Games and Information

MEPS Course

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Karl-Josef Koch

School of Economic Disciplines,
University of Siegen

Contents

In Part II of the course we will look at further ways to classify extensive form games with imperfect information. Moral hazard, adverse selection and signaling will be the elements of structure of games, and we will discuss pooling and separating equilibria, the elements of classification of solutions.

3 Contract Theory

In Part II of the course we will look at further examples and ways to classify these extensive form games with imperfect information.

Moral hazard, adverse selection and signaling will be the structural elements of these games.

Pooling and separating equilibria we will work out as the categories of classification of the solutions.

3.1 Principal Agent Theory

Consider a framework characterized by the following features

- A principal thinks of entering into an agreement with an agent.
- The agent is supposed to perform a particular task.
- The outcome of the agent's activities depends on the effort the agent applies and is affected by random events.
- The principal receives revenues resulting from the agent's activities.
- The agent receives a (wage) payment from the principal.

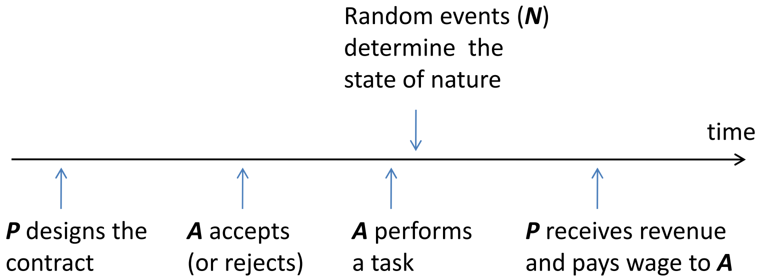
- The principal designs and offers the agent a contract c.
- The agent can reject the contract or accept it.
- If he accepts the offer, he then decides to apply a low or a high level of effort.
- The principal may not know what effort the agent applies. The principal's revenue jointly depends on the unobserved state of nature and the unobserved effort of the agent.
- The agent's pay off depends on revenue.

Representation of a Principal Agent Model

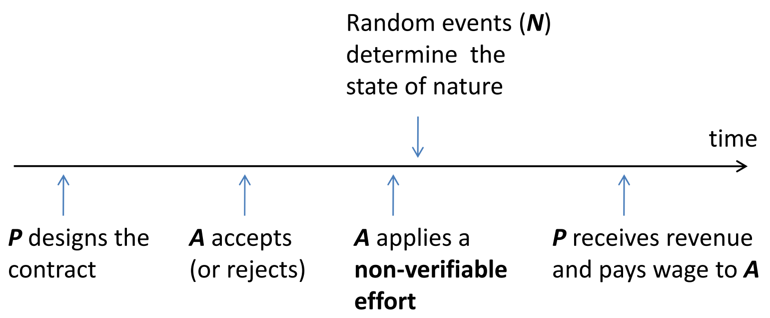
- The elements of the strategic conflict described above fit together in a huge game represented by a game tree.
- However, the nested structure with a continuum of action at each stage and the information structure makes the tree difficult to handle .
- A more appropriate representation sets the standards in the literature.
- It provides a better overview and it helps to classify the different cases of problems.

3.2 Classification of Problems

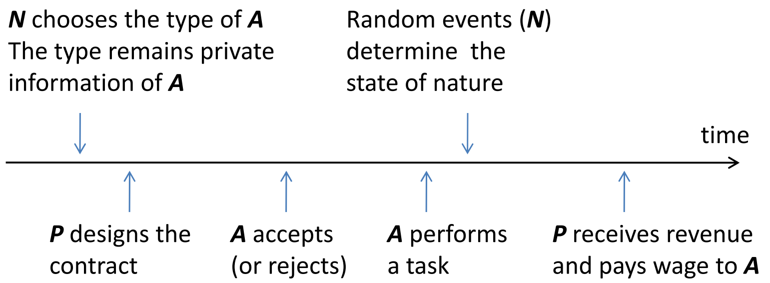
Actions, events and timing of the base model
Let P denote the principal and A the agent.



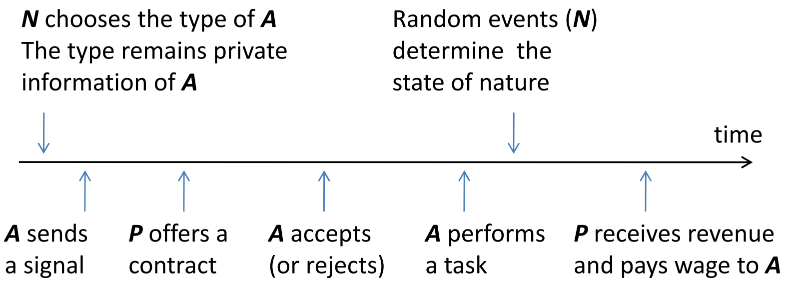
Moral Hazard
The agent's action cannot be observed by the principal (hidden action)



Adverse selection
The agent holds private information from the beginning



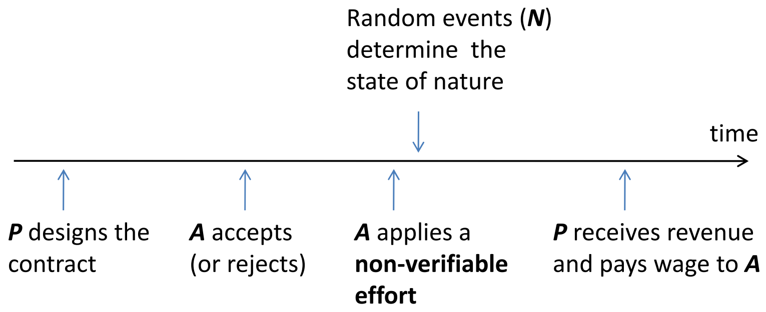
Signalling
The agent can send a signal to the principal



3.3 The Moral Hazard Problem

Hidden action

The agent's action is not observable



Searching for an optimal contract

- There is s strategic conflict between principal and agent.
- The agent's action cannot be observed by the principal.
- The principal infers a rational prediction about the agent's action.
- The principal solves the problem by backward induction.
- The solution is a subgame perfect Nash-equilibrium.

