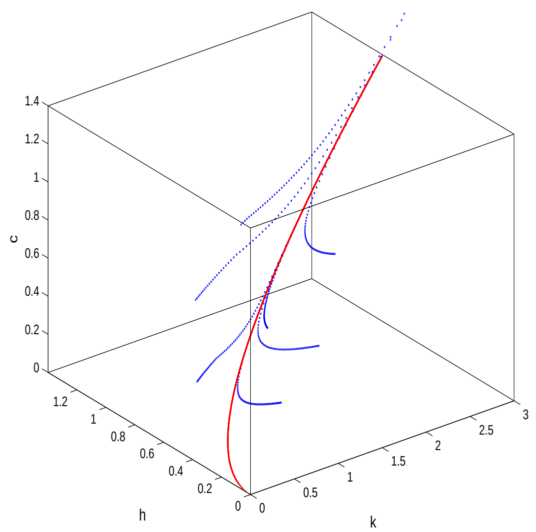
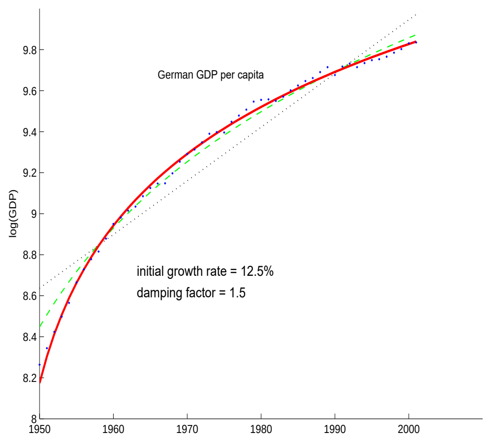
Economic Growth

Master Course WS 2022/23

Prof Dr. Karl-Josef Koch
University of Siegen

$$\max \int_0^\infty \left(\frac{c^{1-\theta} - 1}{1-\theta} \right) \cdot e^{-\rho t}\, dt$$





These notes are more or less a one two one print version of the slides of the course. The lecture provides a lot of additional instructions, information, and discussions beyond the content of the slides.

Contents

1 Introduction **1**
1.1 Goals and Methods 1

2 Introduction **7**
2.1 Goals and Methods 7
2.2 Regularity and Balance 10
2.3 Stability 17
2.4 Convergence 20

1 Introduction

1.1 Goals and Methods

1.1 Goals and Methods

Introduction to Modern Economic Growth

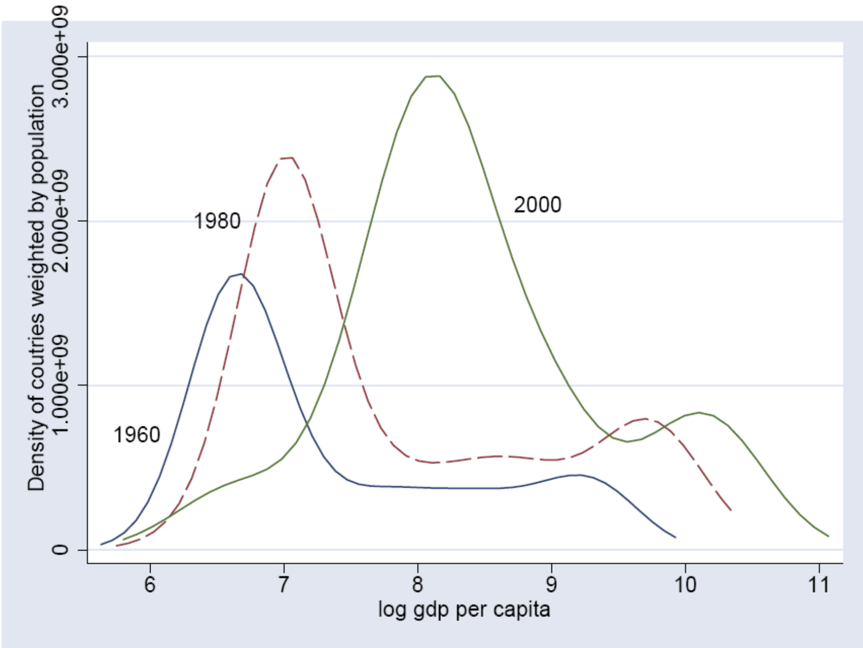


Figure 1.3. Estimates of the population-weighted distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000.

Figure 1: Estimates of the population-weighted distribution of countries according to log GDP per capita (PPP-adjusted) in 1960, 1980 and 2000; source: Acemoglu 2008.

Introduction to Modern Economic Growth

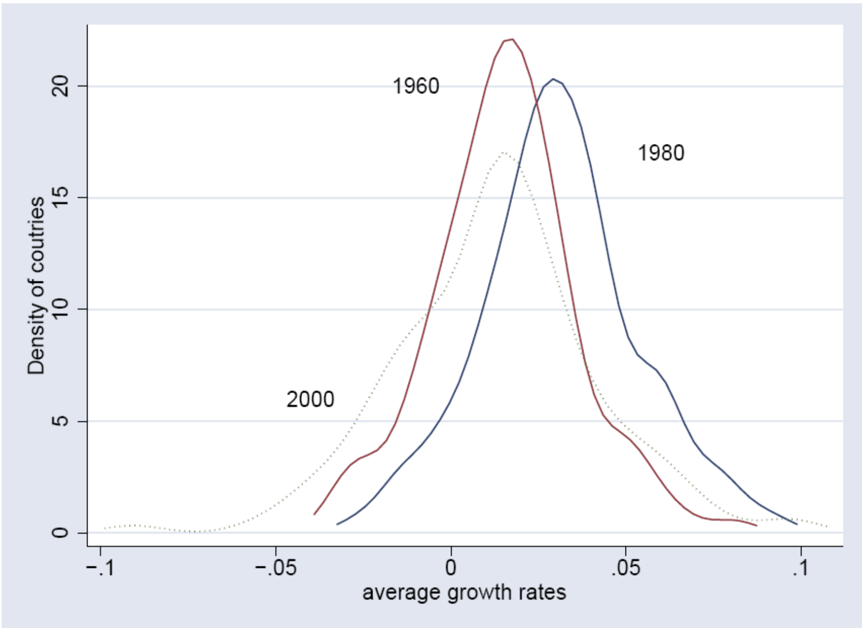


Figure 1.7. Estimates of the distribution of countries according to the growth rate of GDP per worker (PPP-adjusted) in 1960, 1980 and 2000.

Figure 2: Estimates of the distribution of countries according to the growth rate of GDP per worker (PPP-adjusted) in 1960, 1980 and 2000; source: Acemoglu 2008.

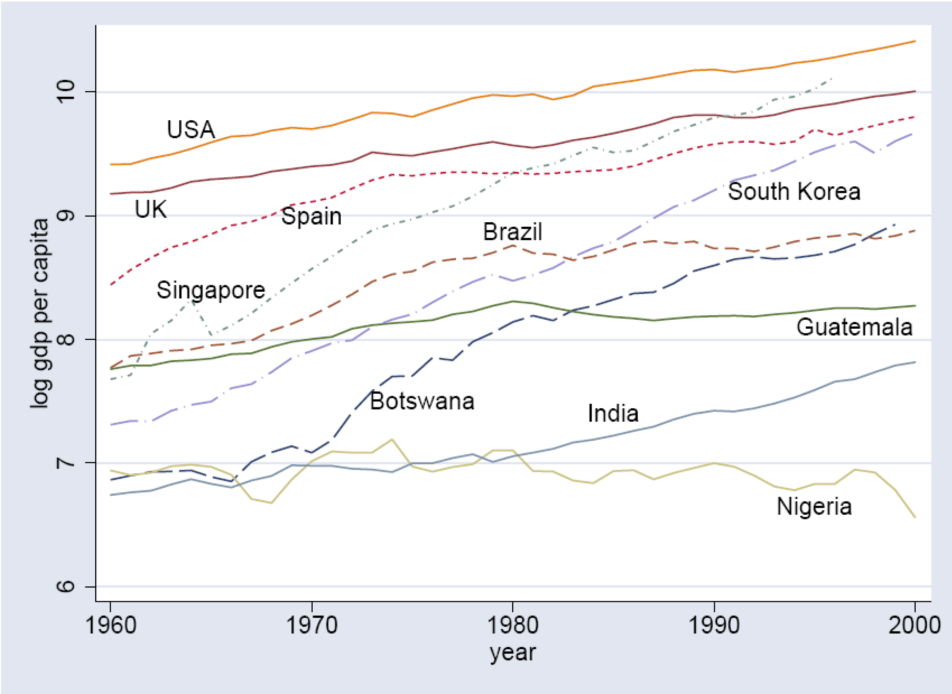


Figure 1.8. The evolution of income per capita in the United States, United Kingdom, Spain, Singapore, Brazil, Guatemala, South Korea, Botswana, Nigeria and India, 1960-2000.

