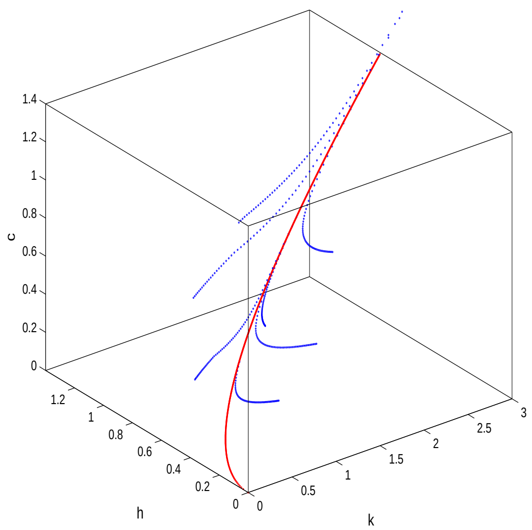
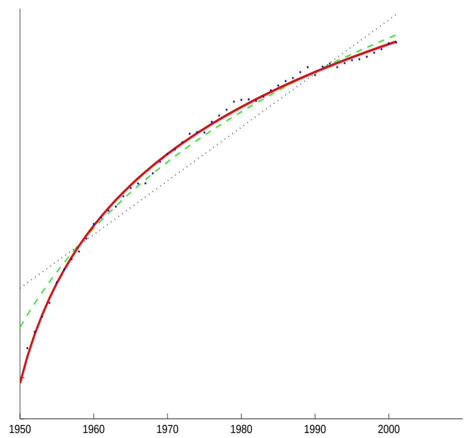
Economic Growth

Master Course WS 2024/25

Prof Dr. Karl-Josef Koch
University of Siegen

$$\max \int_0^\infty \left(\frac{c^{1-\theta} - 1}{1 - \theta} \right) \cdot e^{-\rho t} \, dt$$





These notes are more or less a one two one print version of the slides of the course. The lecture provides a lot of additional information, instructions, and discussions beyond the content of the slides. They are essential although not included in the slides.

Chapter II

Contents

2 Theories of Economic Growth

Robert Solow, who may be regarded as the founder of the research behind this literature, once called his contribution a "parable" and not a theory. In today's diction, I would call it a narrative.

The body of this literature consists of a collection of mathematical models that share a number of basic assumptions, but have different emphases. The results of the analysis of these models point to some connections and mechanisms that are by no means self-evident. In this sense, the contributions deepen and consolidate the understanding of economic growth.

2.1 Capital Accumulation and Maximization of Intertemporal Utility

Optimal growth with exogenous technical progress

This section is based on contributions by

R. Solow (1956), T. Swan (1956), F.P. Ramsey (1928) - D. Cass (1965) - T.C. Koopmans (1965)

Key features are

- Closed economy
- Saving equals investment
- Neoclassical production function
- Exogenous technical progress
- Intertemporal utility maximization

There are three fundamental dynamic effects assumed in this kind of model. Factor supply is changed over time by

- exogenous population growth,
- accumulation of physical capital, and
- exogenous human capital growth.

The three dynamic elements interact and jointly change GDP per capita.

Several issues are addressed by this kind of model

- Accumulation of physical capital changes the endogenous capital intensity, the relation between capital and labor.
- Population growth and exogenous human capital growth affect the capital intensity as well.
- How do savings affect GDP per capita?
- What determines the propensity to save?

One can also discuss balance, stability, and convergence.

The neoclassical macroeconomic production function

$$Y = F(K, L) \qquad \text{e.g. } Y = K^{\alpha} L^{1-\alpha}$$

The marginal product of each factor is positive and decreasing

$$\begin{array}{rcl} \partial F/\partial K > 0 & , & \partial^2 F/\partial K^2 < 0 \\ \partial F/\partial L > 0 & , & \partial^2 F/\partial L^2 < 0 \end{array}$$

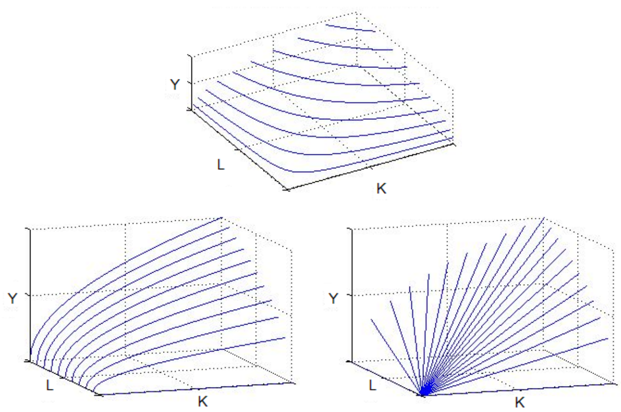
F is linear homogeneous (constant returns to scale)

$$F(\lambda K, \lambda L) = \lambda \cdot F(K, L)$$

and satisfies the Inada conditions

$$\begin{array}{rcl} \lim_{K\to 0}(F_K) & = & \lim_{L\to 0}(F_L) \;=\; \infty \\ \lim_{K\to\infty}(F_K) & = & \lim_{L\to\infty}(F_L) \;=\; 0 \end{array}$$

A neoclassical production function
The pictures below illustrate the properties of a neoclassical production function by curves of constant output, curves of constant input of labor L, and of lines of constant ratios of inputs



The model in terms of intensities
The capital intensity k is the ratio of capital over labor

$$k = K/L$$

Constant returns to scale give rise to expressing output as a function of capital intensities

$$Y = F(K, L) = L \cdot F(K/L, 1) = L \cdot F(k, 1) = L \cdot f(k)$$

Hence, production per capita is given by

$$y = Y/L = f(k)$$

Example: Cobb-Douglas production

$$Y = F(K,L) = K^{\alpha} L^{1-\alpha}$$

yields

$$Y/L = K^{\alpha} L^{-\alpha} = (K/L)^{\alpha}$$

$$\hookrightarrow \quad y = f(k) = k^{\alpha}$$

Marginal productivities

$$\frac{\partial F(K,L)}{\partial K} = \frac{d}{dK}(L \cdot f(K/L)) = L \cdot f'(k)\frac{1}{L} = f'(k)$$

and

$$\frac{\partial F(K,L)}{\partial L} = f(k) - L \cdot f'(k)\frac{K}{L^2} = f(k) - k \cdot f'(k)$$

Hence the following identity holds

$$k \cdot F_K(K,L) + F_L(K,L) = f(k)$$

Example:

$$f(k) = k^{\alpha} \quad \hookrightarrow \quad F_K(K,L) = f'(k) = \alpha k^{\alpha-1} = \alpha f(k)/k$$

$$\hookrightarrow \quad F_L(K,L) = f(k) - \alpha f(k) = (1-\alpha) f(k)$$

Profit maximization

Wage rate w, interest rate r; commodity price index $p = 1$

$$\max_{K,L} \Pi = F(K,L) - (r+\delta) \cdot K - w \cdot L$$

is equivalent to

$$\max_{k,L} \Pi = L\,(f(k) - (r+\delta) \cdot k - w)$$

The solution in terms of L is not determined, but in terms of k

$$\hookrightarrow \quad f'(k) = (r+\delta)$$

Hence r determines the capital intensity k.

For a market equilibrium $\Pi = 0$ must hold, otherwise L equals 0 or ∞

$$\hookrightarrow \quad w = f(k) - (r+\delta)k = f(k) - kf'(k)$$

Maximizing utility

Let $c = C/L$ denote per capita consumption, $u(c)$ utility of c, ρ the discou nt rate of utility, and n the population growth rate. Assume $\rho > n$.

The objective function of consumers is

$$\int_0^{\infty} u(c)\, e^{-(\rho-n)t}\, dt$$

For technical reasons we consider a special case of a utility function: $CRRA$[1]

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$$

Assume: $\theta > 0$.

Note that $u(c) \to \ln(c)$ as $\theta \to 1$.

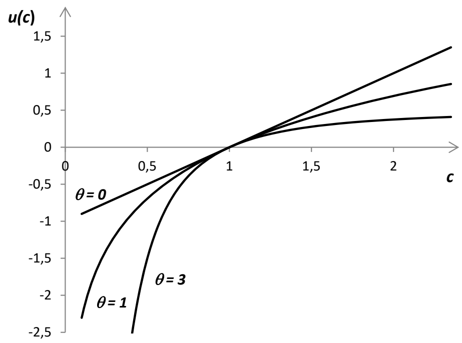
[1] $CRRA$ stands for constant relative risk aversion in expected utility theory. The concept goes back to Arrow (1965) and Pratt(1964)

Intertemporal elasticity of substitution
$\theta = -u''(c) \cdot c/u'(c)$ is called the *relative risk aversion* in the context of decisions under uncertainty.

$$\theta \ \text{large} \longrightarrow \begin{array}{l}\text{strong aversion against}\\ \text{variation of consumption over time.}\end{array}$$

$\sigma = 1/\theta$ is the **intertemporal elasticity of substitution**

Intertemporal elasticity of substitution and smooth consumption
The utility function we consider is strictly concave for $\theta > 0$ and linear in the limit for $\theta \to 0$.



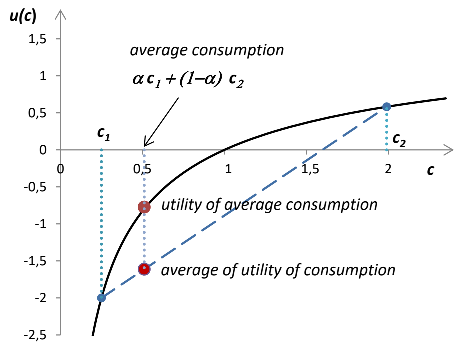
In case of only two periods instead of a time continuum it is obvious what θ implies for average and dispersion of consumption.