The framework

- The agent is supposed to perform a task yielding outcome x in monetary value.
- Consider n different possible outcomes x_i, $i = 1, \ldots, n$.
- They occur randomly with probalities p_i and will be observed by the principal and the agent. We refer to i as the state of nature.
- The probabilities may depend on the level of effort e applied by the agent, $p_i(e)$.
- The contract offers the agent a wage w_i in state i. The wage is not specified conditional on effort because e is private information of the agent.
- The agent has utility $u(w_i)$ of wage and dis-utility $v(e)$ of effort. He prefers high wage and low effort.

Backward induction

- Final stage:
  (i) The principal wants the agent to accept the contract,
  (ii) and the agent chooses a utility maximizing effort level.

  (i) imposes a constraint on the contract called
  participation constraint (**PC**)

  $\sum_{i=1}^{n} p_i(e)u(w_i) - v(e) \geq \underline{U}$

  The agent's expected utility should not fall short of the level he can achieve somewhere else.

  (ii) imposes a further constraint on the contract called
  incentive compatibility constraint (**ICC**)

  e solves: $\max_e \sum_{i=1}^{n} p_i(e)u(w_i) - v(e)$

  Without observing the effort the principal should expect the agent to apply this level of effort.

- First stage: The principal proposes a contract[1] $(w_1, \ldots, w_n)$ based on the possible revenues $x_1 \leq \cdots \leq x_n$ which maximizes the principal's profit (net revenue) subject to the constraints of the second stage [2]:

$$\max_{\{w_1,\ldots,w_n\}} \quad \sum_{i=1}^{n} p_i(e)\,(x_i - w_i)$$

$$\text{s.t. (PC) and (ICC)}$$

- The moral hazard problem of this form can be considered as a sequential game with the principal as a first player, the agent as second player, and nature as third player. (Nature does not pursue its own goal.)
- The strategy set of the principal is the set of all wage schedules $\{(w_1, \ldots, w_n)\}$.
- The strategy set of the agent is the set of all feasible effort levels $\{e\}$.
- The solution is the subgame perfect equilibrium of the corresponding game given by the strategy profile $((w_1, \ldots, w_n), e)$ of principal and agent.

[1] One may include the desired level of effort in the contract. However, it is determined endogenously by the incentive compatibility constraint and is not observable, anyway.

[2] Notice that we consider the case of a risk neutral principal. He considers the expected value of profit, whereas the agent uses utility and dis-utility functions reflecting risk aversion.

The case of two effort levels

- Assume $e \in \{e_L, e_H\}$, x_i ordered from worst to best: $x_1 \leq \cdots \leq x_n$.
- Notation $p_i(e_H) = p_i^H$ and $p_i(e_L) = p_i^L$.
- It seems natural that the principal prefers high effort to low effort. A first assumption in this direction addresses expected revenues. High effort should yield a gain in expected revenues

$$\sum_{i=1}^{n} p_i^H x_i \geq \sum_{i=1}^{n} p_i^L x_i \quad \text{or equivalently} \quad \sum_{i=1}^{n} (p_i^H - p_i^L) x_i \geq 0$$

- However, it is profit that counts for the principal

$$\sum_{i=1}^{n} p_i^H (x_i - w_i) \geq \sum_{i=1}^{n} p_i^L (x_i - w_i)$$

- We rearrange terms and get

$$\sum_{i=1}^{n} (p_i^H - p_i^L) x_i \geq \sum_{i=1}^{n} (p_i^H - p_i^L) w_i$$

  - Obviously, the (weak) inequality holds if wages are equal to revenues wiping out all profits.
  - Furthermore, it holds if high effort yields a gain in expected revenues, and if all wages are identical. In this case the left hand side is positive, and the right hand side is equal to zero.

$$\sum_{i=1}^{n} (p_i^H - p_i^L) w = w \left(\sum_{i=1}^{n} p_i^H - \sum_{i=1}^{n} p_i^L \right) = 0$$

- However, the latter never provides an incentive to supply high effort! Indeed, in this case the expected utility of wage income is independent of effort

$$\sum_{i=1}^{n} p_i(e) u(w) = u(w) \sum_{i=1}^{n} p_i(e) = u(w)$$

  whereas dis-utility is smaller with low effort.
- It remains to find conditions for the probabilities such that the gain in revenues due to high effort can outweigh the gain in expected wage by more than the dis-utility of higher effort.

Summing up we have the following two conditions which must hold for revenues, wages and probabilities of the solution

$$\sum_{i=1}^{n}(p_i^H - p_i^L)x_i \geq \sum_{i=1}^{n}(p_i^H - p_i^L)w_i$$

and

$$\sum_{i=1}^{n}(p_i^H - p_i^L)u(w_i) \geq v(e^H) - v(e^L)$$

We can evaluate the ranking of revenues to get two kinds of conditions such that at least expected revenues raise with higher effort. It then remains to check whether there is enough scope for appropriate wage setting.

