Games and Information

Problems Sets

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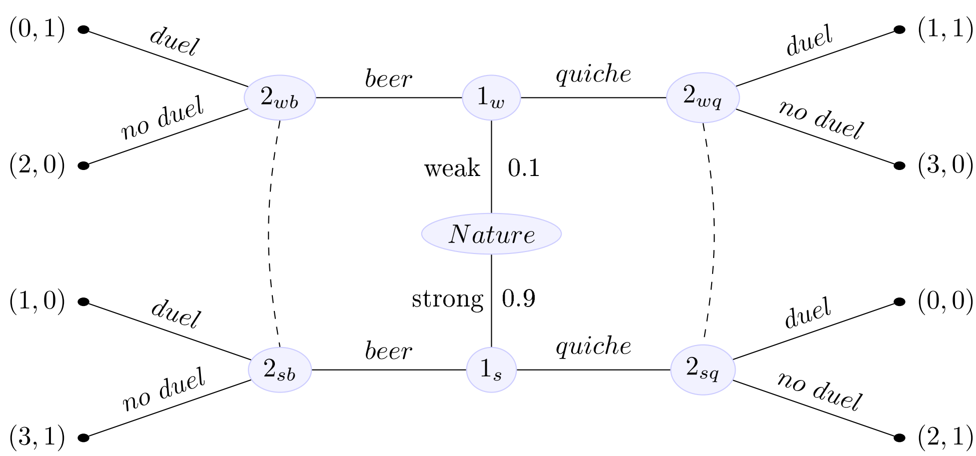
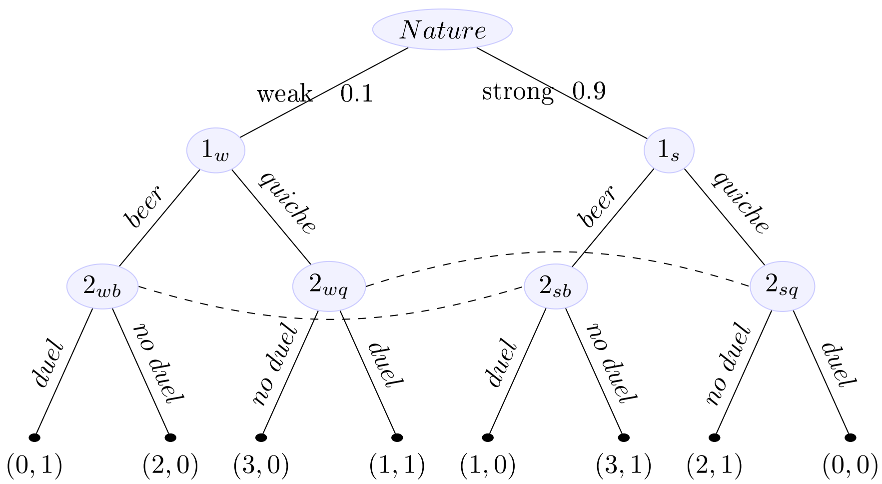
Problem Set 5

Exercise 9

Compare the two games below.





(1) The two game trees look different. I claim they represent the same game. How do you argue to make sure that this is true - or false?

Solution:

- All the elements (players, nodes, priors, actions, and pay-offs) are identical in both versions.
- Nodes that are connected by a branch in one version are connected in the other one as well.

- The order of play is the same in both versions.
- The information sets establish the same structure.

Altogether we can say the topology of both graphs is the same.

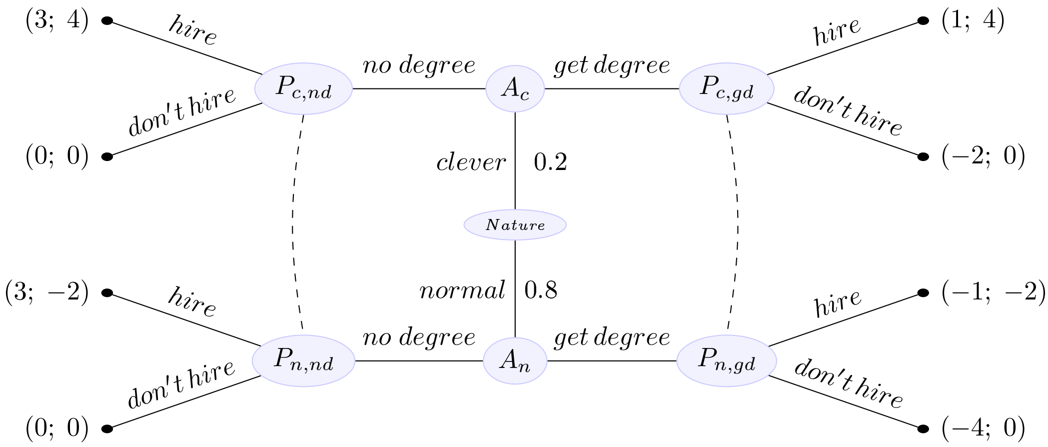
(2) What does the first game tree emphasize more clearly than the second?