For $\theta > 0$ we see

$$c_1 \neq c_2 \quad \Longrightarrow \quad \alpha u(c_1) + (1-\alpha)u(c_2) < u(\alpha c_1 + (1-\alpha)c_2)$$

Average utility of consumption is smaller than utility of average consumption if there is dispersion. In other words, consumers dislike variation of consumption over time.

and smooth consumption



Accumulation of wealth through savings
Let A be the total amount of assets held by households at some time, w the wage rate, r the interest rate, and n the rate of population growth.

The change of assets holdings is equal to total net savings:

$$\dot{A} = wL + rA - C \qquad \text{together with } a = A/L \text{ yields}$$

$$\dot{a} = w + ra - c - na$$

Indeed, from

$$\hat{A} = wL/A + r - C/A = w/a + r - c/a$$

together with $\hat{a} = \hat{A} - n$ we get the desired result.

Intertemporal utility maximization

$$\max_{c(t)} \quad \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta}\, e^{-(\rho - n)t}\, dt$$

subject to the dynamic constraint for a
i.e. $\dot{a} = w + ra - c - na$
and initial condition $a(0) = a_0$

- Notice that we assume $L_0 = 1$ without loss of generality.
- $\rho > 0$ is the *rate of time preference.*

Solution technique: The Maximum Principle of Pontryagin
Define the Hamiltonian function (in current value form)

$$\mathcal{H} = u(c)e^{nt} \; + \; \lambda(w + (r - n)a - c)$$

$\mathcal{H}$ is a function

- of the state variable a,
- the control variable (co-state variable) c, and
- the shadow price λ.

The Maximum Principle yields first order conditions and a transversality condition.

1. $\mathcal{H}_c = 0$ — maximum property
2. $\mathcal{H}_a = -\dot{\lambda} + \lambda\rho$ — Euler equation
3. $\lim_{t\to\infty} e^{-\rho t}\lambda\, a = 0$ — transversality condition

- We know the differential equation for a and initial condition a_0.
- The evolution of a depends on c
- We look for the differential equation for c and the initial condition c_0.
- It is not necessary to determine the evolution of the shadow price λ.

yields

1. $\mathcal{H}_c = c^{-\theta} e^{nt} \; - \; \lambda = 0$ maximum property
2. $\mathcal{H}_a = \lambda\,(r - n) = -\dot{\lambda} + \lambda\rho$ Euler equation
3. $\lim_{t\to\infty} e^{-\rho t} \lambda\, a = 0$ transversality condition

Differentiate (1) with respect to t:

4. $-\theta c^{-(1+\theta)}\,\dot{c}\,e^{nt} \; + \; c^{-\theta} n\, e^{nt} - \dot{\lambda} = 0$

Substitute $-\dot{\lambda}$ from (2) :

5. $-\theta c^{-(1+\theta)}\,\dot{c}\,e^{nt} \; + \; c^{-\theta} n\, e^{nt} \; + \; \lambda(r - n - \rho) = 0$

Substitute $c^{-\theta}\,e^{nt}$ from (1) :

6. $\left(-\theta c^{-1}\,\dot{c} \; + \; (r - \rho)\right)\lambda = 0$

The Keynes-Ramsey-Rule

As λ is positive the last line simplifies to

$$\hat{c} = \frac{1}{\theta}(r - \rho) \qquad \text{the Keynes-Ramsey-Rule}$$

Equilibrium

There is only one asset households can use to invest their savings in: $a = k$ at any point of time.

$$\dot{a} = w + ra - c - na$$

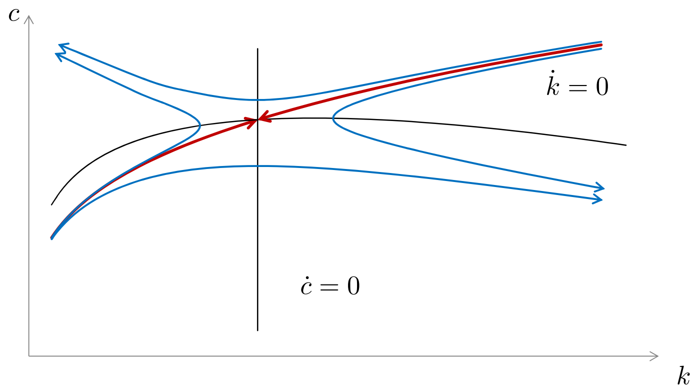
Together with $w + rk = f(k) - \delta k$ this turns into

(1) $\quad \dot{k} = f(k) - c - (n + \delta)k$

The Keynes-Ramsey-rule appears to be

(2) $\quad \dot{c} = \frac{1}{\theta}(f'(k) - \delta - \rho)\, c$

Phase diagram



Technical side notes

Current value and present value form of the Hamiltonian function

The present value form of the Hamiltonian is linked to the current value form by a transformation of coordinates.

$$\mathcal{H}^{present} = e^{-\rho t}\,\mathcal{H}^{current}$$

The shadow price in present values ν turns into the current value shadow price λ by $\lambda = e^{\rho t}\nu$. The derivatives with respect to time are

$$\begin{aligned}\dot{\lambda} &= \rho e^{\rho t}\,\nu + e^{\rho t}\dot{\nu}\\ &= \rho\lambda + e^{\rho t}\dot{\nu}\\ \dot{\lambda} - \rho\lambda &= e^{\rho t}\dot{\nu}\end{aligned}$$

From

$$\frac{\partial\mathcal{H}^{current}}{\partial x} = \dot{\lambda} - \rho\lambda = e^{\rho t}\dot{\nu} \text{ together with } \frac{\partial\mathcal{H}^{present}}{\partial x} = e^{-\rho t}\,\frac{\partial\mathcal{H}^{current}}{\partial x}$$

we get the first order condition for a control variable x from the present value Hamiltonian

$$\frac{\partial\mathcal{H}^{present}}{\partial x} = -\dot{\nu}$$

Transversality in the Ramsey-Cass-Koopmans Model

In terms of the present value shadow price $\nu = e^{-\rho t}\lambda$ the transversality condition is given by

$$\lim_{t\to\infty} a(t)\nu(t) = 0$$

The Euler equation induces a change of ν of form

$$\dot{\nu} = -(r(t) - n)\nu$$

Integration of the Euler equation yields

$$\nu(t) = \nu(0)e^{-\int_0^\infty (r(\tau)-n)d\tau}$$

$\nu(0)$ is equal to $c(0)^{-\theta}$ due to the maximum property. So it is a positive constant and irrelevant for the validity of the transversality condition.

Using the average interest rate

$$\bar{r}(t) = \frac{1}{t}\int_0^\infty r(\tau)d\tau$$

the transversality condition finally reduces to

$$\lim_{t\to\infty} a(t)e^{-(\bar{r}(t)-n)\,t} = 0$$

I.e. in the long run per capita wealth has to grow with a rate smaller than $\bar{r} - n$. We may evaluate the transversality condition explicitly

$$\begin{aligned}\dot{\nu} &= -\nu\,(r-n)\\ &= -\nu\,(f'(k) - \delta - n)\\ \text{and hence}\\ \hat{\nu} &= -(f'(k) - \delta - n)\end{aligned}$$

In the capital accumulation equation we use the maximum property $c^{-\theta}\,e^{-(\rho-n)t} - \nu = 0$ to eliminate c. The time scaled shadow price $\mu = e^{(\rho-n)t}\nu$ with property $\hat{\mu} = (\rho - n) + \hat{\nu}$ makes the dynamics even more transparent.

$$c^{-\theta} = e^{(\rho-n)t}\,\nu = \mu$$

$$\begin{aligned}\dot{k} &= f(k) - c - (n+\delta)k \\ &= f(k) - \mu^{-1/\theta} - (n+\delta)k\end{aligned}$$

We have

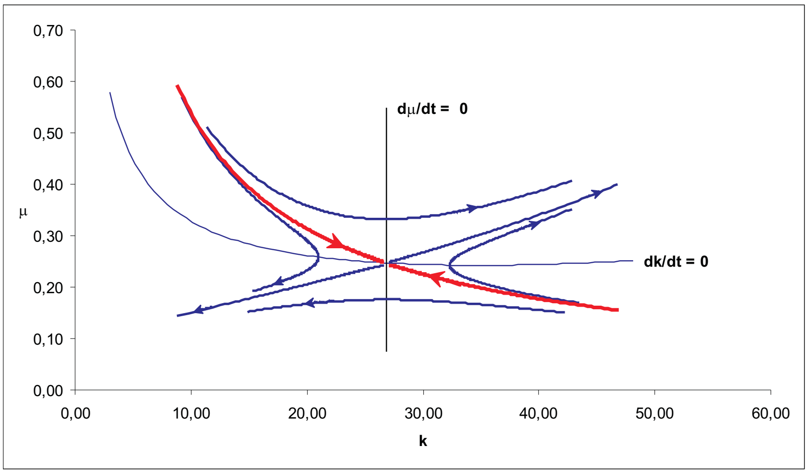
$$\dot{\mu} = (\rho - n)\mu - (f'(k) - \delta - n)\mu = -\mu(f'(k) - \rho - \delta)$$

and hence the system

$$\begin{aligned}\dot{\mu} &= -\mu(f'(k) - \rho - \delta) \\ \dot{k} &= f(k) - (n+\delta)k - \mu^{-1/\theta}\end{aligned}$$

with transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu k = 0$$



Consumption smoothing

$$\max_{c(t)} \int_0^\infty \frac{c^{1-\theta} - 1}{1-\theta} e^{(n-\rho)t}\, dt$$