- An intuitive assumption is that of monotonous likelihood

  $$\frac{p_1^L}{p_1^H} > \frac{p_1^L + p_2^L}{p_1^H + p_2^H} > \ldots$$

  It is not necessary but sufficient for the preference for high effort.

  In particular it implies

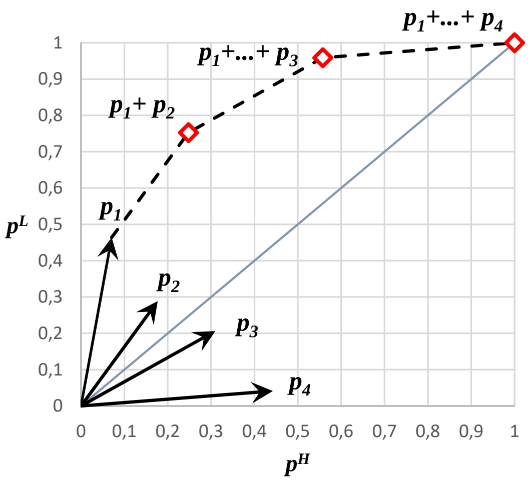
  $$\frac{p_1^L}{p_1^H} > \frac{p_n^L}{p_n^H}$$

The case of two effort levels

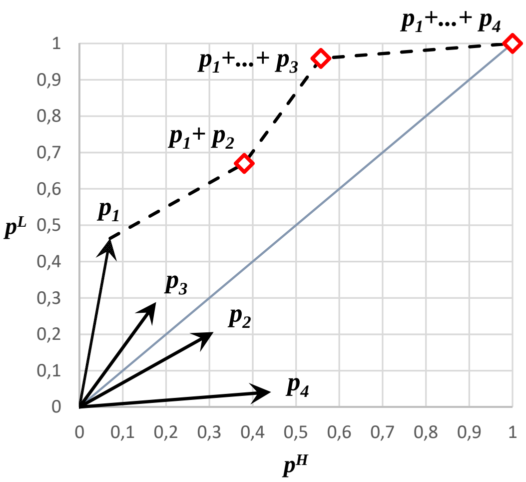
- A weaker but still sufficient assumption is, that p^H stochastically dominates p^L of first order.

  $$p_1^H < p_1^L \text{ , and } p_1^H + p_2^H < p_1^L + p_2^L, \ldots$$

Monotonous Likelihood in Case of Two Effort Levels



Non-monotonous Likelihood in Case of Two Effort Levels



The incentive compatibility constraint is supposed to implement an incentive to supply high effort.

In other words, the expected utility of high effort must be larger than or at least equal to that of low effort.

$$\textbf{(ICC)} \qquad \sum_{i=1}^{n} p_i^H u(w_i) - v(e^H) \geq \sum_{i=1}^{n} p_i^L u(w_i) - v(e^L)$$

$$\text{or equivalently} \qquad \sum_{i=1}^{n} (p_i^H - p_i^L)\, u(w_i) \geq v(e^H) - v(e^L)$$

The latter version of (ICC) means that an increase in expected utility of income due to higher effort has to outweigh the increase of dis-utility.

Searching for an optimal contract $(w_1, \dots, w_n)$

$$\max_{\{w_1,\dots,w_n\}} \quad \sum_{i=1}^{n} p_i^H (x_i - w_i) \qquad \text{s.t. (PC) and (ICC)}$$

Lagrangean function[3]

$$\begin{aligned} \mathcal{L}(w_i, \lambda, \mu) &= \sum_{i=1}^{n} p_i^H (x_i - w_i) + \lambda \left[\sum_{i=1}^{n} p_i^H u(w_i) - v(e^H) - \underline{U} \right] \\ &\quad + \mu \left[\sum_{i=1}^{n} (p_i^H - p_i^L)\, u(w_i) - v(e^H) + v(e^L) \right] \end{aligned}$$

First order conditions and shadow prices
The first order conditions with respect to w_i are

$$-p_i^H + \lambda p_i^H u'(w_i) + \mu (p_i^H - p_i^L) u'(w_i) = 0 \quad \text{for } i = 1, \dots, n$$

Rearrange terms to check whether the constraints are binding

$$\frac{p_i^H}{u'(w_i)} = \lambda\, p_i^H + \mu\, (p_i^H - p_i^L) \quad \text{for } i = 1, \dots, n$$

Summation over i yields $0 < \sum_{i=1}^{n} \frac{p_i^H}{u'(w_i)} = \lambda$

Notice, that all other terms vanish, because the probabilities p_i^H and p_i^L resp. sum up to unity.

We conclude $\lambda > 0$, and therefore (PC) is binding.

The interpretation of the shadow price equation for λ is straight forward.

$$\lambda = \sum_{i=1}^{n} \frac{p_i^H}{u'(w_i)} = \sum_{i=1}^{n} p_i^H \cdot \frac{dw_i}{du}$$

dw_i/du is the marginal wage reduction caused by a shrinking of utility. This marginal effect is small if w_i is small; it is large if w_i is large.

- The sum of these wage reductions weighted by probabilities amounts to an expected profit gain of the principal.
- The relaxation of the binding effect of (PC) as it may be caused by a lower value of $\underline{U}$ allows the principal to reduce the expected utility of the agent.
- The cost reduction raises the principal's expected profit, and this is what λ is supposed to measure!