Figure 3: The evolution of income per capita in the United States, United Kingdom, Spain, Singapore, Brazil, Guatemala, South Korea, Botswana, Nigeria and India, 1960-2000; source: Acemoglu 2008.

Introduction to Modern Economic Growth

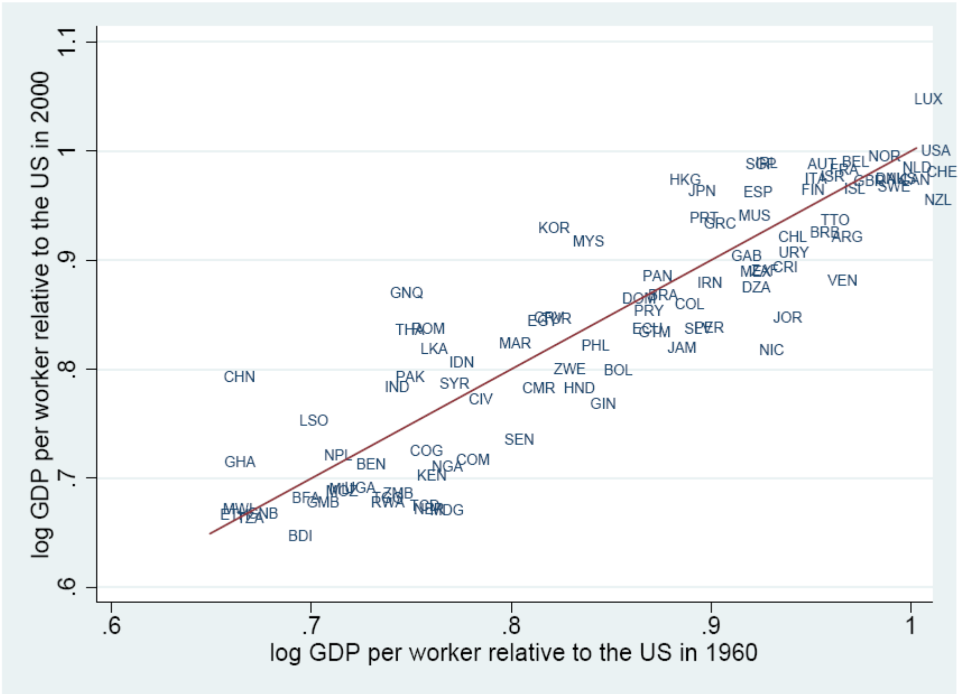


Figure 1.9. Log GDP per worker in 2000 versus log GDP per worker in 1960, together with the 45° line.

Figure 4: Log GDP per worker in 2000 versus log GDP per worker in 1960 together with the 45° line; source: Acemoglu 2008.

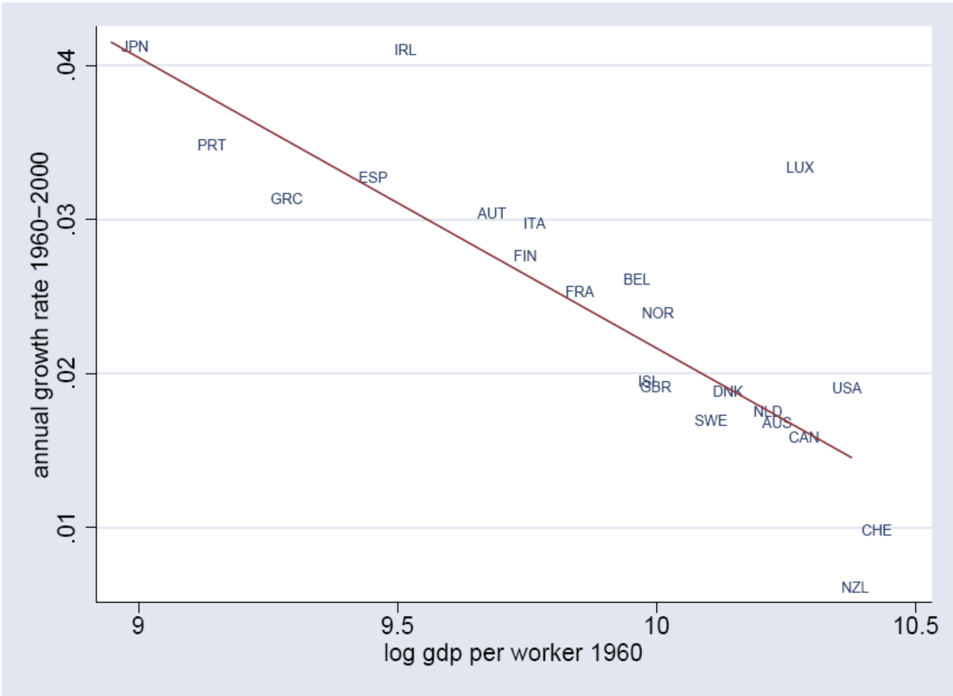


Figure 1.14. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries.

Figure 5: Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries; source: Acemoglu 2008.

INTRODUCTION TO MODERN ECONOMIC GROWTH

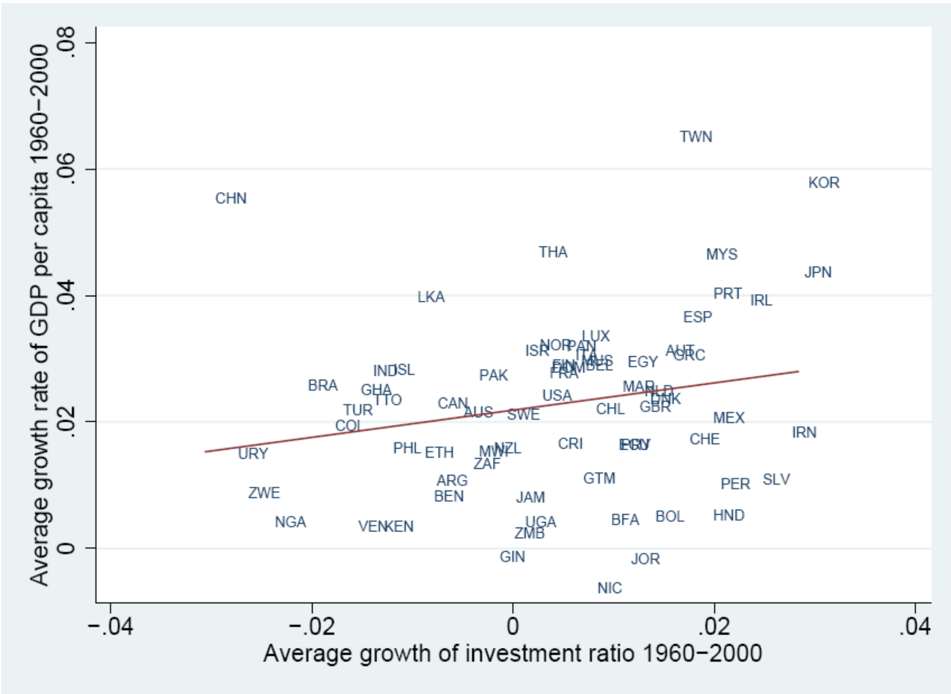


FIGURE 1.15. The relationship between average growth of GDP per capita and average growth of investments to GDP ratio, 1960-2000.