- A large θ means a strong aversion against intertemporal variation of consumption

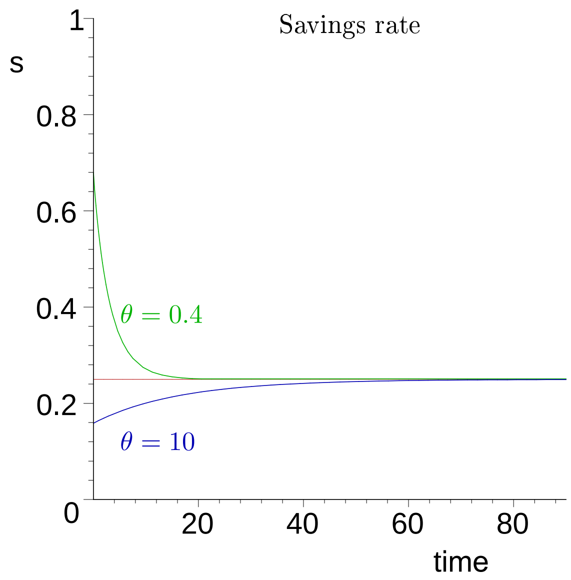
Numerical simulation of the model with the following parameters

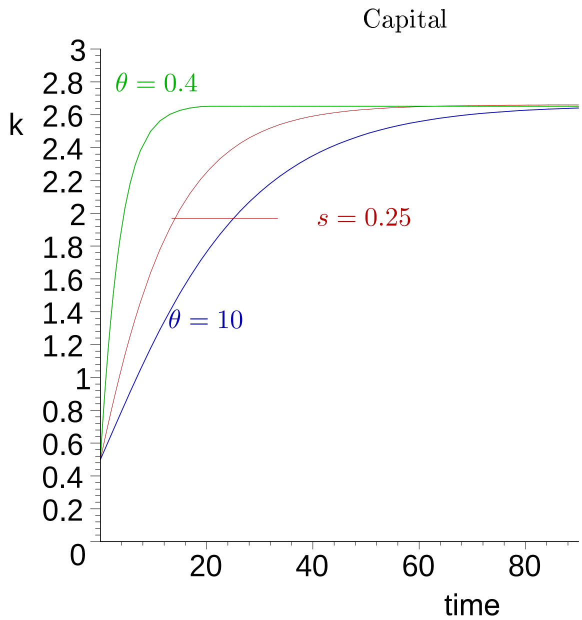
$$\alpha = 0.25\,, \quad n = 0.01\,, \quad \delta = 0.1\,, \quad \rho = 0.02\,, \quad \theta = 0.4\,, \theta = 10 \text{ resp.}$$

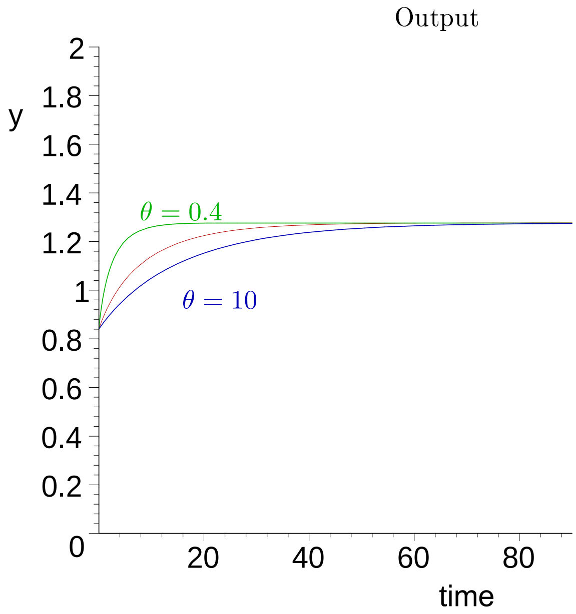
Below we demonstrate the role of the intertemporal elasticity of substitution by comparing simulations for the two values of θ and a constant savings ratio.

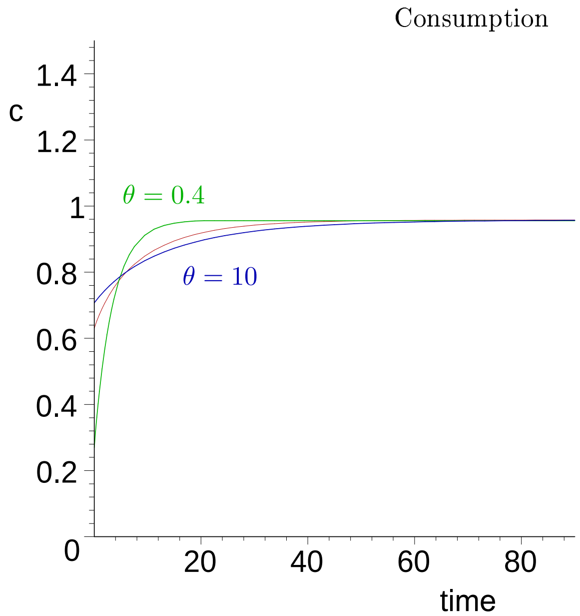
The constant savings ratio is calibrated such that in the long run the same capital intensity k^* is reached with and without intertemporal optimization.

$$\begin{aligned}&\text{from } \hat{c} = 0 \quad \text{we get} \quad (k^*)^{\alpha-1} = (\delta+\rho)/\alpha \\ &\text{from } \hat{k} = 0 \quad \text{we get} \quad s \cdot (k^*)^{\alpha-1} = n + \delta \\ &\text{and hence} \qquad\qquad s = \frac{\alpha(\delta+n)}{\delta+\rho}\end{aligned}$$

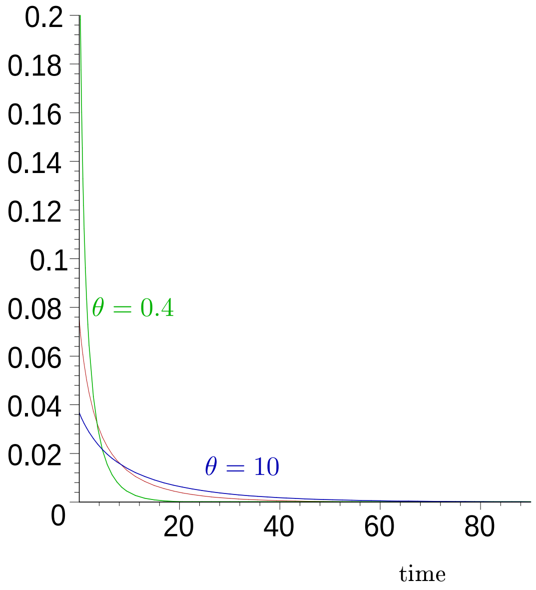

Savings rate
s
1
0.8
0.6
0.4
0.2
0
20
40
60
80
time
θ = 0.4
θ = 10


Capital
k
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
20
40
60
80
time
θ = 0.4
s = 0.25
θ = 10


Output
y
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
20
40
60
80
time
θ = 0.4
θ = 10


Consumption
c
1.4
1.2
1
0.8
0.6
0.4
0.2
0
20
40
60
80
time
θ = 0.4
θ = 10

Per capita output growth rate



Exogenous technical progress
Without technical progress labor as an input factor was simply measured by the number of persons employed

$$\begin{aligned} F(K,L) &= K^\alpha L^{1-\alpha} \\ k &= K/L \\ f(k) &= F(k,1) \end{aligned}$$

In order to add labor augmenting technical progress to the model we define human capital by $H = E \cdot L = E_0 e^{xt} \cdot L$.

E is a scale factor measuring the efficiency of labor and is growing with exogenous rate x. We get

$$\begin{aligned} \tilde{k} &= K/H = k/E = E_0^{-1} e^{-xt} k \\ F(K,H) &= K^\alpha H^{1-\alpha} \\ f(\tilde{k}) &= F(\tilde{k},1) \end{aligned}$$

Exogenous technical progress: The Cobb-Douglas case
Without progress we had

$$F(K,L) = K^\alpha L^{1-\alpha}$$

Labor in efficiency units: $H = E \cdot L$

$$F(K,H) = K^\alpha H^{1-\alpha}$$

The Cobb-Douglas case in relative terms

$$f(k) = k^\alpha$$

or with $k = E\,\tilde{k}$ per efficiency unit of labor

$$f(\tilde{k}) = \tilde{k}^\alpha$$

Exogenous technical progress: Details

$$\dot{K} = Y - C - \delta K$$

Use $\tilde{k} = K/EL$ and get

$$\begin{aligned} \frac{\dot{K}}{EL} &= \frac{Y - C - \delta K}{EL} \\ &= \tilde{y} - \tilde{c} - \delta \tilde{k} \end{aligned}$$

Use the time derivative of $\tilde{k}$

$$\begin{aligned} \dot{\tilde{k}} &= \frac{\dot{K}EL - K\dot{E}L - KE\dot{L}}{(EL)^2} \\ &= \frac{\dot{K}}{EL} - \tilde{k}x - \tilde{k}n \end{aligned}$$

