Games and Information

Problems Sets

Summer Semester 2025

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Problem Set 3

Exercise 4

Recall the concept of stochastic dominance, as presented in the lecture.

(1) What is the formal definition that a distribution p^H for returns $x_1 \leq x_2 \leq \cdots \leq x_n$ stochastically dominates p^L of the first order?

Solution:

$$p_1^H < p_1^L \text{ , and } p_1^H + p_2^H < p_1^L + p_2^L, \ldots \text{and } p_1^H + p_2^H + \cdots + p_{n-1}^H < p_1^L + p_2^L + \cdots + p_{n-1}^L$$

Remember, that we always have $\sum_{i=1}^n p_i^H = \sum_{i=1}^n p_i^L = 1$.

(2) The distributions p^H and p^L give rise to continuous distributions $prob^H$ and $prob^L$ such that $prob^H\{x \leq \bar{x}\}$ is the probability that, according to p^H, the variable x assumes any value less than or equal to $\bar{x}$. Similarly, $prob^L\{x \leq \bar{x}\}$ is defined in relation to p^L.

Relate this definition to the statement

$$prob^H\{x \leq x_i\} < prob^L\{x \leq x_i\} \text{ for all } \; x > 0 \text{ and } x_i,\, i < n$$

Solution:

The definitions of the probabilities together with the ranking of the x_i imply

- $prob^L\{x \leq x_1\} = prob^L\{x = x_1\} = p_1^L$,
- $prob^L\{x \leq x_2\} = prob^L\{x = x_1 \text{ or } x = x_2\} = p_1^L + p_2^L$

  $\vdots$
- $prob^L\{x \leq x_{n-1}\} = prob^L\{x = x_1 \text{ or } \ldots \text{or } x = x_{n-1}\} = p_1^L + \cdots + p_{n-1}^L$
- Notice that the final term of this sequence would be $prob^L\{x \leq x_n\} = 1$.

The corresponding equalities hold for high effort as well. The combination of the statement for low and high effort completes the task.

$$p_1^H < p_1^L \text{ , and } p_1^H + p_2^H < p_1^L + p_2^L, \ldots$$

is equivalent to

$$prob^H\{x \leq x_i\} < prob^L\{x \leq x_i\} \text{ , for all } i < n$$

Exercise 5

A very simple case of the moral hazard setting discussed in the lecture is the one with two effort levels $e^H > e^L$, and two levels of outcome $x_1 < x_2$. The outcome probabilities are p_i^H and p_i^L.

Moreover, let us assume $u(w) = \ln(w)$ and $v(e) = e^2$.

(1) Recall the condition for the principal to have higher expected revenue in case of high effort and reduce the inequality to a simple condition in terms of p_1^L and p_1^H.

Solution:

$$
\begin{aligned}
&p_1^L x_1 + p_2^L x_2 < p_1^H x_1 + p_2^H x_2 \\
&p_1^L x_1 + (1 - p_1^L) x_2 < p_1^H x_1 + (1 - p_1^H) x_2 \\
&p_1^L (x_1 - x_2) + x_2 < p_1^H (x_1 - x_2) + x_2 \\
&(p_1^L - p_1^H)(x_1 - x_2) < 0 \\
&p_1^L > p_1^H \quad \text{as } x_1 - x_2 < 0
\end{aligned}
$$

Of course, due to $n = 2$ the last line is equivalent to $p_2^L < p_2^H$.

(2) Assume the condition of (1) is satisfied, extend the comparison to the case of the principal's profit, and comment on the result.

Solution:

$$
\begin{aligned}
&p_1^L (x_1 - w_1) + p_2^L (x_2 - w_2) < p_1^H (x_1 - w_1) + p_2^H (x_2 - w_2) \\
&(p_1^L - p_1^H)(x_1 - x_2) < (p_1^L - p_1^H)(w_1 - w_2) \\
&x_1 - x_2 < w_1 - w_2 \\
&w_2 < w_1 + x_2 - x_1
\end{aligned}
$$

w_2 can exceed w_1 but only by less than $x_2 - x_1$.

(3) Write down the participation constraint (PC) for the utility and disutility functions specified above assuming that high effort is preferred by the principal.