Figure 6: The relationship between average growth of GDP per capita and average growth of investments to GDP ratio, 1960-2000; source: Acemoglu 2008.

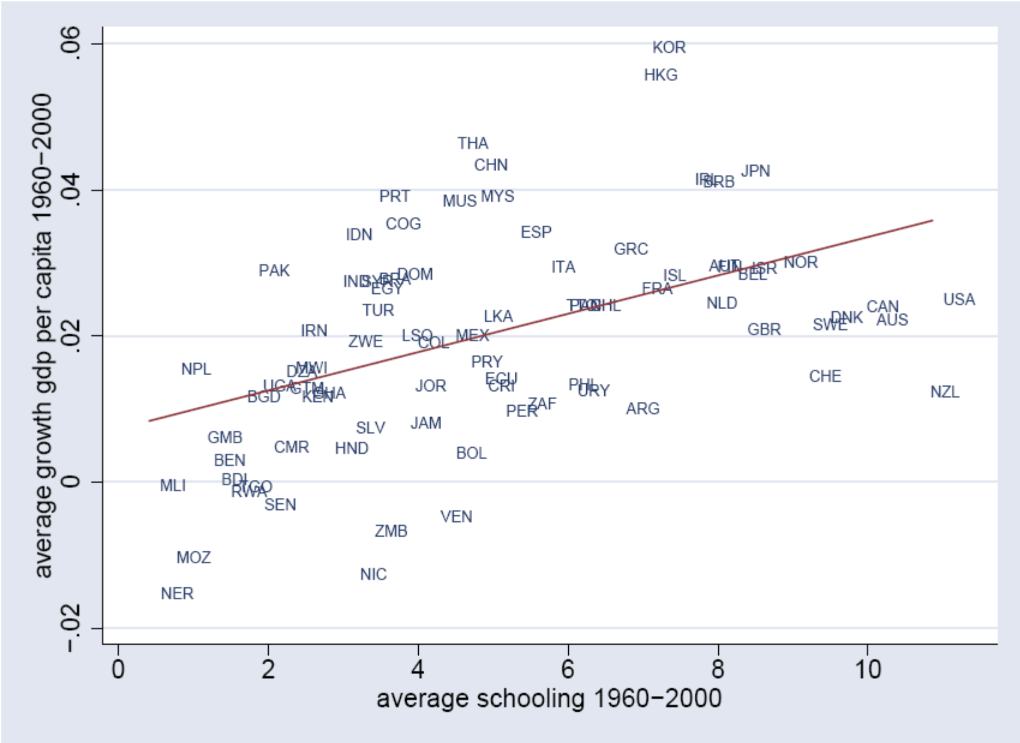


FIGURE 1.16

Figure 7: The relationship between average growth of GDP per capita and average years of schooling, 1960-2000; source: Acemoglu 2008.

2 Introduction

2.1 Goals and Methods

Growth rates in discrete time
Consider a variable changing over time

$$y_{t+1} = (1+g) \cdot y_t$$

We can solve for g, the growth rate of y in year t

$$g = \frac{y_{t+1}}{y_t} - 1$$

Extending the calculation into the future we get

$$\begin{aligned} y_{t+T} &= (1+g) \cdot y_{t+T-1} \\ &= (1+g)^2 \cdot y_{t+T-2} \\ &\vdots \\ &= (1+g)^T \cdot y_t \end{aligned}$$

Mean growth
Based on the data of only the initial year and the final year we can always compute the mean growth rate over T years from t to $t+T$

$$g_{t,t+T} = \left(\frac{y_{t+T}}{y_t}\right)^{1/T} - 1$$

Of course, considering the constant mean growth rate does not exclude that during the period of investigation the *true* growth rate varied. There may have been fluctuation for example due to business cycles or any kind of shocks[1].

Growth rates in continuous time
Assume $y(t)$ is a continuous, differentiable function of time.
At time t we may define the relative change of y by

$$\frac{d\,y(t)}{d\,t}\Big/ y(t) = \gamma(t)$$

We call this the *instantaneous growth rate* of y at t. We may drop the term *instantaneous* if this does not cause any confusion.

In case of a given initial value $y(0)$ and a positive, constant instantaneous growth rate γ the following exponential function solves this equation for all $t > 0$[2]

$$y(t) = y(0) \cdot e^{\gamma t}$$

Growth rates in continuous time
Moreover we find

$$y(t+T)/y(t) = e^{\gamma T}$$

and finally

$$\gamma = \frac{\ln y(t+T) - \ln y(t)}{T}$$

This formula for γ regularly is used in empirical work for technical convenience.[3]

Differential equations
An ordinary one-dimensional differential equation of first order looks as follows[4]

$$\frac{d\,x}{d\,t} = f(x(t))$$

A solution is a function $x(t)$ solving this equation.

Notation

$$\dot{x}(t) = \frac{d\,x}{d\,t} \qquad \hat{x}(t) = \dot{x}(t)/x(t)$$

Linear differential equations

$$\dot{x} = A \cdot x + B$$

with

- A and B being real numbers

[1] We sometimes use explicit notations of growth rates with a time index. Quite often we drop them.

[2] It is easy to show that this is the only solution given $y(0)$. This generalizes easily to the case of a given value $y(T)$ at any positive point of time T due to $y(0) = y(T)/e^{\gamma T}$.

[3] Notice that the discrete time growth rate differs from the instantaneous growth rate. Evaluation of $g = \left(\frac{y_{t+T}}{y_t}\right)^{1/T} - 1$ for the exponential function gives $\gamma = \ln(g+1)$. I.e. γ is a logarithmic approximation of g and both values coincide at $\gamma = g = 0$.

[4] We consider *autonomous* differential equations here. We could allow for non-autonomous differential equations where f directly depends on t.

- A is a quadratic matrix and B a vector of real numbers.

Linear differential equations have exponential solutions.

As the most simple example we consider the one-dimensional ordinary linear differential equation with constant coefficients. The differential equation

$$\dot{x} = ax + b$$

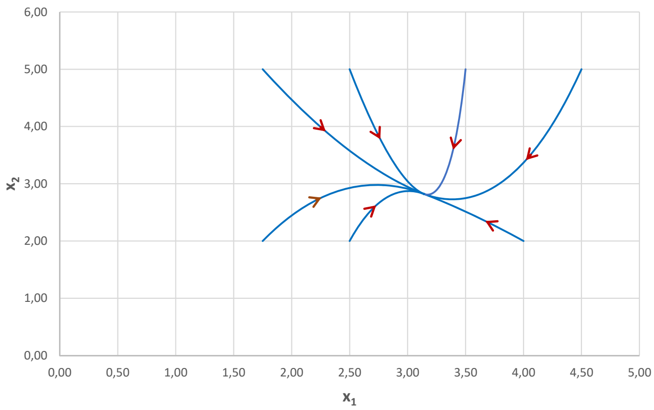
has solution

$$x(t) = (x(0) + b/a)\, e^{at} - b/a$$

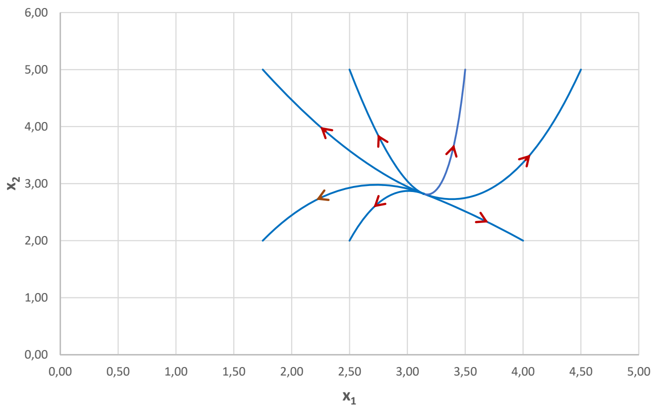
Indeed

$$\begin{aligned} \dot{x}(t) &= \frac{dx(t)}{dt} = (x(0) + b/a)\, a\, e^{at} \\ &= a\,(x(t) + b/a) \\ &= a\, x(t) + b \end{aligned}$$