Substitute to get the accumulation equation for $\tilde{k}$

$$\dot{\tilde{k}} = \tilde{y} - \tilde{c} - (x + n + \delta)\tilde{k}$$

Set up the Hamiltonian to develop the Keynes - Ramsey rule

$$\mathcal{H} = \frac{\tilde{c}^{1-\theta}E^{1-\theta} - 1}{1-\theta} e^{nt} + \lambda \left(\tilde{y} - \tilde{c} - (x + n + \delta)\tilde{k} \right)$$

Compute and evaluate the FOC's

$$\begin{aligned} \mathcal{H}_{\tilde{c}} = 0 \quad &:: \quad \tilde{c}^{-\theta} E^{1-\theta} e^{nt} = \lambda \\ & \quad\;\; \Rightarrow \hat{\lambda} = -\theta \hat{\tilde{c}} + (1-\theta)x + n \\ \mathcal{H}_{\tilde{k}} = -\dot{\lambda} + \lambda\rho \quad &:: \quad \lambda \left(\alpha \tilde{k}^{\alpha-1} - (x + n + \delta) \right) = -\dot{\lambda} + \lambda\rho \\ & \quad\;\; \Rightarrow \hat{\lambda} = -\alpha \tilde{k}^{\alpha-1} + (x + n + \rho + \delta) \end{aligned}$$

Eliminate $\hat{\lambda}$ and solve for the growth rate of $\tilde{c}$:

$$\begin{aligned} -\theta \hat{\tilde{c}} + (1-\theta)x &= -\alpha \tilde{k}^{\alpha-1} + (x + \rho + \delta) \\ \hat{\tilde{c}} &= \frac{1}{\theta} \left(\alpha \tilde{k}^{\alpha-1} - (\delta + \theta x) - \rho \right) \end{aligned}$$

Dynamic implications of exogenous technical progress

The link between the dynamics of the model with stationary equilibrium and the model with exogenous technical progress is now established. In terms of growth rates it can be demonstrated through the example of k and c

$$\tilde{c} = c\, E_0^{-1} e^{-xt},\ \tilde{k} = k\, E_0^{-1} e^{-xt} \quad \Rightarrow \quad \hat{\tilde{k}} = \hat{k} - x\,,\ \hat{\tilde{c}} = \hat{c} - x$$

or the other way around

$$c = \tilde{c}\, E_0 e^{xt},\ k = \tilde{k}\, E_0 e^{xt} \quad \Rightarrow \quad \hat{k} = \hat{\tilde{k}} + x\,,\ \hat{c} = \hat{\tilde{c}} + x$$

Differential equations

$$\begin{aligned}\dot{\tilde{k}} &= \tilde{k}^{\alpha} - \tilde{c} - (n+\delta+x)\tilde{k} \\ \dot{\tilde{c}} &= \frac{1}{\theta}\left(\alpha\tilde{k}^{\alpha-1} - (\delta+\rho+\theta x)\right)\tilde{c}\end{aligned}$$

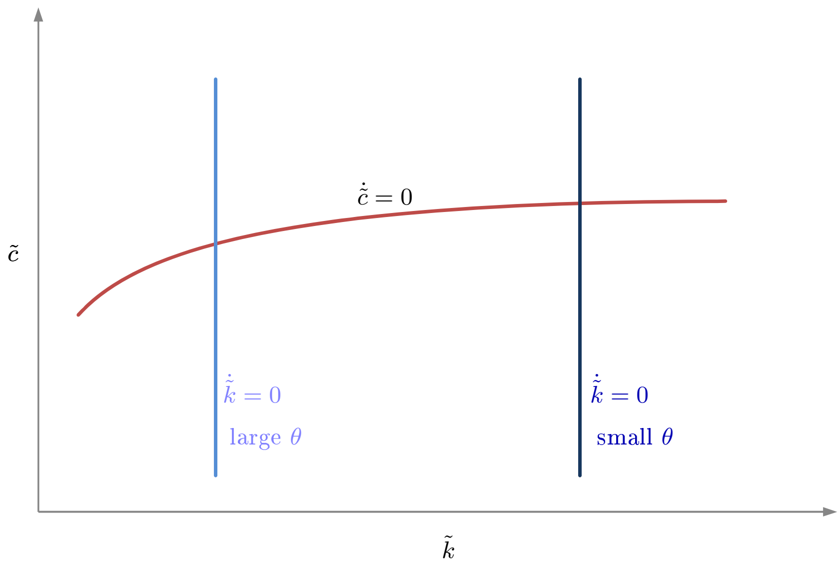
Stationary equilibria and the intertemporal elasticity of substitution

- The stationary value of $\tilde{k}$ depends on θ as $\dot{\tilde{c}}$ depends on θ
- Through $\tilde{k}^*$ the stationary value of $\tilde{c}$ depends on θ too.[2]

Stationary equilibria

$$\begin{aligned}\tilde{k}^* &= \left(\frac{\delta+\rho+\theta x}{\alpha}\right)^{\frac{1}{\alpha-1}} \\ \tilde{c}^* &= \left(\tilde{k}^*\right)^{\alpha} - (n+\delta+x)\tilde{k}^* \\ &= \left(\left(\tilde{k}^*\right)^{\alpha-1} - (n+\delta+x)\right)\tilde{k}^* \\ &= \left[\frac{\delta+\rho+\theta x}{\alpha} - (n+\delta+x)\right]\tilde{k}^*\end{aligned}$$

Different values of θ result in different saddle points. The respective stable manifolds are not shown in the picture below.

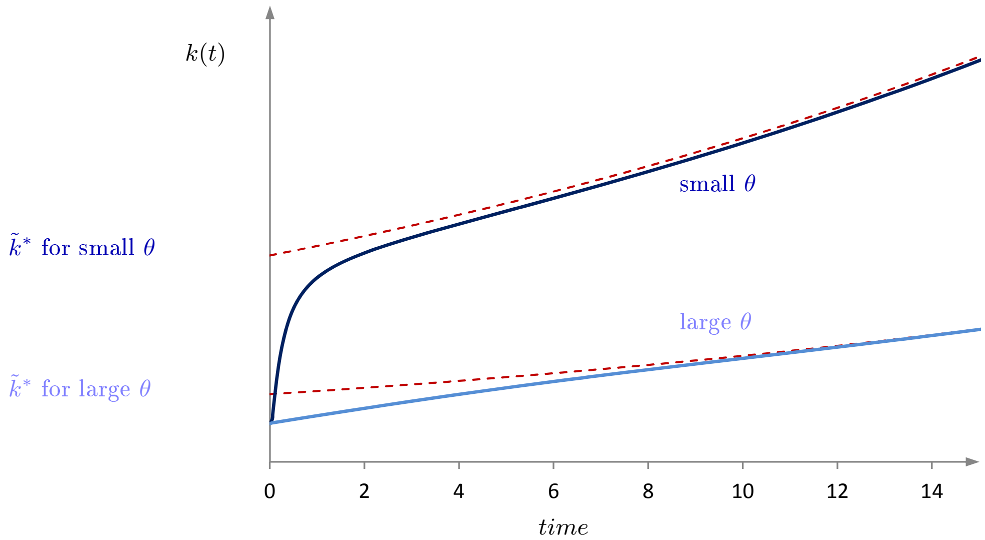


[2]In order that an equilibrium exists ρ must be large enough: $\rho > n + (1-\theta)x$.

Going back to true per capita variables
There is a persistent level effect of growth with different intertemporal rates of substitution due to (exogenous) technical progress. It can be demonstrated by the following inspection of capital intensities k.

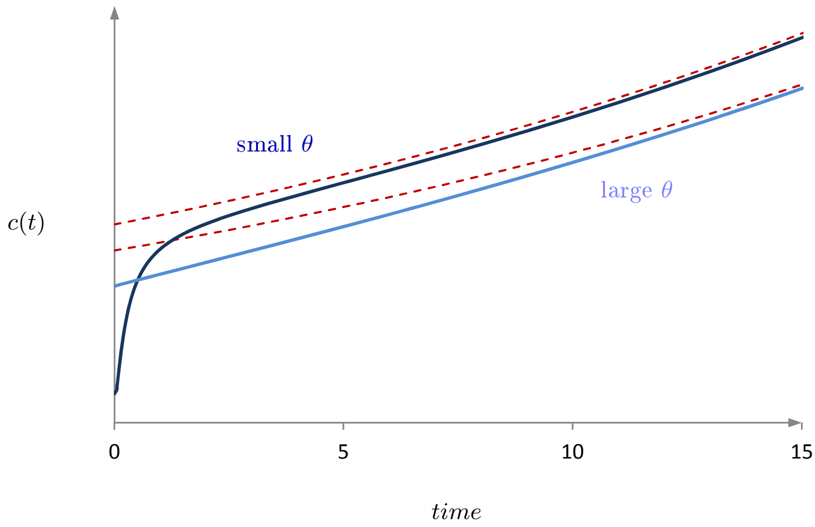
- Assume $E_0 = 1$. A different value of E_0 would only rescale all results.
- At $t = 0$ all variables in efficiency units and in per capita units coincide as $E_0 e^{x0} = 1$.
- In particular this holds for $\tilde{k}^*$ and the corresponding $k(t)$ in balanced growth. Recall that $\tilde{k}^*$ is larger if θ is smaller. Now, let time advance continuously. $\tilde{k}(t)$ will stay at the equilibrium level whereas $k(t)$ will start to grow with rate x.