[3] The arguments of the Lagrangean function always are the elements of the contract together with all shadow prices. In order to simplify notation we drop the arguments in what follows.

In order to check whether ICC is binding we assume the opposite, i.e. μ is equal to zero. Then the first order conditions reduce to $\lambda u'(w_i) = 1$ and hence the wage schedule w has to be flat.

But then (ICC)

$$\sum_{i=1}^{n} (p_i^H - p_i^L)\, u(w_i) \geq v(e^H) - v(e^L)$$

reduces to $0 \geq v(e^H) - v(e^L)$ which is wrong by assumption.

We conclude $\mu > 0$, and therefore (ICC) is binding as well.

- Although both constraints are binding as before, the optimal contract creates the incentives to make the agent supply high effort.
- The principal does not have to pay higher wages on average, he only has to offer a differentiated wage schedule instead of full insurance.

Characterization of the differentiated wage schedule

Recall the first order conditions and divide by p_i^H

$$\frac{1}{u'(w_i)} = \lambda + \mu \, \frac{p_i^H - p_i^L}{p_i^H} \quad \text{for } i = 1, \ldots, n$$

- The term
$$\frac{p_i^H - p_i^L}{p_i^H} = 1 - \frac{p_i^L}{p_i^H}$$
measures the relative change of probabilities due to an increase in effort from low to high.
- p_i^L / p_i^H is called the likelihood ratio due to a decrease of effort.
- Due to first order stochastic dominance the term one minus likelihood ratio is negative for $i = 1$, changes sign for some i, and is positive for $i = n$.
- The shadow prices λ and μ measure the value the principal can assign to a relaxation of the corresponding constraint.

Recall the first order conditions devided by p_i^H again

- If $p_i^H = p_i^L$ for some i, the wage rate is independent of probabilities and solves

$$u'(w_i) = 1/\lambda$$

  It depends on the shadow price of the participation constraint (PC).

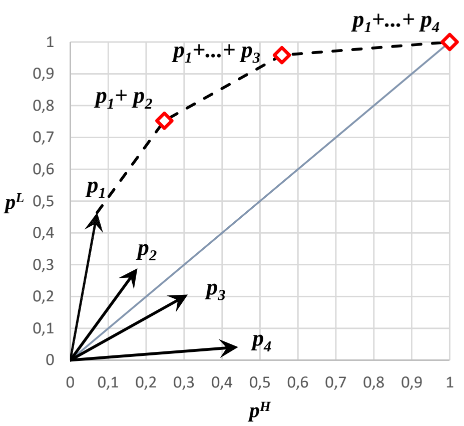
- The smaller the likelihood ratio (i.e. the less revealing a high outcome w.r.t. the effort devoted to the job), the larger the right hand side of the FOC above

  $\hookrightarrow$ the larger w_i must be.If the right hand side is larger, the **marginal utility** must be smaller, and therefore the wage must be larger!

- Remember: $x_1 \leq x_2 \leq x_3 \ldots$
- Use notation $p_i = (p_i^H, p_i^L)$.
- Stochastic dominance: p_1, $p_1 + p_2$, $p_1 + p_2 + p_3$ are located above the diagonal in a p_i^H, p_i^L-diagram.
- The likelihood ratios are the slopes of the vectors p_i.
- w_i increases with i, if and only if the likelihood ratio decreases with i.

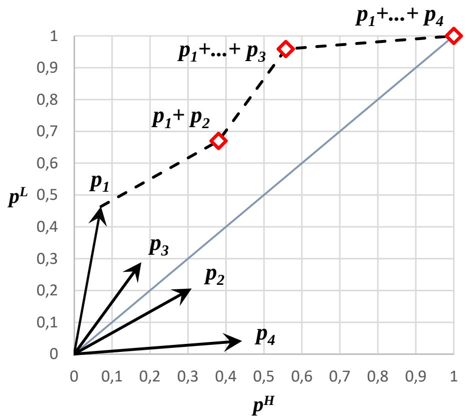
The pictures below illustrate different cases of monotonous and non-monotonous relationships between revenues and wages.