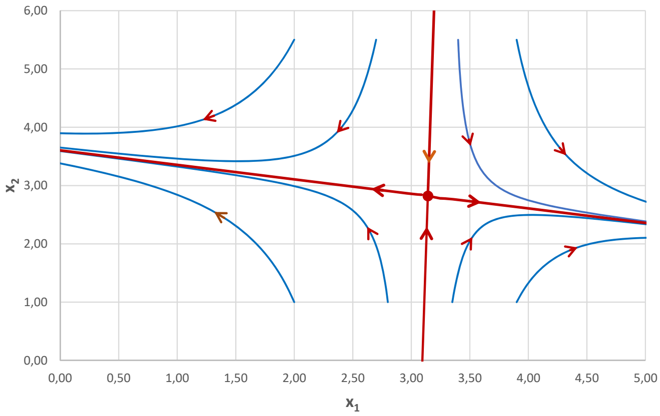
Phase diagrams



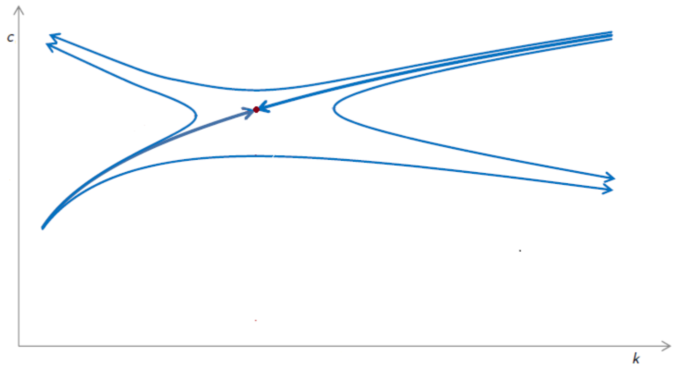
Stable equilibrium of a linear differential equation: a sink



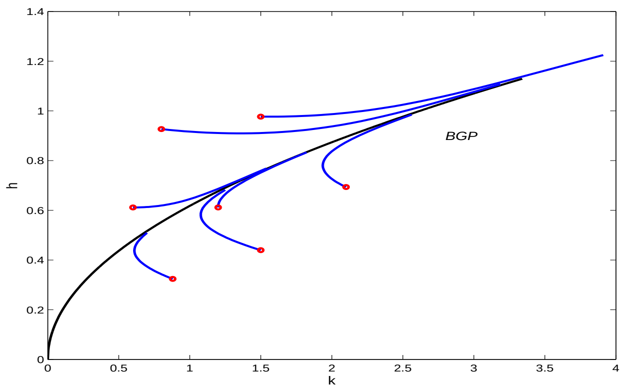
Unstable equilibrium of a linear differential equation: a source



Fifty-fifty: a saddle



Optimal growth with a saddle point



Endogenous optimal growth with a saddlepoint stable balanced growth path

2.2 Regularity and Balance

1.2 Regularity and Balance

Regularity

- What is a reasonable model for the description of economic growth in the long run?
- Are growth rates per capita constant in the long run?

- What is the difference between adjusted dynamics and long run dynamics?
- From the theoretical point of view could be concluded, how long the adjustment can take and which long run trends can be observed: convergence speed (see chapter 1.3).

- Economic models should be developed in a flexible shape to enable the explanation of different trends in the long run.
- Important is the development of methods for the description of different trends in the long run.

Regularity in a European comparison

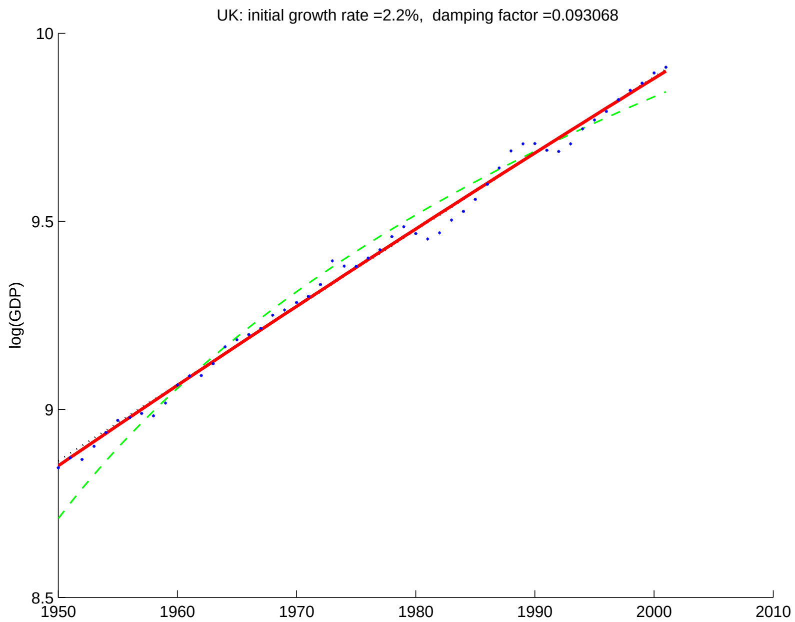
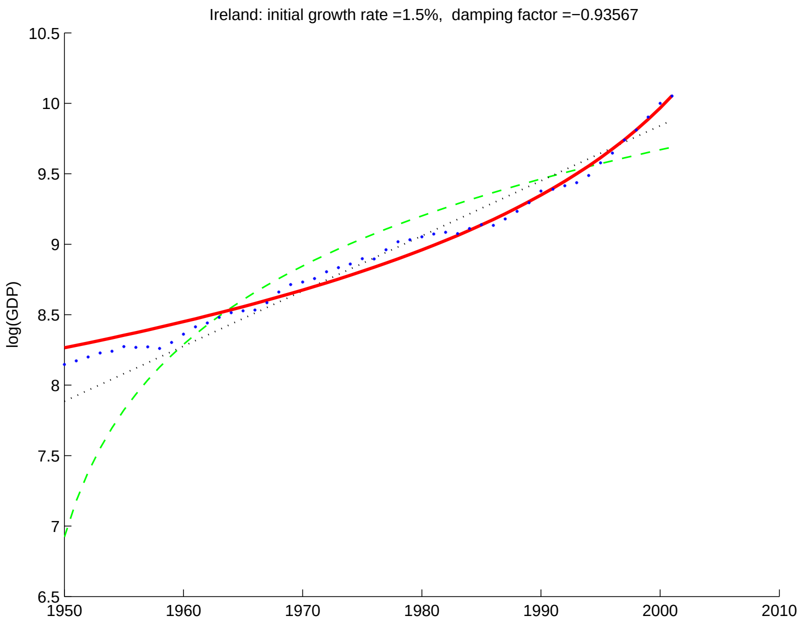


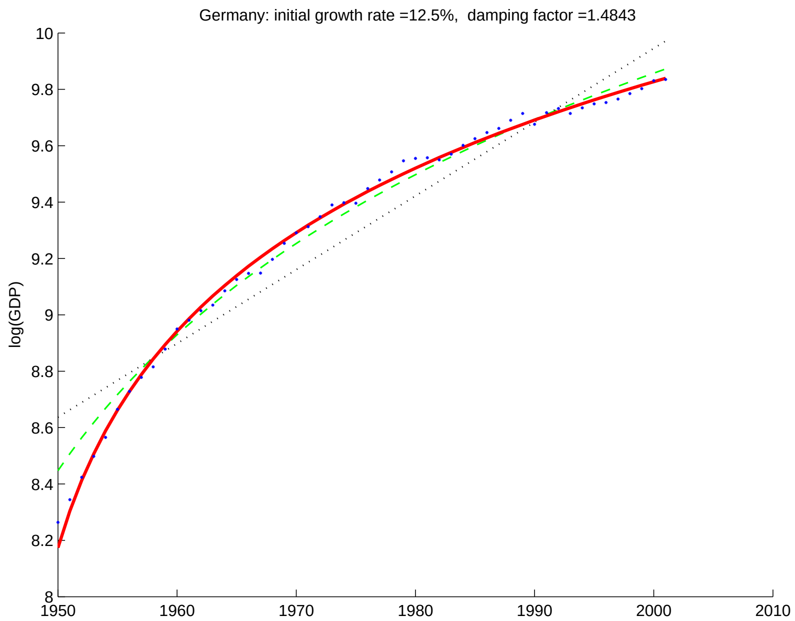
Figure 8: source: Own calculation on the basis of World Historical Data