Capital intensity with exogenous progress



The broken lines in red color depict exponential growth with rate x starting from the respective levels $\tilde{k}^*$ for different θ.

Consumption with exogenous progress



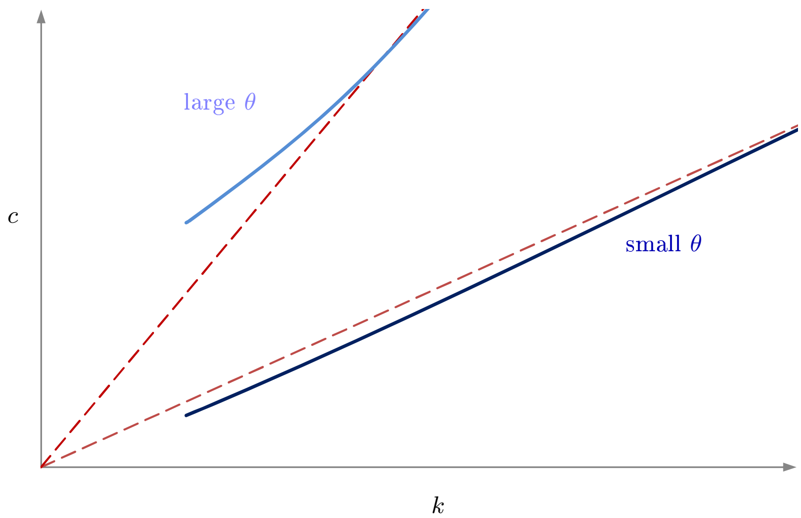
Capital and consumption with exogenous progress
Recall the relation between $\tilde{k}^*$ and $\tilde{c}^*$

$$\tilde{c}^* = \left[\frac{\delta + \rho + \theta x}{\alpha} - (n + \delta + x) \right] \tilde{k}^*$$

Multiplication of both sides with $E_0 e^{xt}$ turns the saddlepoint condition for $(\tilde{k}, \tilde{c})$ into a balanced growth condition for $(k(t), c(t))$.

Notice that we can omit t at the (stationary) saddlepoint, but keep it in the relation for balanced growth (with positive growth rate x)

$$c(t) = \left[\frac{\delta + \rho + \theta x}{\alpha} - (n + \delta + x) \right] k(t)$$



2.2 Human Capital Accumulation: The Approach of Uzawa and Lucas

In his 1957 paper Robert Solow already introduced the idea of exogenous labor augmenting technical progress. We discussed the the importance of this engine of growth in an extension of the Ramsey-Cass-Koopmans model. Lucas(1988) draws on Uzawa (1965) when he endogenizes this idea as a second engine of growth.

The questions to be asked in the analysis of this model will be

- What are the technical implications of scale effects in this model?
- Does time preference play a similar role as in the Ramsey-Cass-Koopmans model?
- How do initial endowments affect transition and long run performance?

Model framework

The model is in particular characterized by the following properties

- no growth of population
- physical capital K and human capital H are accumulated
- constant economies of scale concerning the accumulation of human capital[3]
- economies of scale of physical capital are endogenous and depend on the development of human capital
- u decision of households regarding the application of human capital

$$\begin{aligned} Y &= AK^{\alpha}(uH)^{1-\alpha}, \qquad 0 < \alpha,\, u < 1 \\ \dot{K} &= Y - C - \delta K, \quad K(0) > 0 \\ \dot{H} &= B(1-u)H - \delta H, \qquad B > 0, \end{aligned}$$

$$U_0 = \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt$$

4

$$\mathcal{H} = \frac{c^{1-\theta} - 1}{1-\theta} + \lambda(Y - C - \delta K) + \mu(B(1-u)H - \delta H)$$

Define $L = uH$ to denote labor employed in production measured in efficiency units, and use the following shorthands.

$$\begin{aligned} MPK &= \frac{dY}{dK} = \alpha Y/K = \alpha APK = \alpha(uH/K)^{1-\alpha} \\ MPL &= \frac{dY}{dL} = (1-\alpha)Y/uH \end{aligned}$$

The marginal productivity of a factor is equal to the product of elasticity and average productivity.

[3]Lucas allows for externalities of individual human capital on the aggregate human capital without changing the outcome of the model substantially.

[4]Notice that we can drop the size of the population from the objective function because there is no population growth in this model.

The maximum principle (current value)

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} + \lambda(Y - C - \delta K) + \mu(B(1-u)H - \delta H)$$

$$\left.\begin{array}{ll} (1) & \mathcal{H}_c = 0 \hookrightarrow c^{-\theta} = \lambda \\ (2) & \mathcal{H}_u = 0 \hookrightarrow \lambda MPL \cdot H - \mu B \cdot H = 0 \end{array}\right\} \text{Max. property}$$

$$\left.\begin{array}{ll} (3) & \mathcal{H}_k = -\dot{\lambda} + \lambda\rho \quad \hookrightarrow \\ & \quad \lambda(MPK - \delta) = -\dot{\lambda} + \lambda\rho \\ & \quad \hat{\lambda} = \rho + \delta - MPK \\ (4) & \mathcal{H}_H = -\dot{\mu} + \mu\rho \quad \hookrightarrow \\ & \quad \lambda MPL \cdot u + \mu(B(1-u) - \delta) = -\dot{\mu} + \mu\rho \\ & \quad \hat{\mu} = \rho + \delta - B(1-u) - \frac{\lambda}{\mu} MPL \cdot u \end{array}\right\} \text{Euler-eq.}$$

with (2) we get

(2′)
$$\frac{\lambda}{\mu} MPL = B$$

and hence with (4)

(4′)
$$\hat{\mu} = \rho + \delta - B(1-u) - Bu = \rho + \delta - B$$

Furthermore, computing growth rates in (1) we get

$$\hat{\lambda} = -\theta \hat{C}$$

Finally, from (3)

$$\hat{C} = \frac{1}{\theta}(MPK - \delta - \rho) \qquad \text{Keynes-Ramsey rule}$$

The (long run) marginal productivity of capital does not only depend on a certain value of u^*. To determine u^*, the differential equation of u is needed.

Calculation of $\hat{u}$

Turn (2') into growth rates

$$\hat{\lambda} - \hat{\mu} + \widehat{MPL} = 0$$

Then recall (3) and (4') and combine them to get another expression for the difference of shadow price growth rates. Alltogether this yields:

$$MPK - B = \widehat{MPL}$$

Turn MPL into growth rates

$$\begin{aligned} \widehat{MPL} &= \hat{Y} - \hat{u} - \hat{H} \\ &= \alpha\hat{K} + (1-\alpha)\hat{u} + (1-\alpha)\hat{H} - \hat{u} - \hat{H} \\ &= \alpha\hat{K} - \alpha\hat{u} - \alpha\hat{H} \\ &= MPK - \alpha C/K - \alpha\delta - \alpha\hat{u} - \alpha(B(1-u) - \delta) \end{aligned}$$

Hence

$$\begin{aligned} MPK - B &= MPK - \alpha C/K - \alpha\delta - \alpha\hat{u} - \alpha(B(1-u) - \delta) \\ \alpha\hat{u} &= B - \alpha B(1-u) - \alpha C/K \\ \hat{u} &= \frac{B}{\alpha} - B(1-u) - C/K \end{aligned}$$

The growth rates of k, h, c and u
We get the following growth rates in per capita terms[5]

$$
\begin{aligned}
\hat{k} &= APK - c/k - \delta \\
\hat{h} &= B(1-u) - \delta \\
\hat{c} &= \frac{1}{\theta}(MPK - \delta - \rho) \\
\hat{u} &= \frac{B}{\alpha} - B(1-u) - c/k
\end{aligned}
$$

Along the balanced growth path we conclude

- $\hat{c} =$ constant implies MPK, and hence APK are constant.
- $\hat{k} =$ as well as APK are constant implies that $\hat{k} = \hat{c}$.
- This implies $u =$ constant, i.e. $\hat{u} = 0$.
- Together with $APK =$ constant this yields $\hat{k} = \hat{h}$.

The balanced growth path is a straight line as $\hat{k} = \hat{h} = \hat{c}$[6]

Determination of the common rate of balanced growth γ of k, h and c

$$
\begin{aligned}
0 &= \frac{B}{\alpha} - B(1-u) - \frac{c}{k} && \text{(from } \hat{u}) \\
&= \frac{B}{\alpha} - \gamma - \delta - \frac{c}{k} && \text{(from } \hat{h}) \\
&= \frac{B}{\alpha} - APK \qquad \text{or} \quad B = \alpha APK && \text{(from } \hat{k})
\end{aligned}
$$

the rate of return on human capital investments is equal
to the rate of return on investments in physical capital

$$\gamma = \frac{1}{\theta}(B - \delta - \rho) \qquad \text{(from } \hat{c})$$

Computation of u^*
Solve $\hat{h} = \gamma$ for $1 - u^*$ and substitute γ:

$$
\begin{aligned}
1 - u^* &= \frac{1}{B}(\gamma + \delta) \\
&= \frac{1}{B}\left(\frac{1}{\theta}(B - \delta - \rho) + \delta\right) \\
&= \frac{1}{\theta} - \frac{1}{\theta B}(\rho + (1-\theta)\delta)
\end{aligned}
$$

Computation of $(c/k)^*$ at balanced growth
$\hat{u} = 0$ yields

$$\left(\frac{c}{k}\right)^* = \frac{B}{\alpha} - B(1 - u^*)$$

[5] As there is no population growth there is no difference between nominal and per capita growth rates (e.g.: $\hat{K} = \hat{k}$).
[6] The externalities of individual human capital Lucas allows for yield different growth rates of k and h.