Solution: In the first version the separation according to the action of player one into a left side of the tree and a right side is stressed. In the analysis this helps to focus on the question whether this action, the action of player one, can be interpreted as a signal. On the other hand, the sequence of moves plays a lesser role in the representation.

In the second version, the freedom of player one to choose between two moves, regardless of nature's random decision, has a greater weight in the presentation.

Exercise 10

A version of Spence's Education Game



Player A can be seen as the agent in this game, player P as the principal.

(1) List the strategies of the agent and identify pooling and separating strategies!

Solution:[1]

$$\begin{aligned}
\sigma_1^A &= (\textit{get degree} \,|\, \textit{clever},\ \textit{get degree} \,|\, \textit{normal}) \\
\sigma_2^A &= (\textit{get degree} \,|\, \textit{clever},\ \textit{no degree} \,|\, \textit{normal}) \\
\sigma_3^A &= (\textit{no degree} \,|\, \textit{clever},\ \textit{get degree} \,|\, \textit{normal}) \\
\sigma_4^A &= (\textit{no degree} \,|\, \textit{clever},\ \textit{no degree} \,|\, \textit{normal})
\end{aligned}$$

σ_1 and σ_4 are pooling strategies; σ_2 and σ_3 are separating. Intuitively, σ_2^A appears to be the most plausible strategy of the agent.

(2) What are the Bayesian beliefs associated with these strategies?

Solution: The principal does not know the type of agent. He forms beliefs according to the respective strategies of the agent. In case of pooling strategies there is no updating: The Bayesian beliefs are equal to the priors on the path of the strategy. The separating strategies are fully informative leading to Bayesian beliefs equal to one or zero in the obvious way.

Let μ_i denote the principal's beliefs if he assumes the agent uses strategy σ_i^A. Each μ_i consists of beliefs for each information set, i.e. beliefs in case he observes *no degree*, and beliefs in case he observes *get degree*, $\mu_i = (\mu_i^{\textit{no degree}}, \mu_i^{\textit{get degree}})$. And in turn, each $\mu_i^{\textit{action}}$ consists of two probabilities. Here, let the first entry denote the probability that the agent is *clever*, and let the second stand for the case of a *normal* agent.

Below you find the Bayesian beliefs and the undetermined beliefs off the respective paths. Below we use shorthands *nd*, *gd* for *no degree, get degree*, and *nBb* for *no*

[1] We could use a short notation where the first element denotes the action if the agent is clever, and the second element denotes the action in case the agent is *normal*. However, we would have to remember the meaning of the short hands all the time. The standard explicit notation is easier to handle.

Bayesian beliefs.

$$\begin{aligned}
\mu_1 = (\mu_1^{nd},\ \mu_1^{gd}) &= (\ nBb\ ,\ (0.2;\ 0.8)\ ) \\
\mu_2 = (\mu_2^{nd},\ \mu_2^{gd}) &= (\ (1;\ 0)\ ,\ (0;\ 1)\ ) \\
\mu_3 = (\mu_3^{nd},\ \mu_3^{gd}) &= (\ (0;\ 1)\ ,\ (1;\ 0)\ ) \\
\mu_4 = (\mu_4^{nd},\ \mu_4^{gd}) &= (\ (0.2;\ 0.8)\ ,\ nBb\ )
\end{aligned}$$

(3) What are the best responses of the principal based on Bayesian beliefs?

