c\end{array}$$



Along the optimal solutions the growth rates of capital intensity are



- The growth rate converge to zero, as no exogenous technical progress is considered.

$$\hat{k} = f(k)/k - c/k - (n+\delta)$$

- The reason: The rate of marginal return to capital converges goes from infinity down to zero, if the capital intensity increases from zero to infinity (Inada-conditions)

The assumptions of the AK-model (Jones und Manuelli 1992)

- Physical capital and human capital are perfect substitutes
$\hookrightarrow$ denote K as aggregate of both kinds of capitals

- the production function:

$$Y = AK \quad A > 0$$

- in intensities

$$y = f(k) = Ak$$

- marginal product of (aggregated) capital

$$f'(k) = A = r + \delta\,, \qquad \text{also } r = A - \delta$$

$\hookrightarrow$ sustainable willingness to invest

Optimal growth
The differential equations of the Ramsey-Cass-Koopmans-model become

$$\begin{array}{lrcl} (1) & \dot{k} & = & (A - n - \delta)k - c \\ (2) & \dot{c} & = & \frac{1}{\theta}(A - \delta - \rho)\,c \end{array}$$

$$\begin{array}{lrcl} (1) & \dot{k} & = & (A - n - \delta)k - c \\ (2) & \dot{c} & = & \frac{1}{\theta}(A - \delta - \rho)\,c \end{array}$$

- γ_c is always constant
- $\gamma_k = A - n - \delta - c/k$ is constant, if and only if
  $\gamma_k = \gamma_c$
- hence

$$\gamma_c = \gamma_k = \gamma_y = \frac{1}{\theta}(A - \delta - \rho)$$

- to get $\gamma_c = \gamma_k$, c must be chosen optimally:

$$\gamma_k = (A - n - \delta) - c/k = \frac{1}{\theta}(A - \delta - \rho)$$

- Hence optimal growth requires

$$(c/k)^* = A - n - \delta - \gamma_k^* = \frac{\theta - 1}{\theta}(A - \delta) + \frac{\rho}{\theta} - n =: \varphi$$

The enhanced AK-model (Jones und Manuelli 1990)

- In the short run there is low substitutability, in the long run perfect substitution
- production function:
$$Y = AK + BK^{\alpha}L^{1-\alpha} \quad A, B > 0$$
- in intensities
$$y = Ak + Bk^{\alpha}$$
- The marginal productivity of the (aggregated) capital converges to the value of the simple AK-model
$\hookrightarrow$ sustainable willingness to invest
- $\gamma_k = f(k)/k - c - (n+\delta) = A - k^{\alpha-1} - n - \delta - c/k$

Further generalisation (Jones und Manuelli 1990)

- The produktion function on basis of constant elasticity of substitution(CES):
$$\begin{aligned} Y &= A\left(a(bK)^{\psi} + (1-a)[(1-b)L]^{\psi}\right)^{1/\psi} \\ & \quad 0 < a, b < 1, \quad \psi < 1 \end{aligned}$$