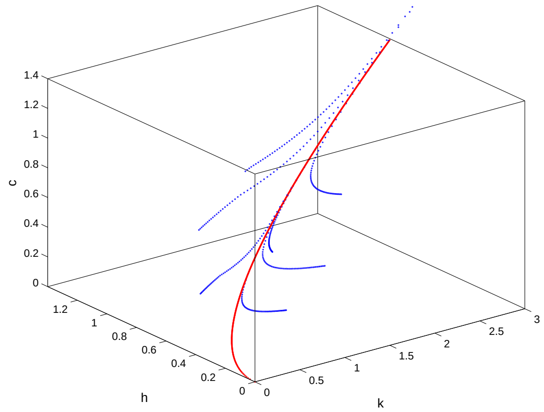
The computation of $(k/h)^*$ at balanced growth

The comparison of the Keynes-Ramsey rule and the final equation for γ leads to $MPK = B$ and hence[7]

$$\alpha A \left(\frac{uh}{k}\right)^{1-\alpha} = B$$

$$\left(\frac{k}{h}\right)^* = \left(\frac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}} u$$

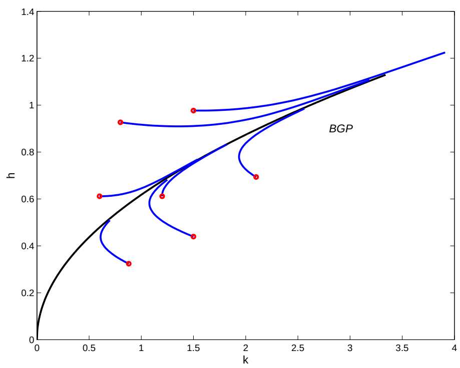
Imbalance of initial states



For the calculation of this picture $u(t)$ was chosen optimally.

State variables and optimal decision

The adjustment can be depicted in terms of the level of state variables. The control variables c and u are chosen optimally.



State variables and optimal decision

- The differences in the adjustment are only totally comprehensible, if the dependence on time is taken into account.

[7]The externalities of individual human capital mentioned before imply different growth rates of k and h and therefore the balanced growth path is no longer a straight line! Instead it is a smooth curve of form $k^\mu = h$ for some positive constant μ. In the computations for the pictures in the remainder of this section I used a μ between zero and 1.

- The adjustment to the balanced growth path must be observed in the short and long run. Additionally intertemporal effects should be investigated.
- In the short run the acceptance of regulatory measures is an important point of view.
- In the long run aspects of sustainability play an important role.
- In total the intergenerational balance is important.

Imbalanced initial conditions of growth
A state (k, h) completely determines the optimal path starting from this state. This includes the near future and the very far. This means, there are different aspects we can focus on when we compare states with each other.

The fact that any optimal path approaches the balanced growth path *(BGP)* does not make all paths alike. And, most obviously, two different states on the balanced growth path will grow apart - although along the *(BGP)*!

Comparison of different (initial) states involves comparison of optimal growth paths. It involves the solution of optimal growth problems by numerical methods.

Criteria for classification of imbalanced states

1. **Convergence**: Economies with different initial states may undergo different transition but become more and more equal in the long run. In terms of state variables and therefore with respect any aspect concerning their remaining future they converge.
2. **Initial utility**: Consider the initial utility $u(c(0))$ of an economy starting to grow optimally from state (k, h). Different optimal initial utility is important for the current generation, but is no indicator for superior long run performance. Catching up and overtaking may happen along optimal growth paths.
3. **Discounted utility**: Discounted utility $\int_0^\infty u(c(t))e^{-\rho t}\, dt$ is the objective function for optimal growth in this model. It comprises initial utility as well as long run performance. In a particular way integration with discounting combines criterion (1) and (2).

Classification of imbalanced states
We distinguish three concepts of classification.

1. Convergence classes Two states i und j belong to the same **convergence class** if

$$\lim_{t \longrightarrow \infty} \parallel (k_i(t), h_i(t)) - (k_j(t), h_j(t)) \parallel = 0$$

The states of two initially different economies belonging to the same convergence class are identical in the long run. As a consequence the optimal controls converge as well. The economies align to each other.

2. **initial utility classes** Two states i and j belong to the same **initial utility class** if

$$u(c_i(0)) = u(c_j(0))$$

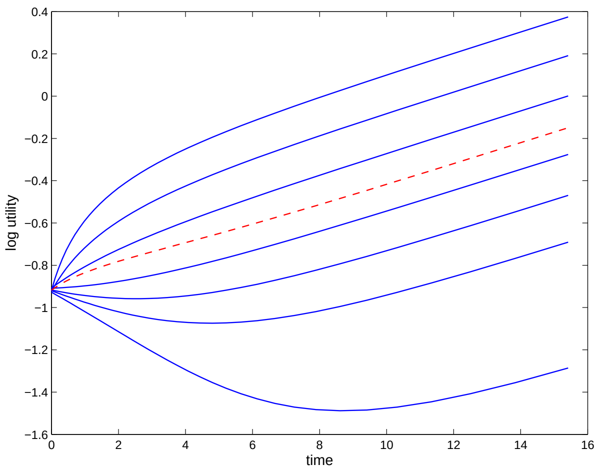
The states of two initially different economies belonging to the same initial utility class imply the same optimal initial consumption, and of course the same utility of consumption. However the consumption paths will develop towards different levels.

3. **discounted utility classes** Two states i and j belong to the same **discounted utility class** if

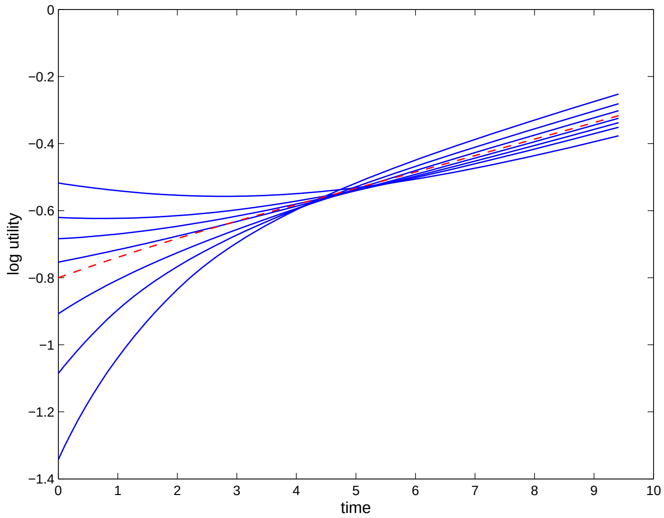
$$\int_0^\infty u(c_i(t))e^{-\rho t}\ dt = \int_0^\infty u(c_j(t))e^{-\rho t}\ dt$$

The growth paths of two economies belonging to the same discounted utility class are different in the short and the long run. But the integral over all generations´ utilities are identical.

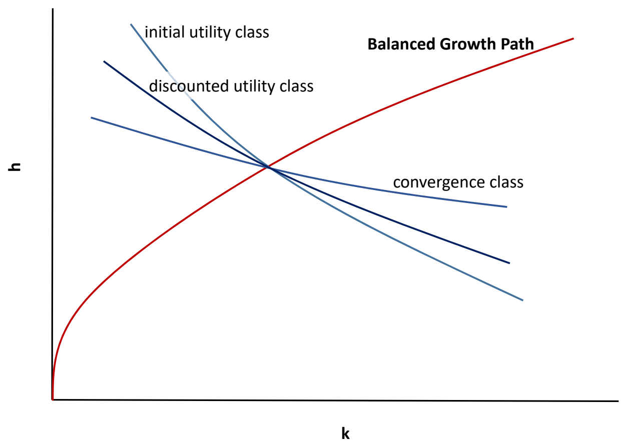
The following two graphs depict the utility in a logarithmic scale towards time. Several initial states are evaluated with respect to one of the mentioned classifications. Each picture evaluates states belonging to one of the classes. It is straight forward to find out which classes are depicted.[8]



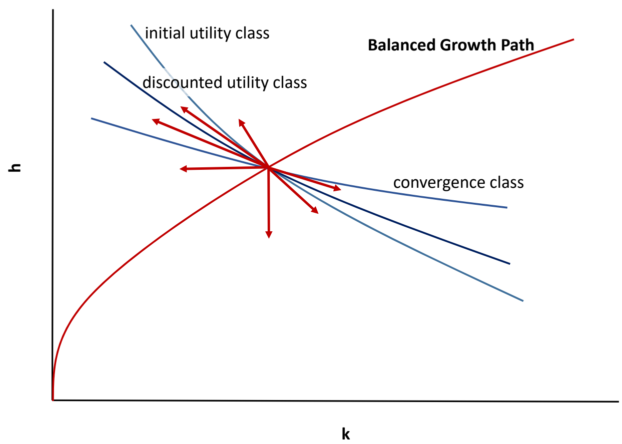
[8]Hint: One is an initial utility class, one a discounted utility class.



The following graph depictures the classes relative to a common point of reference. Call it the base point of comparison.



Shocks can have very different short run and long run effects. This becomes most obvious when we consider a combination of positive and negative shocks in k and h, respectively.



A shock is a sudden change in state variables. It can be characterized by the relative change of state variables and its intensity.