Regularity in a European comparison



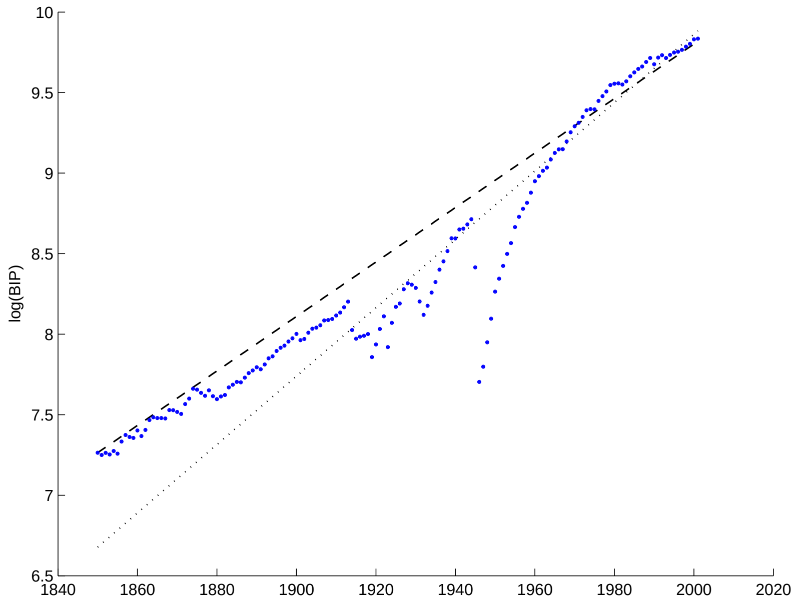
source: Own calculation on the basis of World Historical Data

Regularity in a European comparison



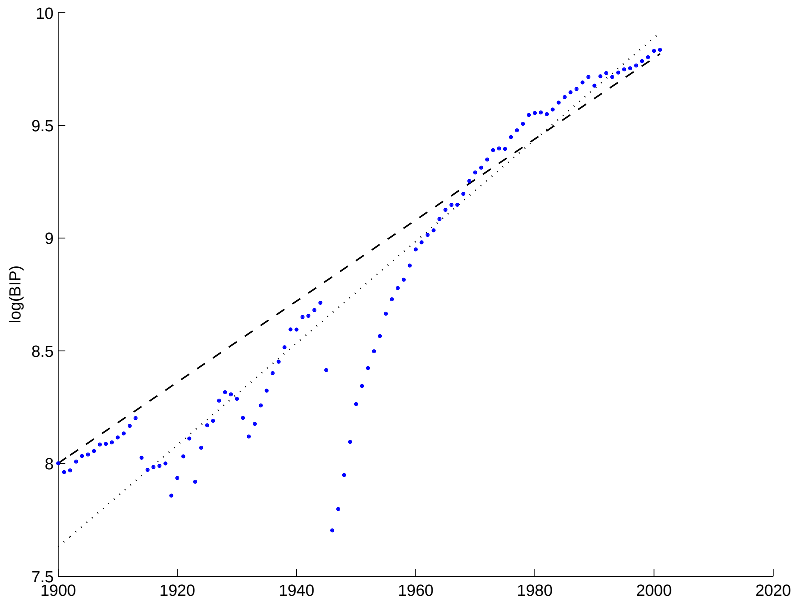
source: Own calculation on the basis of World Historical Data

Regularity and time horizon: German data



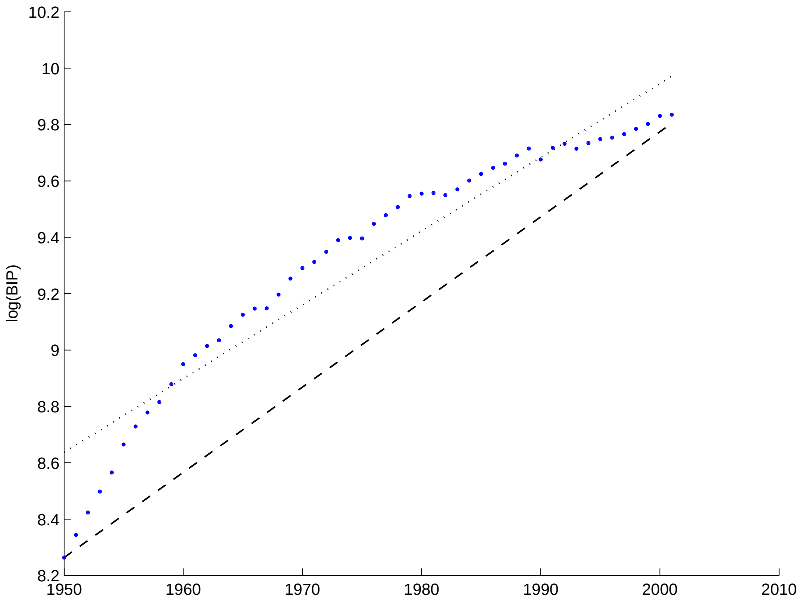
source: Own calculation on the basis of World Historical Data

Regularity and time horizon: German data



source: Own calculation on the basis of World Historical Data

Regularity and time horizon: German data



source: Own calculation on the basis of World Historical Data

The descriptive model for regular growth

$$x(t) = \left(a_0^\beta + a \cdot \beta \cdot t\right)^{1/\beta}$$

Take the derivative of $x(t)$ with respect to time t

$$\dot{x}(t) = \frac{1}{\beta}\left(a_0^\beta + a \cdot \beta \cdot t\right)^{(1-\beta)/\beta} \cdot a \cdot \beta$$

which can be expressed in terms of $x(t)$

$$\dot{x}(t) = a\, x(t)^{1-\beta} \quad \text{a Bernoulli differential equation}$$

Similarly we get the second derivative

$$\ddot{x}(t) = a(1-\beta)\, x(t)^{-\beta}\, \dot{x}(t)$$

Definition of first and second order growth rates

$$g_1 = \dot{x}/x \qquad g_2 = \dot{g}_1/g_1$$

We calculate

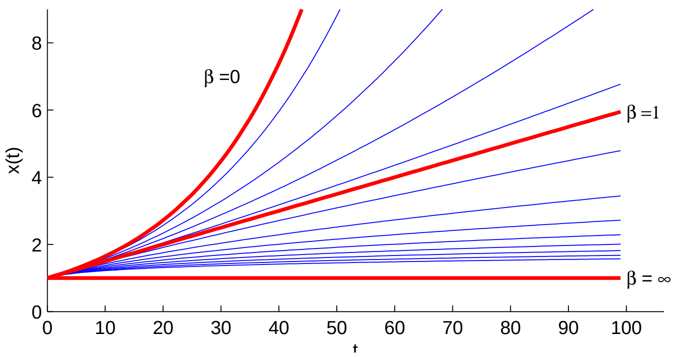
$$\begin{aligned} g_2 &= \frac{\ddot{x}\cdot x - \dot{x}\dot{x}}{x^2} \Big/ \frac{\dot{x}}{x} = \frac{\ddot{x}}{\dot{x}} - \frac{\dot{x}}{x} \\ &= a(1-\beta)x(t)^{-\beta} - ax(t)^{-\beta} \\ &= a(-\beta)x(t)^{-\beta} \\ &= -\beta g_1 \end{aligned}$$

Definition:

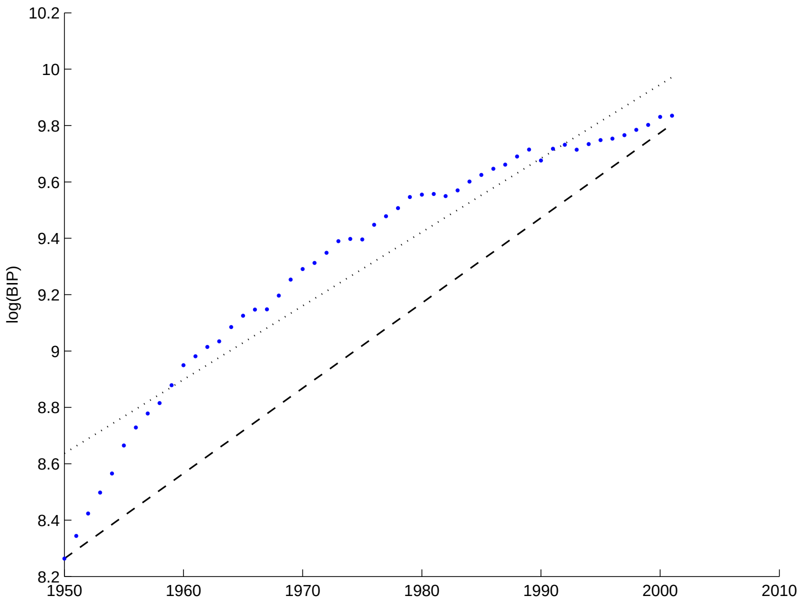
Growth of $x(t)$ is regular if $g_2 = -\beta g_1$ for some constant β.