The case of monotonous likelihood ratio



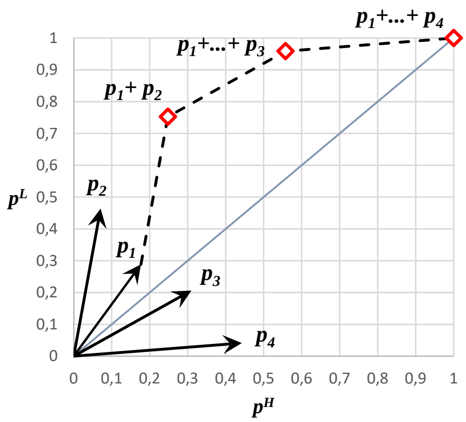
$w_1 < w_2 < w_3 < w_4$

The case of a non-monotonous likelihood ratio I



$w_1 < w_3 < w_2 < w_4$

The case of a non-monotonous likelihood ratio II



$w_2 < w_1 < w_3 < w_4$

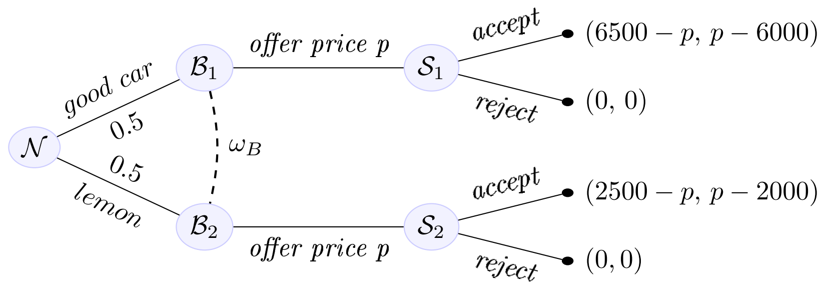
Final remarks on moral hazard

- The analysis can be extended to the case of a continuous effort variable in a straight forward manner (cf. Macho-Stadler and Pérez-Castrillo, 2001).
- We considered the case of hidden action and not that of hidden information (cf. Hart and Holmström, 1987).
- There are interesting recent empirical investigations on moral hazard (e.g. Hoppe and Schmitz, 2018).
- The main conclusion is that risk can be shared through a contract and at the same time incentives can be implemented to reduce the risk.

- Nevertheless, although the starting point of reasoning is the the assumption of a lack of information, designing "good" contracts requires a lot of basic information about the environment, preferences ect.

3.4 The Adverse Selection Problem

The Lemons' Problem I
In a used car market the seller $\mathcal{S}$ knows the quality of the cars the buyer $\mathcal{B}$ does not. We say: The seller holds private information from the beginning of the game.

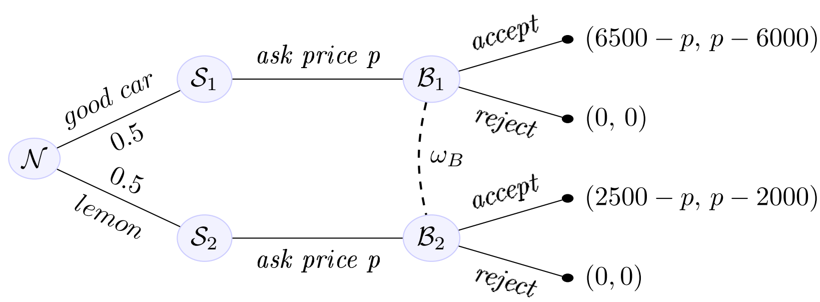


The analysis shows that good cars don't sell in this market!

View of buyer and seller on the market

- If $p < 2000$ there is no deal because the offer is below the least value of a car. This is obvious for the prospective buyer and of course for the seller.
- If $p \geq 2000$ the seller is in a comfortable situation. He will accept the offer or reject according to his valuation of the car. At a price between 2000 and 6000 he will only agree to sell a bad car.
- Yet, a price above 2500 pays for the prospective buyer only if he expects to get a good car. As the seller will not sell a good car below 6000, a price between 2500 and 6000 yields a negative pay-off for the prospective buyer because he can get a bad car, only. A price of 6000 or more opens up the chance for a good car. But the risk to get a bad car pulls down the expected pay-off to 4500 and therefore he should not offer a price of 6000 or more.

The Lemons' Problem II
In this second version we assume the seller $\mathcal{S}$ calls up a price by placing a price tag behind the wind screen of the car. $\mathcal{B}$ can only *accept* or *reject*.



Again the analysis shows that good cars don't sell in this market!

Motivation: Where and how does adverse selection crop up?

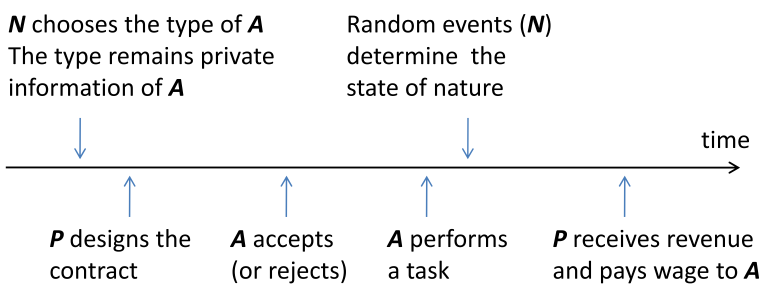
- A driver knows more about his driving habits than the insurance company.
- By the insurance contract he wants to shift his risk to the insurance. It is the business concept of insurance companies to share (buy) risks.
- However, the insurance may not be willing to contract with this particular driver, because she estimates the risk too high.
- Even worse, careful drivers who are willing to pay a high premium may look suspicious to the insurance manager: "The more you are willing to pay for for insurance the higher may be your risk!"

A Principal Agent Model of Adverse Selection

- We consider a bilateral relationship of principal and agent
- The agent holds private information relevant to the (contractual) relationship.
- Typical examples are the agent's (true) qualification,the agent's attitude towards risk,the quality of a service he or she can offer ...
- The principal designs a contract
- ...

Adverse selection

The agent holds private information from the beginning



A formal model of adverse selection

- The expected revenue is $R(e) = \sum_i p_i(e)x_i$
- The principal is risk neutral with expected pay-off equal to expected profit $R(e) - w$
- We consider two types of agents with different dis-utility of effort $kv(e)$
  - with $k = 1$ for type 1, and

- with $k > 1$ for type 2.

- The agents are risk averse with utility $U(w, e) = u(w) - kv(e)$
- We call the agent of type 1 the *good agent*, G, and the agent with the larger unwillingness to work hard (type 2) the *bad agent*, B.
- The respective utility functions are labeled as U^G or U^B.

