Economic Growth

Problems Sets

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Problem Set 1 with Solutions

Labor is one of the most important factors of production. So let's take a look at some simple models of population growth. Let $N(t)$ be the size of the population, which can change over time. Let time run from zero to infinity, and $N_0 = N(0)$ the initial size of N.

Exercise 1 Assume that N changes with t in the following way

$$\dot{N}(t) = \nu N(t) \quad \text{or in short notation} \quad \dot{N} = \nu N$$

(1) What is the growth rate $\hat{N}$ of N?
Solution:

$$\hat{N} = \frac{\dot{N}}{N} = \nu$$

(2) Is $N(t) = N_0\, e^{\nu \cdot t}$ the function matching the dynamic assumption, i.e. solving the differential equation?
Solution:

$$\dot{N}(t) = \frac{d\,N(t)}{dt} = N_0\,\nu\, e^{\nu \cdot t} = \nu N(t)$$

Comment: This is the case of exponential population growth with a constant growth rate. The process is driven by a linear differential equation.

Exercise 2 Now assume that N changes with t in a different way

$$N(t) = N^* + (N_0 - N^*)e^{-\beta t}$$

with some positive constant β.

(1) What is the growth rate?
Solution:

$$\dot{N}(t) = -\beta(N_0 - N^*)e^{-\beta t} = -\beta\,(N(t) - N^*)$$

and hence

$$\hat{N}(t) = \frac{\dot{N}(t)}{N(t)} = -\beta \frac{N(t) - N^*}{N(t)}$$

Here the growth rate is not constant, unless $N^* = 0$ or $N(t) = N^*$. Furthermore, the growth rate is positive if $N(t) < N^*$ and negative if $N(t) > N^*$. $N(t)$ approaches N^* from below or from above. N^* is a stable equilibrium.

(2) What is the differential equation $\dot{N} = \dots$ of this process?

Solution:

$$\dot{N}(t) = -\beta \left(N(t) - N^* \right)$$

Comment: This is the case of linear differential equation augmented by a constant shift (βN^*).

(3) Describe the difference between the first model and the second in words!

Solution: In the first case the size of population grows exponentially without limits up to infinity. In the second case the population is bounded growing or shrinking monotonically from N_0 to N^*. N^* is the limit value and the limit point is stable. Empirically the first model is unlikely to be appropriate for long periods of time. A constant positive growth rate may be a good approximation of real data for a limited period of time only.

Exercise 3 An empirically plausible formal dynamic model of N matching recent data is the logistic model of the following form

$$N(t) = \frac{N_{max}}{1 + \frac{N_{max} - N_0}{N_0} e^{-\nu t}}$$

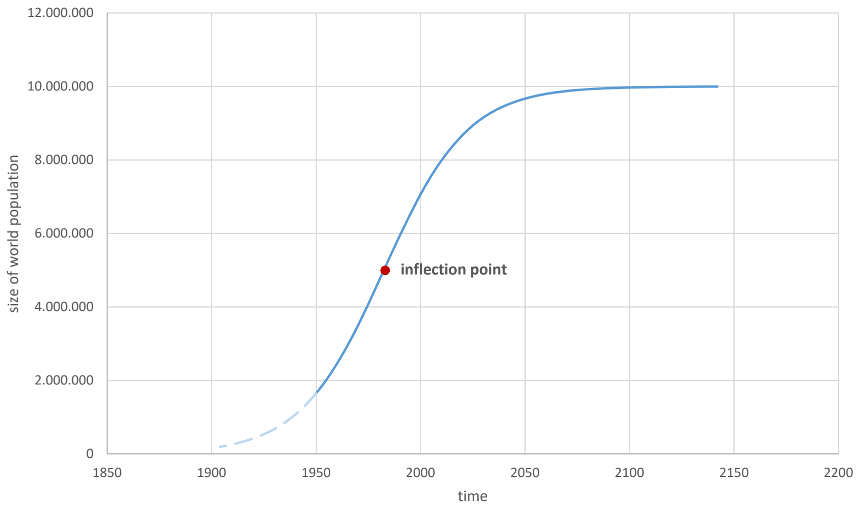
with initial value N_0 as before and a maximum value N_{max}

(1) What is the limit of $N(t)$ for $t \longrightarrow \infty$?

Solution: With $t \longrightarrow \infty$ the term $e^{-\nu t}$ goes to zero, because ν is positive. Hence $N(t)$ converges to N_{max}.

(2) Draw a rough sketch of $N(t)$!

Solution:



(3) Show that $N(t)$ as given above solves the following differential equation

$$\dot{N} = \nu N \left(1 - \frac{N}{N_{max}} \right)$$

Solution:

$$\begin{aligned}
\dot{N} = \frac{dN(t)}{dt} &= -\frac{N_{max}}{\left(1 + \frac{N_{max}-N_0}{N_0}e^{-\nu t}\right)^2} (-\nu) \frac{N_{max} - N_0}{N_0} e^{-\nu t} \\
& \text{According to the formula for } N(t) \text{ we substitute} \\
& \text{the expressions with the exponential function.} \\
& \text{First we substitute the quadratic term} \\
&= \nu \frac{N^2}{N_{max}} \frac{N_{max} - N_0}{N_0} e^{-\nu t} \\
& \text{Then we substitute the remaining term} \\
&= \nu \frac{N^2}{N_{max}} \left(\frac{N_{max}}{N} - 1 \right) \\
&= \nu N \left(1 - \frac{N}{N_{max}} \right)
\end{aligned}$$

(4) What is the limit of $\hat{N}(t)$ for $t \longrightarrow \infty$?

Solution: From the differential equation in (3) we get a simple formula for the growth rate

$$\hat{N} = \nu \left(1 - \frac{N}{N_{max}} \right)$$

We already know that $N(t)$ converges to N_{max} with $t \longrightarrow \infty$. Hence the growth rate converges to zero.

Comment: For $N(t)$ close to zero the growth rate is close to ν like in the exponential model! In other words, the logistic process is almost exponential at the beginning. But growth is becoming increasingly weaker and after an inflection point it is fading out towards stationarity in the long run.