The prototype of regular growth

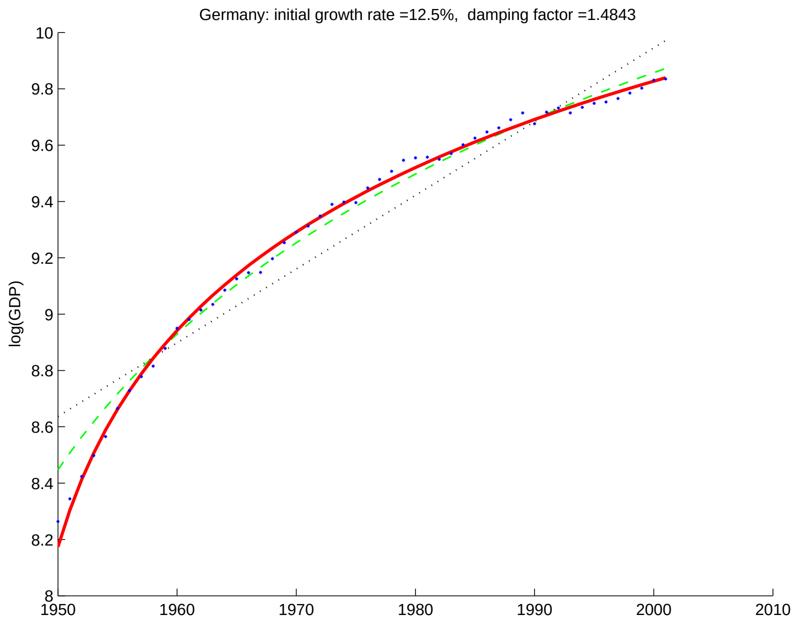
$$x(t) = \left(a_0^\beta + a \cdot \beta \cdot t\right)^{1/\beta}$$



Growth in Germany

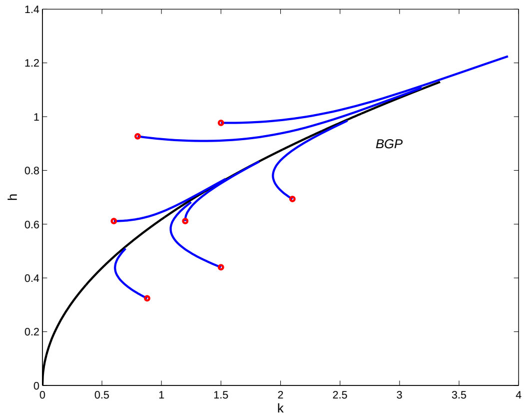


source: Own calculation on the basis of World Historical Data

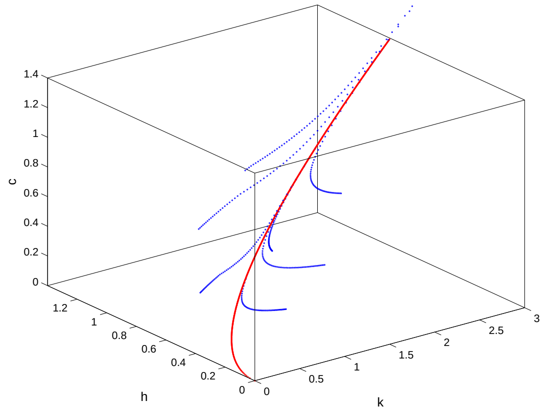


source: Own calculation on the basis of World Historical Data

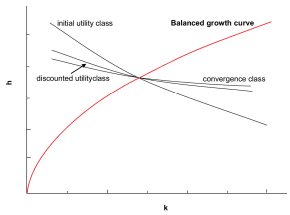
Balance and imbalance



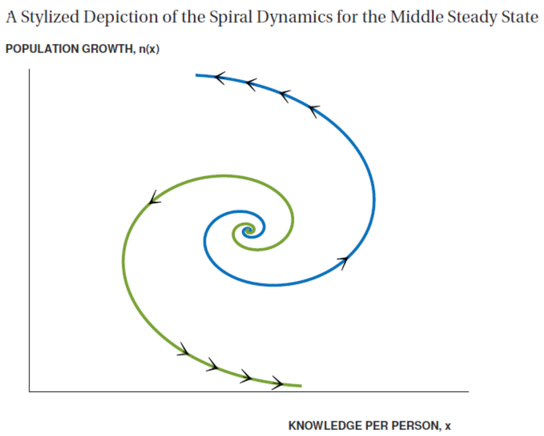
State variables and optimal consumer decision



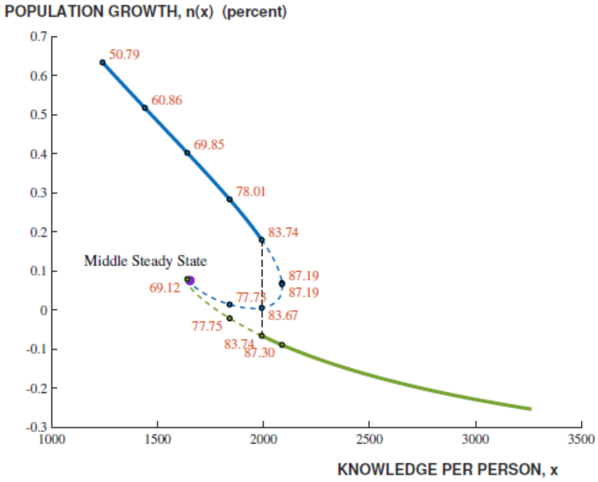
Classification of states



Shrinking Population I



Shrinking Population II



2.3 Stability

Dynamic models sometimes have rest points, also called stationary points. I.e. they have values of variables where the rules of dynamics allow the system to stay.

A simple linear model with a stationary point is the following:

$$\dot{x} = -\beta(x - x^*) \quad \text{for some positive constant parameter } \beta$$

At $x = x^*$ the right hand side is equal to zero, i.e. the system is stationary. We may call this the dynamic equilibrium.

If the system is pushed out of equilibrium by some shock, the question arises whether x will return to equilibrium. A check of the sign of the right hand side indicates stability.

But even without referring to the mathematical literature on stability analysis we can say a little bit more.

half-life

Will it take a long time for the system to return to equilibrium?

The solution

$$x(t) = x^* + (x(0) - x^*)e^{-\beta t}$$

converges to x^* with $t \to \infty$.