It can be evaluated with respect to the criteria underlying the definition of classes defined above.

- Negative shocks in either single state variable have negative effects in terms of all criteria.
- A combination of shocks to physical as well as human capital (e.g. due to a war and subsequent foreign aid) may yield different results when the evaluation is based on different criteria.

- Consider a balanced state as point of reference. Call it the base point of comparison.
- Draw the curves of the three classes through the base point. They decompose the (k, h) plane into different segments.
- After the shock the state may be above some and below other curves. I.e. the shock improves the economy with respect to some critaria and damage it with respect to others.

2.3 Endogenous Technical Progress: Models of Romer, Jones, and Others

A number of authors have contributed to the analysis of a particular class of models focusing on endogenous technical progress. Among them: Charles Jones (1995), Paul Romer (1990), Christian Groth, Karl-Josef Koch and Thomas Steger (2010).

Key ingredients of the models are

- population growth with a constant rate $n \geq 0$, hence $L = L(0)e^{nt}$
- K, L capital and labor
- A stock of "knowledge"
- u labor allocation

The growth engines

- Capital is accumulated in the usual way, but without depreciation.
- Via research and development a share of labor leads to technical progress, which compensates for decreasing marginal productivity of capital. Hence a permanent incentive of investment is given.

The combination of returns to scale effects leads to balanced growth.

The dynamic model

$$Y = A^{\sigma} K^{\alpha} (uL)^{1-\alpha}, \qquad \sigma > 0, \quad 0 < \alpha < 1, \tag{1}$$
$$\dot{K} = Y - cL, \quad K(0) > 0 \tag{2}$$
$$\dot{A} = \gamma A^{\varphi} (1-u) L, \qquad \gamma > 0, \varphi \leq 1, \ A(0) > 0 \tag{3}$$

Notice the following features

- The neoclassical Cobb-Douglas production function is augmented by a dynamic technology factor which can increase productivity; only a share u of labor is used in production.
- The remaining share $(1-u)$ of labor is used in R&D; the growth rate of A depends on the level of technology if $\varphi < 1$.

- The standard CRRA utility of Ramsey-Cass-Koopmans with $\theta > 0$ and $\rho > n$ is used

$$U_0 = \int_0^\infty \frac{c^{1-\theta}-1}{1-\theta} L_0\, e^{(n-\rho)t} dt,$$

- Cases studied
  - $\varphi = 1, \rho > n = 0, \quad \hookrightarrow \quad g_c = (1/\theta)\,[\sigma\gamma L/(1-\alpha) - \rho)]$
    Romer (1990), endogenous growth
  - $\varphi < 1, \rho > n > 0, \quad \hookrightarrow \quad g_c = n/(1-\varphi)$
    Jones (1995), semi-endogenous growth
  - $\varphi < 1,\ \rho > n = 0 \quad \hookrightarrow$ stagnation

Solution of Romer's version of the model: $\varphi = 1, \rho > n = 0$

The present value per capita version of the Hamiltonian is based on utility per capita, the production function $y = A^\sigma k^\alpha u^{1-\alpha}$ and the capital accumulation $\dot{k} = y - c$:

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta}\, e^{-\rho t} \; + \; \lambda(y-c) \; + \; \mu\gamma A^\varphi (1-u)L$$

$$\left.\begin{array}{ll} (1) & \mathcal{H}_c = c^{-\theta}\, e^{-\rho t} \; - \; \lambda = 0 \\ (2) & \mathcal{H}_u = \lambda(1-\alpha)y/u \; - \; \mu\gamma A^\varphi L = 0 \end{array}\right\} \text{Maximum principle}$$

$$\left.\begin{array}{ll} (3) & \mathcal{H}_k = \lambda\alpha\, y/k = -\dot{\lambda} \\ (4) & \mathcal{H}_A = \lambda\sigma\, y/A \; + \; \mu\gamma\varphi A^{\varphi-1}(1-u)L = -\dot{\mu} \end{array}\right\} \text{Euler-equations}$$

From (1) and (3) the Keynes-Ramsey-rule is obtained:

$$\begin{array}{ll} (1) & c^{-\theta}\, e^{-\rho t} \; - \; \lambda = 0 \\ (3) & \lambda\alpha\, y/k = -\dot{\lambda} \end{array}$$

differentiate (1) with respect to t:

$$(5) \qquad -\theta c^{-(1+\theta)}\, \dot{c}\, e^{-\rho t} \; - \; c^{-\theta} \rho\, e^{-\rho t} - \dot{\lambda} = 0$$

substitute $-\dot{\lambda}$ from (3), divide by $c^{-\theta}\, e^{-\rho t}$, notice (1) and solve with respect to $\hat{c}$

$$(6) \qquad g_c = \hat{c} = \frac{1}{\theta}\,(\alpha\, APK - \rho)$$

APK is the average productivity of capital, y/k, and $\alpha\, APK$ equal to the marginal productivity of capital MPK.

Balanced growth means: the growth rates g_c, g_u, g_k, g_A and hence also g_y are constant. We conclude:

- $g_u = 0$
- $g_A = \gamma(1-u^*)L$ (due to the $\varphi = 1$ in $\dot{A}$)
- g_λ is constant (due to (1) $c^{-\theta}\, e^{-\rho t} - \lambda = 0$)
- $g_y = g_k$ (due to (3) $\lambda\alpha\, y/k = -\dot{\lambda}$)
- $g_y = \sigma g_A + \alpha g_k$ (due to $y = A^\sigma k^\alpha u^{1-\alpha}$)
  $g_y = \frac{\sigma}{1-\alpha}\, g_A$
- g_μ is constant (due to (2) $\lambda(1-\alpha)y/u = \mu\gamma A^\varphi L$)

With a little extra effort a complete solution can be found. □

A special case

In the case of $\rho = n$ the utility integrals in general do not converge. This can be cured by the use of a different objective function. The analysis leads to regular growth (Groth, Koch, Steger, 2010).

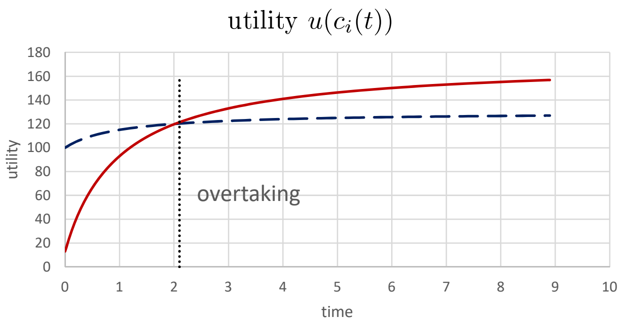
The appropriate optimality criterion is *catching-up*: a feasible growth path $(K^*, A^*, c^*, u^*)_{t=0}^{\infty}$ is *catching-up*-optimal, if

$$\liminf_{t\to\infty} \left(\int_0^t \frac{(c^*)^{1-\theta} - 1}{1-\theta} d\tau - \int_0^t \frac{c^{1-\theta} - 1}{1-\theta} d\tau \right) \geq 0$$

for all feasible paths $(K, A, c, u)_{t=0}^{\infty}$.

Overtaking

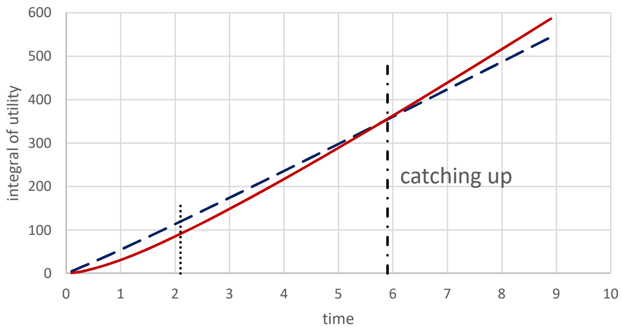
We compare two growth paths by their utility profile $c_1(t)$ (solid line), $c_2(t)$ (broken line). $c_1(t)$ starts on a lower level but overtakes $c_2(t)$ after roughly 2 years in terms of instantaneous utility. After overtaking the dominance of $c_1(t)$ is permanent.



Catching up

The disadvantage of $c_1(t)$ compared to $c_2(t)$ accumulated during the first years is compensated roughly during the following four years. Catching up takes place.

utility integrals $\int_0^t u(c_i(\tau))d\tau$



Further Questions

To solve such a model is not really the main purpose of the exercise. The implementation of ideas about particular engines of growth in such a way that the model is tractable is a challenge. Knowing that we can solve the model allows us to raise further questions :

- How does the particular structure affect the accumulation and allocation of resources?
- How do initial conditions affect transition and long run performance?
- What kind of policy interventions does the analysis suggest?

We gave a partial answer to the first question, and we will take a step to answer the second one in the next model! To answer the other third one on goes beyond the scope of the course.

2.4 Product Differentiation: The Romer Model

Product differentiation

- In modern industrial economics product differentiation plays an important role.
- Early papers concerning that are written by Spence '76 and Dixit& Stiglitz '77.
- In the literature of Economic Growth Paul Romer has introduced product differentiation with some publications ('87, '90).
- An important monograph about international spillovers in this context was published by Grossman & Helpman '91.

