Economic Growth

Problems Sets

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Problem Set 1

Exercise 1

Digression: The SIR-model of Mathematical Epidemiology

Three real variables S, I, N are relevant measures in the context of a pandemic

S is the number of susceptible individuals,

I the number of infectious individuals,

N the size of the population

Moreover, c is the infection rate, ω the recovery rate.[1]

$$\begin{aligned} \dot{I} &= c\,I\,S + \omega\,I \\ \dot{S} &= -c\,I\,S - \omega\,I \\ N &= I + S \end{aligned}$$

Verify:

(1) N is constant over time!

(2) The differential equations above are interdependent. Although the differential equation for I looks linear in I, it is actually quadratic.

(3) The system of differential equations $(\dot{I},\ \dot{S})$ can be transformed into two very similar unconnected, quadratic differential equations. Using a short hand $A = c\,N - \omega$ one of them is

$$\dot{I} = (A - c\,I)\,I$$

(4) If I takes the value I_0 at time zero the following function is the solution for $I(t)$.

$$I(t) = \frac{I_0 e^{At}}{1 + \frac{c}{A} I_0 \left(e^{At} - 1\right)}$$

(5) Find the solution for $S(t)$.

(6) For $t \to \infty$ the number of infected I will converge to $N - \omega/c$.

[1] The inverse of these rates is the average time it takes to get infected or to recover.

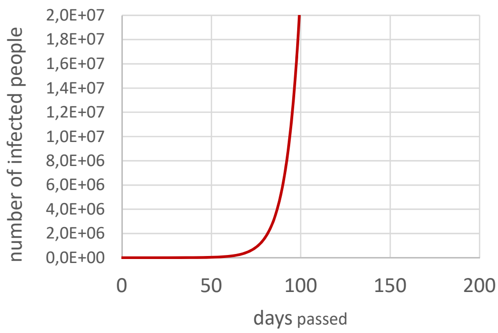

Figure 1: Exponential spreading

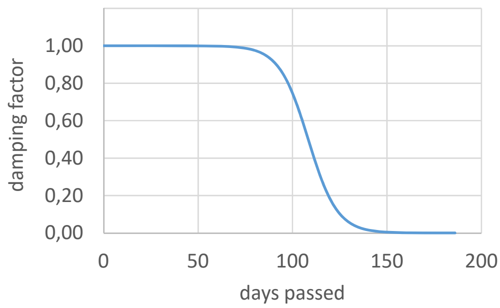

Figure 2: Damping of the exponential spreading

Let us put some numbers to the model get a feeling for what can happen in such a system.

$$c = 2 \cdot 10^{-9}, \qquad \omega = 0,03$$

The right hand side of the differential equation decomposes into two effects.

- The positive linear part $A\,I$ of $\dot{I}$ causes exponential growth due to spreading of the virus. This makes up the numerator of the solution.
- This linear part drives the infecion process if there are no limiting obstacles for the virus to spread. Various reaactions may cause a damping of spreading of the virus if the share of infected people is high.
- The negative quadratic part $-c\,I^2$ causes damping of the exponential growth due to saturation making up the denominator.
- The result may be almost exponential spreading at the beginning of the pandemic whereas spreading may come to a halt before the share of infected people reaches 100%.

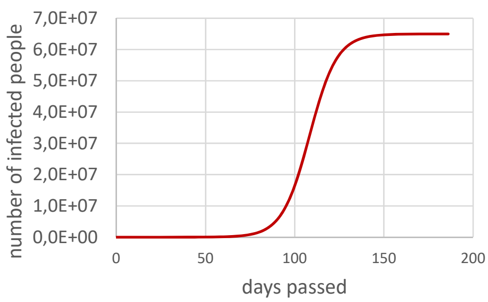

Figure 3: The SIR-model of mathematical epidemiology