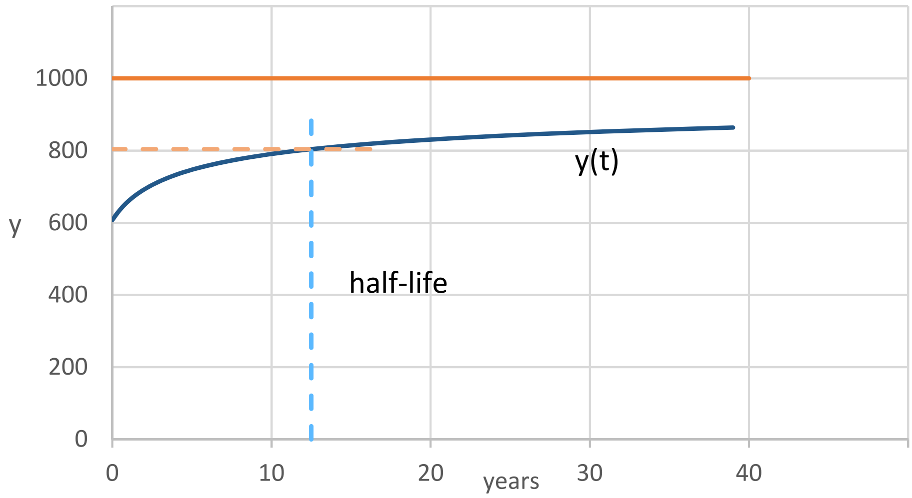
Heads up: It is correct to use the term "converge" in this situation but in the next chapter we use "convergence" in a different sense.

Half-life is the time required to halve the distance from a starting point $x(0)$ to $x^* = x(\infty)$.

We can compute this length of time:

$$\begin{aligned} x(t) - x^* &= (x(0) - x^*)/2 \\ (x(0) - x^*)e^{-\beta t} &= (x(0) - x^*)/2 \\ e^{\beta t} &= 2 \\ \beta t &= ln(2) \\ t &= ln(2)/\beta \end{aligned}$$

As you can see, half-life is inversely proportional to the absolute value of the convergence rate from x to x^*;
The proportionality factor is $ln(2)$.



The convergence coefficient
In the framework of the Solow-Swan model there is growth of the capital intensity

$$\dot{k} = sf(k) - (n+\delta)k$$

and hence the growth rate is

$$\gamma = \hat{k} = sf(k)/k - (n+\delta)$$

k in the long run converges to a constant value k^*, solving the following equation

$$sf(k)/k = (n+\delta)$$

.

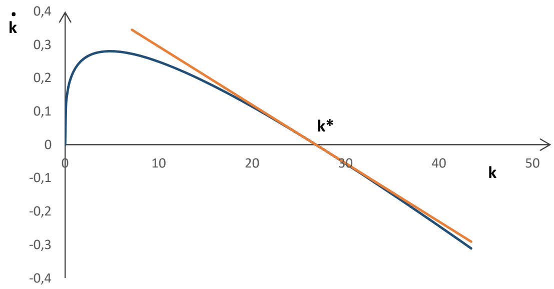
Approximation

- The differential equation is non-linear.
- Mathematical theory suggests to simplify the stability analysis by linear approximation. This is sufficient to characterize the convergence behavior near k^*

We compute the Taylor approximation of $\dot{k}$ near $k = k^*$:

$$\begin{aligned} \dot{k} &\approx [sf(k^*) - (n+\delta)k^*] + [sf'(k^*) - (n+\delta)]\,(k-k^*) \\ &= [sf'(k^*) - (n+\delta)]\,(k-k^*) \\ &= -\beta\,(k-k^*) \\ &\quad \text{with } \beta = -(sf'(k^*) - (n+\delta)) \end{aligned}$$

Notice that one can show that β is positive!



Example: Cobb Douglas production function $f(k) = k^{\alpha}$

- We obtain $f'(k^*) = \alpha(k^*)^{\alpha-1} = \alpha f(k^*)/k^*$.
- Recalling $sf(k^*)/k^* = (n+\delta)$ we have

$$\beta = (n+\delta) - s\alpha f(k^*)/k^* = (1-\alpha)(n+\delta)$$

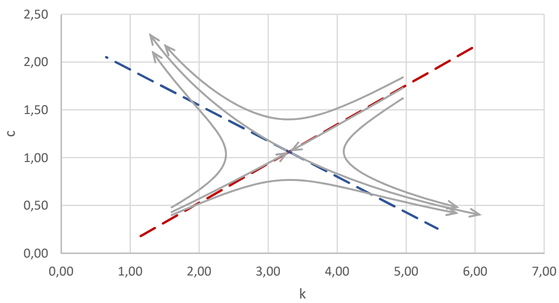
Although we started by approximating the nominal change $\dot{k}$ we can interpret the result in terms of the growth rate $\hat{k}$

$$\begin{aligned} \dot{k} &\approx -\beta\,(k-k^*) \\ \hat{k} &\approx -\beta\,\frac{k-k^*}{k} \end{aligned}$$

- The nominal change approximately decomposes into the negative of the product of the convergence rate and the deviation from equilibrium.

  Equivalently
- The growth rate approximately decomposes into the negative of the product of the convergence rate and the relative deviation from equilibrium.

- Linear approximation of a system near a rest point: Jacobian matrix, eigenvalues and eigenvectors



- Stable Manifold Theorem, Hartman-Grobman Theorem (Lawrence Perko 2001, Differential Equations and Dynamical Systems)

2.4 Convergence

Growth regressions
Assume efficiency of labor is given by $E(t)$, and is growing with constant exogenous rate g.
Redefine

$$k(t) = K(t)/E(t)L(t)$$

to measure capital intensity incorporating efficiency of labor.
Per capita output now is given by the augmented production function production

$$y(t) = E(t)\, f(k(t))$$

- y is still measured in per capital terms.
- k is measured incorporating efficiency of labor.
- In the augmented Solow-Swan model k will grow like per capita capital of a population growing with rate $g + n$.
- In the long run the growth rate of k will be equal to zero, whereas y will be growing with rate g.

Recall the Solow-Swan growth equation now augmented by efficiency growth

$$\widehat{k(t)} = sf(k(t))/k(t) - \delta - g - n$$

The growth rate of $\hat{y}$ relates to $\hat{k}$ as follows[5]

$$\begin{aligned} \widehat{y(t)} &= g + \widehat{f(k(t))} \\ &= g + \varepsilon_f(k(t))\,\widehat{k(t)} \end{aligned}$$

Approximation
In order to compute the Taylor approximation of $\hat{k}$ w.r.t. $\log k$ we need to compute the derivative of $f(k)/k$ w.r.t. $\log k$:

$$\begin{aligned} \frac{d\,\frac{f(k)}{k}}{d\,log(k)} &= \frac{d\,\frac{f(k)}{k}}{d\,k} \cdot k \\ &= \frac{f'(k)\cdot k - f(k)}{k^2} \cdot k \\ &= \frac{f'(k)\cdot k - f(k)}{k} \\ &= \left(\frac{f'(k)\cdot k}{f(k)} - 1\right) f(k)/k \\ &= (\varepsilon_f(k) - 1) f(k)/k \end{aligned}$$

[5]

$$\begin{aligned} \frac{d\,\log(f(k(t))}{d\,t} &= \frac{1}{f(k(t))} f'(k(t))\,\dot{k}(t) \\ &= \frac{k(t)}{f(k(t))} f'(k(t))\,\frac{\dot{k}(t)}{k(t)} \\ &= \varepsilon_f(k(t))\,\hat{k}(t) \end{aligned}$$

Approximation of input growth

$$\begin{aligned}\hat{k}(t) &\approx (sf(k^*)/k^* - \delta - g - n) \\ &\qquad + s(\varepsilon_f(k^*) - 1)\frac{f(k^*)}{k^*}(\log k(t) - \log k^*) \\ &\approx -(1 - \varepsilon_f(k^*))(\delta + g + n)(\log k(t) - \log k^*)\end{aligned}$$

Approximation of per capita output growth

$$\begin{aligned}\hat{y}(t) &\approx g + \widehat{f(k(t))} \\ &\approx g - \varepsilon_f(k^*)(1 - \varepsilon_f(k^*))(\delta + g + n)(\log k(t) - \log k^*) \\ &\approx g - (1 - \varepsilon_f(k^*))(\delta + g + n)(\log y(t) - \log y^*)\end{aligned}$$

and in the case of a Cobb-Douglas production function

$$\approx g - (1 - \alpha)(\delta + g + n)(\log y(t) - \log y^*)$$

Barro regression
This equation gives rise to a discrete time regression with an $\mathcal{N}(0, \sigma_\varepsilon^2)$ distributed error term. γ denotes the GDP growth rate.

$$\gamma_{i,t} = b^0 + b^1 \log y_{i,t} + \varepsilon_{i,t}$$

It is not difficult to see that the left hand side can be replaced by the average growth rate over a number of years. In that case the error term is the corresponding accumulated error.