Solution:

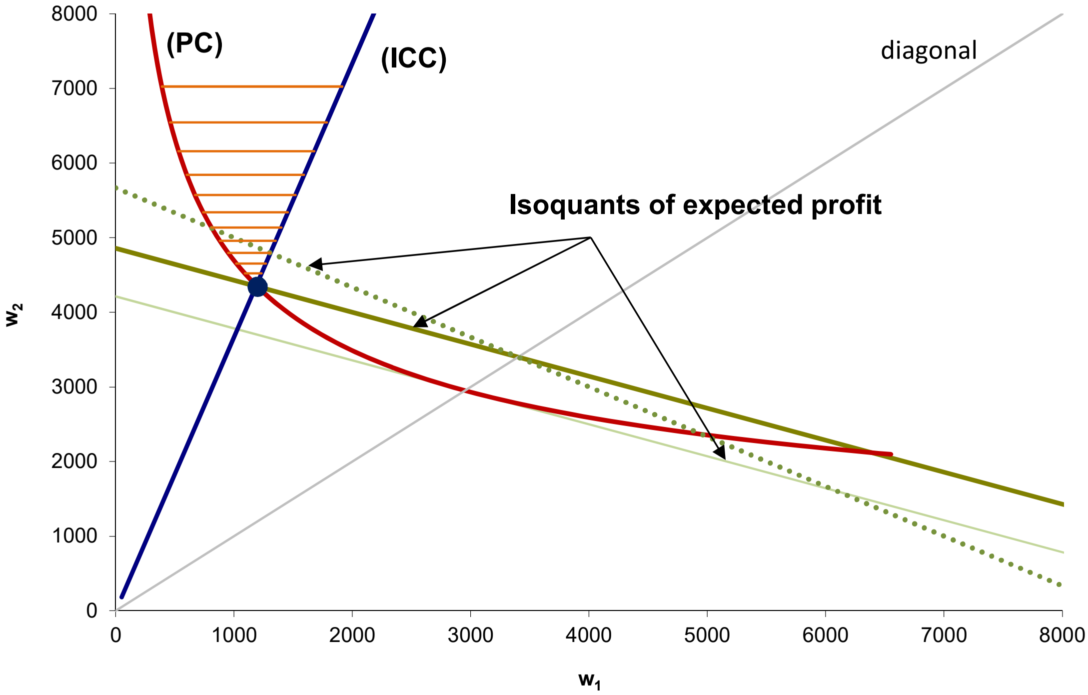
$$p_1^H \ln(w_1) + p_2^H \ln(w_2) - {e^H}^2 \geq \underline{U}$$

(4) Write down the incentive compatibility constraint (ICC) for the utility and disutility functions specified above, and reduce it to a simple form.

Solution:

$$
\begin{aligned}
&p_1^H \ln(w_1) + p_2^H \ln(w_2) - (e^H)^2 \geq p_1^L \ln(w_1) + p_2^L \ln(w_2) - (e^L)^2 \\
&p_1^H \ln(w_1) - p_1^H \ln(w_2) - (e^H)^2 \geq p_1^L \ln(w_1) - p_1^L \ln(w_2) - (e^L)^2 \\
&0 \geq (p_1^L - p_1^H) \ln(w_1) - (p_1^H - p_1^L) \ln(w_2) + \left(e^H\right)^2 - \left(e^L\right)^2 \\
&0 \geq (p_1^L - p_1^H)(\ln(w_1) - \ln(w_2)) + \left(e^H\right)^2 - \left(e^L\right)^2 \\
&\ln(w_2) - \ln(w_1) \geq \frac{\left(e^H\right)^2 - \left(e^L\right)^2}{p_1^H - p_1^L} \\
&\frac{w_2}{w_1} \geq exp\left(\frac{\left(e^H\right)^2 - \left(e^L\right)^2}{p_1^H - p_1^L}\right)
\end{aligned}
$$

(5)



Solution:

The graph with wages w_1, w_2 on the axes depicts:

- The convex participation constraint on the basis of high effort chosen. A contract below the curve will be rejected by the agents.
- The incentive compatibility constraint which is linear for the case of logarithmic utility of wages. To the right of the line agents will choose low effort.
- Hatching of the area where (PC) and (ICC) are hold with inequality.
- Three isoquants of profits of the principal. The thick solid one through the intersection of (PC) and (ICC) is based on high effort probabilities of outcome. I.e.: The slope is $-p_1^H/p_2^H$. It corresponds to the highest level of profit which can be reached satisfying (PC) and (ICC). The blue dot is the optimal contract.

- The thin line with slope $-p_1^H/p_2^H$ as well is added to demonstrate that the iso-profit lines based on high effort probabilities and the (PC) curve which is based on these probabilities as well have the same slope on the diagonal. The dotted line has slope $-p_1^L/p_2^L$. It is steeper than the other lines.