Growth with differentiated intermediates

Sectors of goods production

$$Y_i = AL_i^{1-\alpha} \sum_{j=1}^{N} X_{ij}^{\alpha}$$

- X_{ij} intermediate, specialized in i, differentiated in j.
- We can call $X_i = \left(\sum_{j=1}^{N} X_{ij}^{\alpha}\right)^{1/\alpha}$ the index of intermediates in sector i. It measures the common impact of all intermediates so that $Y_i = AL_i^{1-\alpha} X_i^{\alpha}$.
- The marginal product of X_{ij} is independent of $X_{ij'}$.
- The unit costs of production of X_{ij} is assumed to be equal to one.
- The costs of development of a new variant is assumed to be equal to η.

Economies of scale by diversification
Assumption for the purpose of demonstration: only one sector, equal quantities of all variants $X_{ij} = X$

- $$\begin{aligned} Y &= AL^{1-\alpha} \sum_{j=1}^{N} X^{\alpha} \\ &= AL^{1-\alpha} \cdot N \cdot X^{\alpha} = AL^{1-\alpha} \cdot N^{1-\alpha} \cdot (NX)^{\alpha} \end{aligned}$$

- Twice the number of variants along with half the quantity used of each variant leaves the total quantity of intermeditaes NX unchanged, but leads to a larger output through $N^{1-\alpha}$.

- On the other hand, the marginal rate of production of X is decreasing.

Profit maximization in sector i

$$\Pi_i = Y_i - wL_i - \sum_{j=1}^{N_i} p_j X_{ij}$$

1. $\frac{\partial \Pi_i}{\partial X_{ij}} = 0 \quad \hookrightarrow \quad \frac{\partial Y_i}{\partial X_{ij}} = p_j$

   $\alpha A L_i^{1-\alpha} X_{ij}^{\alpha-1} = p_j$

   $\alpha A L_i^{1-\alpha} / p_j = X_{ij}^{1-\alpha}$

   $X_{ij} = L_i (A\alpha/p_j)^{1/(1-\alpha)}$

   demand for X_{ij} conditional on L_i

$$\Pi_i = Y_i - wL_i - \sum_{j=1}^{N_i} p_j X_{ij}$$

2. $\frac{\partial \Pi_i}{\partial L_i} = 0 \quad \hookrightarrow \quad \frac{\partial Y_i}{\partial L_i} = w$

   $(1-\alpha) Y_i / L_i = w$

   wage equation

Innovation and company value

- Idea: pricing (p_j) determines the demand for X_{ij} and hence the company value.
- Present company value
$$V(0) = \int_0^{\infty} (p_j - 1) X_j \, e^{-r\nu} d\,\nu$$
- Current company value
$$V(t) = \int_t^{\infty} (p_j - 1) X_j \, e^{-r(\nu - t)} d\,\nu$$
- The current company value is equal to the present value, if prices and quantities are constant over time
$$V(t) = \int_0^{\infty} (p_j - 1) X_j \, e^{-r\nu} d\,\nu$$

- Total demand of X_j (clearing of labor market)

$$\begin{aligned} X_j &= \sum_i L_i (A\alpha/p_j)^{1/(1-\alpha)} \\ &= L(A\alpha/p_j)^{1/(1-\alpha)} \end{aligned}$$

- Company value yields:

$$V(t) = \int_t^\infty (p_j - 1) L (A\alpha/p_j)^{1/(1-\alpha)} \, e^{-r(\nu - t)} d\nu$$

- Maximization of the company value by price setting period by period.

$$\max_{p_j} (p_j - 1) L (A\alpha/p_j)^{1/(1-\alpha)}$$

or

$$\max_{p_j} (p_j - 1) p_j^{-1/(1-\alpha)}$$

FOC: $p_j^{-1/(1-\alpha)} - (p_j - 1)\frac{1}{1-\alpha} p_j^{-1/(1-\alpha)} \frac{1}{p_j} = 0$

$(1-\alpha)p_j = p_j - 1$

$-\alpha p_j = -1 \quad \hookrightarrow p_j = 1/\alpha > 1$

- p_j is time independent and because of that, also is X_j.
- The demand for X_j:

$$X_j = L A^{1/(1-\alpha)} \alpha^{2/(1-\alpha)}$$

- The company value for time independent price setting is:

$$V(t) = \frac{1-\alpha}{\alpha} L A^{1/(1-\alpha)} \alpha^{2/(1-\alpha)} \int_0^\infty e^{-r\nu} d\nu$$

- Market access, i.e. development of new X_j, is carried out as long as $V(t) = \eta$.
- Constant company value implies a constant interest rate r.
- Then the aggregate discount factor is

$$\begin{aligned} \int_0^\infty e^{-r\nu} d\nu &= \left[-1/r e^{-r\nu}\right]_0^\infty \\ &= -(-1/r) = 1/r \end{aligned}$$

- $V(t) = \eta \quad \hookrightarrow r = L/\eta \, A^{1/(1-\alpha)} \frac{1-\alpha}{\alpha} \alpha^{2/(1-\alpha)}$

Utility maximization of households

- Intertemporal utility maximization

$$\int_0^\infty \frac{c^{1-\theta}-1}{1-\theta} e^{-\rho t}\, dt$$

- Keynes-Ramsey-rule

$$\begin{aligned}\gamma_c &= \frac{1}{\theta}(r-\rho)\\ &= \frac{1}{\theta}\left(L/\eta A^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)}-\rho\right)\end{aligned}$$

- γ_c is always constant, i.e. there is always balanced growth. We suppose that $\gamma_c > 0$

Remaining growth rates

- Assumption: There are m sectors with identical structure.
- For all i, j we use $X_{i,j}$ to denote the quantity of intermediate j used in sector i.
- $X_{i,j} = X_{i,j'}$ due to the symmetry of the model. So we can drop the sector index i
  For all i, j We get: $X_{ij} = X_i = X/Nm$ and
- $L_i = L/m$.
- Because of $Y_i = AL_i^{1-\alpha}X_i^\alpha N$ and $X_i = A^{1/(1-\alpha)}L_i\alpha^{2/(1-\alpha)}$ we get

$$\begin{aligned} Y_i &= A^{1/(1-\alpha)}L_i\alpha^{2\alpha/(1-\alpha)}N\\ \text{and on aggregate}\quad Y &= A^{1/(1-\alpha)}L\alpha^{2\alpha/(1-\alpha)}N\end{aligned}$$

  and hence

$$\gamma_Y = \gamma_N$$

- Y allows for three kinds of use
  - consumption C
  - production factor for the manufacture of intermediates in the amount of NX
  - effort for R& D in the amount of $\eta\dot{N}$
- Adding up yields:

$$\begin{aligned} Y &= C + NX + \eta\dot{N}\\ &= C + NX + \gamma_N\eta N\\ Y/N &= C/N + X + \gamma_N\eta\end{aligned}$$

  hence: $\gamma_C = \gamma_N$

- $$\gamma = \gamma_N = \gamma_Y = \gamma_C$$
- $$\gamma = \frac{1}{\theta}\left(L/\eta A^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)} - \rho\right)$$

Efficiency growth processes

- In this model - as in all models, discussed up to now - households maximize intertemporal utility.
- In that framework the interest rate depends on allocations in all markets.
- But the production decisions are taken in a framework of monopolistic competition.
- Hence a utility maximum of households should not be expected!

Intertemporal optimization by a planer

- Still the integral over total discounted utility shall be calculated.
- Production possibilities and resource constraints have to be taken into account.
- The market allocation mechanisms are not considered.
- Market power of certain players is unconsidered.

The Hamiltonian

$$\mathcal{H} = \frac{c^{1-\theta}-1}{1-\theta} + \mu\frac{1}{\eta}(AL^{1-\alpha}NX^{\alpha} - Lc - NX)$$

FOC

$$\begin{aligned}
\mathcal{H}_c = 0 \quad &:: \quad c^{-\theta} = \mu L/\eta \\
\mathcal{H}_X = 0 \quad &:: \quad \mu/\eta(\alpha AL^{1-\alpha}NX^{\alpha-1} - N) = 0 \\
\mathcal{H}_N = -\dot{\mu} + \rho\mu \quad &:: \quad \frac{\mu}{\eta}(AL^{1-\alpha}X^{\alpha} - X) = -\dot{\mu} + \rho\mu
\end{aligned}$$

Solve $\mathcal{H}_X = 0$ for X:

$$X = \alpha^{\frac{1}{1-\alpha}}A^{\frac{1}{1-\alpha}}L$$

From $\mathcal{H}_c = 0$

$$c^{-\theta}\eta/L = \mu$$

we get

$$-\theta\gamma_c = \gamma_\mu$$

Divide $\mathcal{H}_N$ by μ and solve for γ_μ

$$\gamma_\mu = \rho - \frac{1}{\eta}\left(AL^{1-\alpha}X^\alpha - X\right)$$

Together with the prior results we get

$$\begin{aligned}\gamma_c &= \frac{1}{\theta}\left(\frac{1}{\eta}(AL^{1-\alpha}X^\alpha - X) - \rho\right) \\ &= \frac{1}{\theta}\left(\frac{1}{\eta}LA^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{1/(1-\alpha)} - \rho\right)\end{aligned}$$

Comparison of market solution and welfare optimum

$$\gamma_c^{market} = \frac{1}{\theta}\left(\frac{1}{\eta}LA^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{2/(1-\alpha)} - \rho\right)$$

$$\gamma_c^{opt} = \frac{1}{\theta}\left(\frac{1}{\eta}LA^{1/(1-\alpha)}\frac{1-\alpha}{\alpha}\alpha^{1/(1-\alpha)} - \rho\right)$$

$$\gamma_c^{market} < \gamma_c^{opt}$$

The market growth rate is suboptimal due to monopolistic competition.