β convergence
The analysis of the Solow-Swan model suggests $g = b^0 + b^1 \log y^*$ if g and y^* are identical in all the countries considered.

Moreover, the right hand side is larger (smaller) than g if y is smaller (larger) than y^*.

In particular it suggests the hypothesis $b^1 < 0$. It reflects the fact that countries with higher GDP should have smaller growth rates.

In the literature the test of this hypothesis is called test for *unconditional β convergence*.

σ convergence
In the Barro regression we can approximate the growth rate by the difference of log GDP and rearrange the equation

$$\begin{aligned}\log y_{i,t} - \log y_{i,t-1} &= b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t} \\ \log y_{i,t} &= b^0 + (1 + b^1) \log y_{i,t-1} + \varepsilon_{i,t}\end{aligned}$$

This gives rise to a difference equation for the variance σ of y and σ_η of the the error terms

$$\sigma_t^2 = (1 + b^1)^2\, \sigma_{t-1}^2 + \sigma_\eta^2$$

The observation and discussion of this issue goes back to a paper by Danny Quah (Quah 1993, The Scandinavian Journal of Economics).

σ versus β convergence

- If $b^1 < 0$ the growth process is called β convergent and y^* is stable.
- If $\sigma_t^2 < \sigma_{t-1}^2$ the cross section analysis of the growth process is called σ convergent.
- $\sigma_t^2 < \sigma_{t-1}^2$ can only hold only if $b^1 < 0$.
- If the growth process is only modestly β-convergent, σ-convergence may be violated due the dominance of σ_η^2.

Conditional β convergence
Country specific regression

$$\gamma_{i,t} = b_i^0 + b^1 \log y_{i,t} + \varepsilon_{i,t}$$

General formal estimation equation
If b_i^0 is decomposed into a number of specific economic effects, the regression is called a *f*ormal growth regression. The components mays be disaggregate accumulation activities, research and development expenditure, institutional efficiency, and the like.

General informal estimation equation
The generalized regression without a structural decomposition of country specific effects is called informal regression.

With a vector of further explaining variables $X_{i,t}$ and a vector of coefficients b^0 the model takes the form

$$g_{i,t} = X_{i,t}' b^0 + b^1 \log y_{i,t} + \varepsilon_{i,t}$$

A remark on stability and convergence
In economic literature the terms *s*tability and *c*onvergence are not always clearly separated. From a formal point of view the terms should be used the following way

- If the solutions of a differential equation converge to a particular dynamic equilibrium point (at least locally), the equilibrium is called (locally) stable.
- If a sample of initially different values become less different over time, the sample may be called convergent.

The Barro regression of a large sample of countries

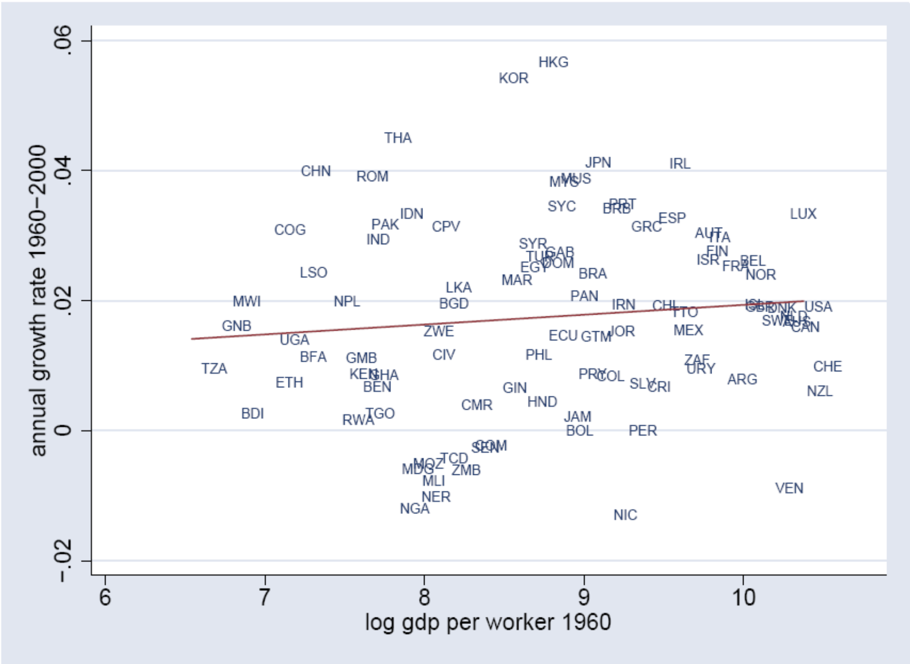


Figure 1.13. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for the entire world.

Quelle: Acemoglu 2008

The Barro regression of a selection of countries:

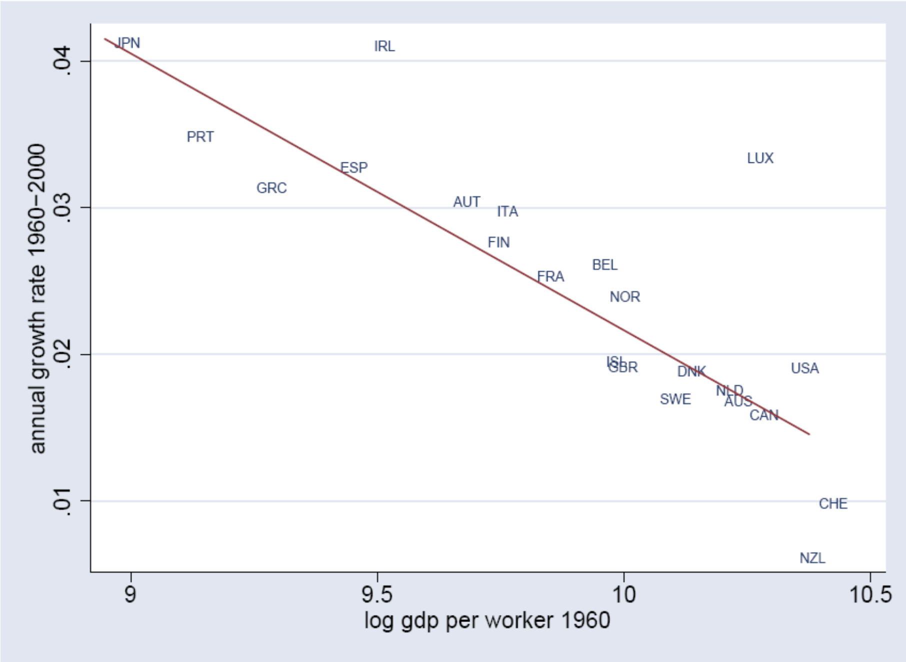


Figure 1.14. Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries.

Quelle: Acemoglu 2008

Stability and Convergence

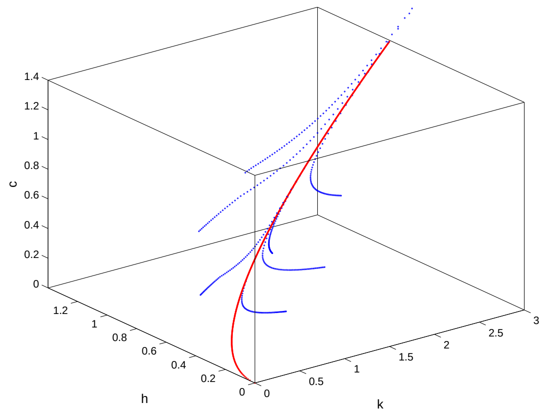


Figure 9: Optimal growth paths of the Lucas-Uzawa model

- At the core of this model there is a particular path with constant growth rates, the *balanced growth path (BGP)*.
- All other optimal paths starting from unbalanced initial states converge towards this path.Non-optimal paths will miss it: The BGP is **saddlepoint stable**.
- Some ptimal growth paths **converge** pointwise to one another, others don't.