3.4.1 Symmetric information as a case of reference

- Assume for a moment the principal could identify the type of the agent.

$$\max_{(e,w)} \Pi(e, w) = R(e) - w \quad \text{s.t.} \;\; u(w) - kv(e) \geq \underline{U} \quad \text{(PC)}$$

From total differentials along curves of constant profit or constant utility we get

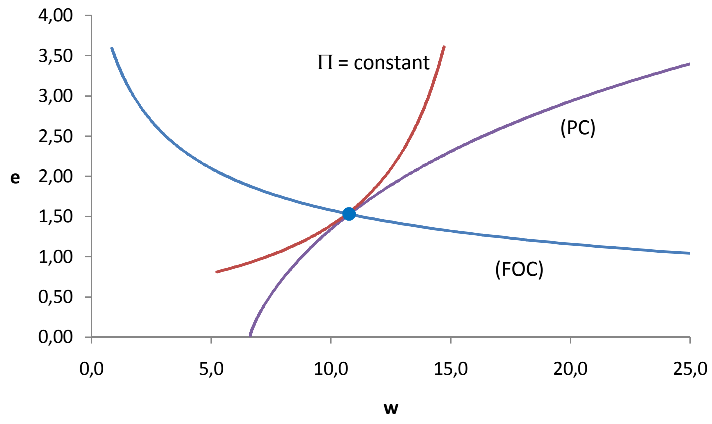
$$R'(e)\, de - dw = 0\,, \quad u'(w)\, dw - kv'(e)\, de = 0$$

- The first order condition of the principals constrained utility maximization equates the principal's and the agent's rate of substitution between w and e.

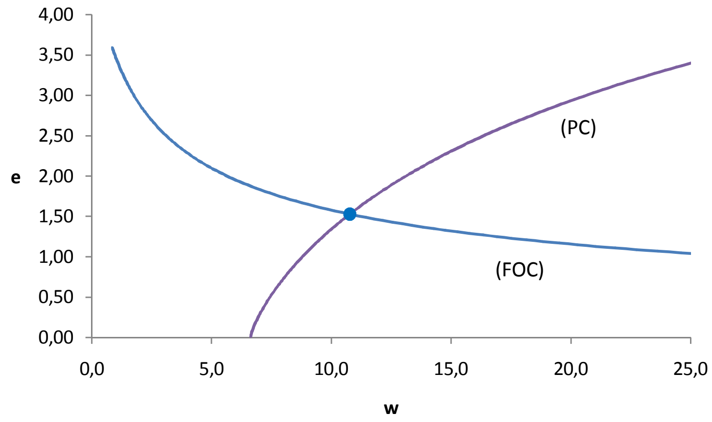
$$R'(e) = \frac{kv'(e)}{u'(w)} \qquad (FOC)$$

The marginal loss of the principal due to a rise of wage is equal to one. View the left hand side as $R'(e)/1$ to make it more obvious that the marginal revenue actually is a rate of substitution.

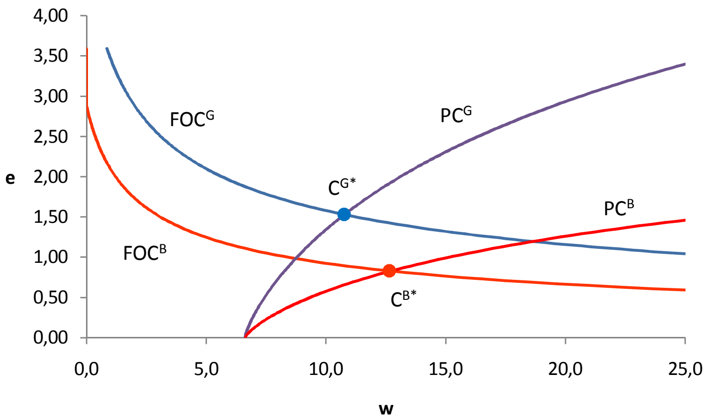
- Notice that the first order condition is independent of the agent's outside option or reservation utility $\underline{U}$.
- The point of tangency of the participation constraint (PC) and the appropriate level curve of the principal's expected profit function determines the optimal contract C^*.
- Hence, (FOC) has to be satisfied by the optimal contract.
- (FOC) is the curve of all points of tangency of constant profit curves and curves of constant utility for matching levels.
- In the picture we have chosen the utility level $\underline{U}$ which corresponds to the (PC)-curve.



One of the three equations or curves is redundant. We drop the isoquant of the principal's expected profit, because the level of expected profit is unknown a priori.



Optimal contracts in case of symmetric information



- We combine the respective pictures for the two different types G and B.
- Obviously $U^G(C^{B^*}) > U^G(C^{G^*})$.

3.4.2 Asymmetric Information

Asymmetric information and self-selection

A problem arises, if the principal does not know the type of agent he is facing.

- If G has the choice, he will choose C^{B^*}, the contract which is not designed for him.
- Hence, the expected profit of the principal will fall due the lack of information.
- We say, the menu (C^{G^*}, C^{B^*}) is not self-selective, but it is pooling [4].