Solution: The strategies of the principal are

$$\begin{aligned}
\sigma_1^P &= (hire \,|\, no\,degree,\ hire \,|\, get\,degree) \\
\sigma_2^P &= (hire \,|\, no\,degree,\ don't\,hire \,|\, get\,degree) \\
\sigma_3^P &= (don't\,hire \,|\, no\,degree,\ hire \,|\, get\,degree) \\
\sigma_4^P &= (don't\,hire \,|\, no\,degree,\ don't\,hire \,|\, get\,degree)
\end{aligned}$$

If the agent uses a pooling strategy, the principal's pay-off is independent of the choice between σ_1^A or σ_4^A. The choice between pooling on *no degree* and pooling on *get degree* has no influence on the principal's pay-offs. It only matters for the principal's pay-off whether he hires a *clever* or a *normal* agent.

Using the priors the expected pay-off from *hire* is $0.2 \cdot 4 - 0.8 * 2 = -0.8$ whereas *don't hire* yields 0. I.e. $\sigma_4^P = (don't\,hire \,|\, no\,degree,\ don't\,hire \,|\, get\,degree)$ is the best response to any pooling strategy. Pooling leads to a break down of the job market.

The principal definitely wants to avoid hiring a *normal* agent. Whether the agent has acquired a *degree* or not is irrelevant. However, *hiring* an agent bears a high risk to get a *normal* agent and a negative pay-off. Therefore, *don't hire* is the best response no matter what the agent's action is. Only useful, reliable information about the type of the agent can change this outcome.

If the agent uses a separating strategy σ_2^A or σ_3^A, the principal will identify the type of agent. The best response to σ_2^A is $\sigma_3^P = (don't\,hire \,|\, no\,degree,\ hire \,|\, get\,degree)$ in order to hire only *clever* agents. The principal's expected pay-off now is positive and equal to 0.8.

Similarly, the best response to σ_3^A is $\sigma_2^P = (hire \,|\, no\,degree,\ don't\,hire \,|\, get\,degree)$. Again, the principal's expected pay-off is positive and equal to 0.8. For the principal this less plausible separating strategy of the agent is fine. The only thing that matters for him is the information transmitted by the strategy.

Summing up the results on best responses of the principal, σ^{P*}, to the strategies of the agent we have

$$\begin{aligned}
\sigma^{P*}(\sigma_1^A) &= \sigma_4^P \\
\sigma^{P*}(\sigma_2^A) &= \sigma_3^P \\
\sigma^{P*}(\sigma_3^A) &= \sigma_2^P \\
\sigma^{P*}(\sigma_4^A) &= \sigma_4^P
\end{aligned}$$

(4) Find the perfect Bayesian equilibria of the game?

Solution:

- As the principal will never hire the agent, if agents use a pooling strategy, the agent should pool on *no degree*, i.e. σ_4^A to get a zero pay-off instead of a negative one. This establishes the pooling equilibrium $(\sigma_4^A;\ \sigma_4^P)$ with a break down of the job market.
- The plausible separating strategy σ_2^A has best response σ_3^P. The principal hires on the basis of a degree shown by the agent. In turn this has best response σ_2^A. We have a separating equilibrium $(\sigma_2^A;\ \sigma_3^P)$ where the principal hires only clever agents.
- The non-plausible separating strategy σ_3^A has best response σ_2^P. Despite of the irritating strategy of the agent the principal will identify the type of agent correctly and hire only clever agents. But in turn the best response of the agent to σ_2^P is to pool on *no degree*, i.e. σ_4^A. Hence, the non-plausible separating strategy σ_3^A cannot be an equilibrium-strategy.

Altogether we have a pooling equilibrium $(\sigma_4^A;\ \sigma_4^P)$ and a separating equilibrium $(\sigma_2^A;\ \sigma_3^P)$.

(5) Explain the role of signaling in this game!

Solution: The pay-offs in the pooling equilibrium are equal to zero in any case for agents as well as for the principal.

The pay-offs of the separating equilibrium are random depending on nature's move. In case the agent is clever, agent and principal get a positive pay-off. In case of a normal agent both get a zero pay-off. The expected pay-off of both of them is positive.

Hence, the separating equilibrium is Pareto-superior compared to the pooling equilibrium. Agents should understand that as well as principals. If the agent is clever he should take the chance, get a degree, and get the job. The principal should understand that nobody wants the pooling equilibrium and trust the signal. They should assume that only clever agents take the degree, and therefore they should hire agents if and only if they show a degree.