A menu of contracts $\{C^G, C^B\} = \{(e^G, w^G), (e^B, w^B)\}$ is self-selective, if

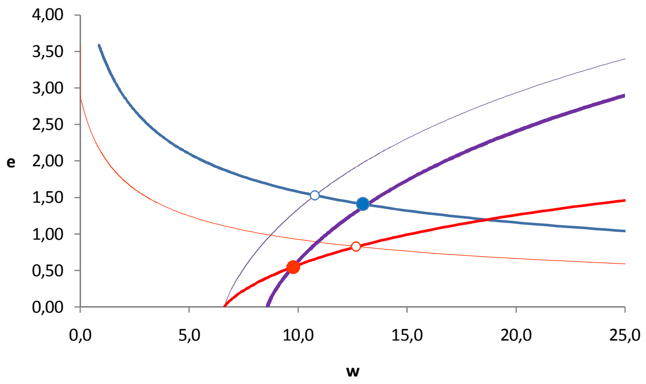
$$\begin{array}{rcll} u(w^G) - v(e^G) & \geq & u(w^B) - v(e^B) & \quad (SSC^G) \\ u(w^B) - kv(e^B) & \geq & u(w^G) - kv(e^G) & \quad (SSC^B) \end{array}$$

In comprehensive notation we may write

$$\begin{array}{rcll} U^G(C^G) & \geq & U^G(C^B) & \quad (SSC^G) \\ U^B(C^B) & \geq & U^B(C^G) & \quad (SSC^B) \end{array}$$

3.4.3 Self-selection

- The picture below shows a self-selective menu of contracts



- The hollow nodes depict the symmetric information optimal menu of contracts.
- A self-selective menu of contracts must violate some of the optimality conditions of this menu.
- The solid nodes form a self-selective (weakly separating) menu.

[4] We use the term *self-selective* emphasising that we focus on the agents choice. An equilibrium with a self-selective menu of contracts then is called *separating equilibrium*.

The picture above suggests

- (PC) is binding for agents of type B. In other words, bad agents receive their reservation utility $\underline{\mathrm{U}}$.
- Good agents receive extra utility, $U > \underline{\mathrm{U}}$. For them (PC) is not binding.
- The contract for agents of type G is efficient (FOC).

The amount of extra utility of good agents and the degree of inefficiency of the contract of bad agents have to be balanced as to maximize the expected profit of the principal.

3.4.4 Formal analysis in case of asymmetric information

- In order to find the optimal contract the principal has to know the respective likelihood of facing a good or a bad agent. Let q be the share of good agents around, $(1-q)$ the share of bad agents.
- The principal's problem is to find the optimal menu of contracts $\{C^G, C^B\} = \{(e^G, w^G), (e^B, w^B)\}$.

$$\begin{aligned}
\max_{\{C^G, C^B\}} \quad & q\Pi(e^G, w^G) + (1-q)\Pi(e^B, w^B) & \\
s.t. \quad & \text{good agents receive at least } \underline{U} \text{ from } C^G & (PC^G) \\
& \text{bad agents receive at least } \underline{U} \text{ from } C^B & (PC^B) \\
& \text{good agents prefer } C^G \text{ to } C^B & (SSC^G) \\
& \text{bad agents prefer } C^B \text{ to } C^G & (SSC^B)
\end{aligned}$$

Redundancy of good agents' participation constraint
The system of constraints contains a typical redundancy. It it implies

$$U^G(C^G) \quad \geq \quad \underbrace{U^G(C^B) \quad > \quad U^B(C^B)}_{\text{larger dis-utility of bad agents}} \quad \geq \quad \underline{U}$$

This is exactly what was claimed above: good agents receive some extra utility.

Another implication
Good agents will be required to supply higher effort.

$$\begin{aligned} v(e^G) - v(e^B) &\leq u(w^G) - u(w^B) && \text{due to } (SSC^G) \\ &\leq k\left(v(e^G) - v(e^B)\right) && \text{due to } (SSC^B) \end{aligned}$$

By assumption k is larger than one. Hence this can only hold, if $v(e^G) - v(e^B) \geq 0$ and thus $e^G \geq e^B$.

Moreover:

- If effort levels were equal, $e^G = e^B$, wages should be equal as well, $w^G = w^B$.
- If effort levels of good agents are higher, $e^G > e^B$, wages must be higher as well, $w^G > w^B$.

Searching for the optimal contract

$$\begin{aligned} \max_{\{C^G, C^B\}} \quad & q\Pi(e^G, w^G) + (1-q)\Pi(e^B, w^B) \\ s.t. \quad & u(w^B) - k\, v(e^B) \geq \underline{U} && (PC^B) \\ & u(w^G) - v(e^G) \geq u(w^B) - v(e^B) && (SSC^G) \\ & u(w^B) - kv(e^B) \geq u(w^G) - kv(e^G) && (SSC^B) \end{aligned}$$

Let λ, μ and δ be the Lagrange multipliers of (PC^B),(SSC^G), and (SSC^B) respectively.

The Lagrangean function

$$\begin{aligned} \mathcal{L} = \; & q\Pi(e^G, w^G) + (1-q)\Pi(e^B, w^B) \\ & + \lambda\,(\, u(w^B) - k\, v(e^B) - \underline{U}) \\ & + \mu\,(u(w^G) - v(e^G) - u(w^B) + v(e^B)) \\ & + \delta\,(u(w^B) - kv(e^B) - u(w^G) + kv(e^G)) \end{aligned}$$

First order conditions

$$\begin{aligned} -q + \mu u'(w^G) - \delta u'(w^G) = 0 && (\mathcal{L}_{w^G}) \\ -(1-q) + \lambda u'(w^B) - \mu u'(w^B) + \delta u'(w^B) = 0 && (\mathcal{L}_{w^B}) \\ qR'(e^G) - \mu v'(e^G) + \delta k v'(e^G) = 0 && (\mathcal{L}_{e^G}) \\ (1-q)R'(e^B) - \lambda k v'(e^B) + \mu v'(e^B) - \delta k v'(e^B) = 0 && (\mathcal{L}_{e^B}) \end{aligned}$$

The constraints must hold with complementary slackness.

Rearrange thr first order conditions

$$\begin{aligned}
\mu - \delta \;&=\; q \,/\, u'(w^G) \qquad (\mathcal{L}_{w^G})\\
\lambda - \mu + \delta \;&=\; (1-q) \,/\, u'(w^B) \qquad (\mathcal{L}_{w^B})\\
\mu - \delta k \;&=\; qR'(e^G) \,/\, v'(e^G) \qquad (\mathcal{L}_{e^G})\\
\lambda k - \mu + \delta k \;&=\; (1-q)R'(e^B) \,/\, v'(e^B) \qquad (\mathcal{L}_{e^B})
\end{aligned}$$

Binding versus non-binding (SSC)
From

$$\mu - \delta k \;=\; \frac{qR'(e^G)}{v'(e^G)} \qquad (\mathcal{L}_{e^G})$$

we conclude that $\mu > 0$ because the right hand side is positive and shadow prices cannot be negative.

- Hence (SSC^G), the constraint corresponding to μ, is binding.
- More or less obviously this implies that (SSC^B) is not binding and therefore $\delta = 0$

Binding versus non-binding (SSC)
From $\mu > 0$ we recall

$$u(w^G) - u(w^B) = v(e^G) - v(e^B)$$

We use this to check whether SSC^B is binding

$$\begin{aligned}
& u(w^B) - kv(e^B) - u(w^G) + kv(e^G)\\
= \;& v(e^B) - v(e^G) - kv(e^B) + kv(e^G)\\
= \;& (k-1)(v(e^G) - v(e^B))
\end{aligned}$$

Both factors are positive as $k > 1$ and $e^G > e^B$, and therefore $\delta = 0$.

The information rent of good agents

$$\begin{aligned}
u(w^G) - v(e^G) \;&=\; u(w^B) - v(e^B) \qquad (SSC^G)\\
&=\; u(w^B) - kv(e^B) + (k-1)v(e^B)\\
&\geq\; \underline{U} + (k-1)v(e^B)
\end{aligned}$$

- The utility of good agents exceeds that of bad agents.
  (PC^G) is not binding.
- An optimal menu of contracts cannot offer extra utility to both types of agents. Hence, (PC^B) must be binding.

I.e. good agents receive a reward called *information rent* equal to $(k-1)v(e^B)$.

(SSC^G) and (FOC^G) and the first order conditions of $\mathcal{L}$
With $\mu > 0$ and $\delta = 0$ we get from ($\mathcal{L}_{w^G}$) and ($\mathcal{L}_{e^G}$)

$$\frac{1}{u'(w^G)} = \frac{R'(e^G)}{v'(e^G)}$$

Solve for R' to get the standard form of the efficiency condition

$$\frac{v'(e^G)}{u'(w^G)} = R'(e^G)$$

I.e., (SSC^G) is binding corresponds to (FOC^G).

Reduced first order conditions of optimal contracts
From the first order conditions with respect to w we get an equation for the shadow price λ.

$$\left.\begin{array}{rcll} \mu &=& q \,/\, u'(w^G) & (\mathcal{L}_{w^G}) \\ \lambda - \mu &=& (1-q) \,/\, u'(w^B) & (\mathcal{L}_{w^B}) \end{array}\right\} +$$

$$\lambda \;=\; q \,/\, u'(w^G) + (1-q) \,/\, u'(w^B) \;>\; 0$$

From ($\mathcal{L}_{w^B}$) we get $-\mu = (1-q) \,/\, u'(w^B) - \lambda$.

Finally, in ($\mathcal{L}_{e^B}$) we can eliminate the shadow prices μ and λ using the formula above and the equation for λ we have just derived from ($\mathcal{L}_{w^G}$) and ($\mathcal{L}_{w^B}$).

$$\begin{aligned} \lambda k - \mu &= (1-q)R'(e^B) \,/\, v'(e^B) \qquad (\mathcal{L}_{e^B}) \\ \lambda k + \frac{(1-q)}{u'(w^B)} - \lambda &= \frac{(1-q)R'(e^B)}{v'(e^B)} \\ \frac{(1-q)k}{u'(w^B)} + \frac{q(k-1)}{u'(w^G)} &= \frac{(1-q)R'(e^B)}{v'(e^B)} \\ \frac{kv'(e^B)}{u'(w^B)} + \frac{q(k-1)}{(1-q)} \frac{v'(e^B)}{u'(w^G)} &= R'(e^B) \end{aligned}$$

The distortion of the optimal contract

$$\Delta = \frac{q(k-1)}{(1-q)} \frac{v'(e^B)}{u'(w^G)} = R'(e^B) - \frac{kv'(e^B)}{u'(w^B)}$$

takes a positive value. As it is zero along the FOC^B curve, and as it decreasing with w^B the optimal contract C^B must be on the PC^B curve to the lower left of the FOC^B curve. In other words, e^B and w^B both must be smaller than in the first best contract for bad agents.

3.4.5 Adverse Selection: Final conclusion

Final check for optimality
The menu characterized above may be dominated by the first best contract for good agents. I.e.

- The principal may decide to offer the first best contract C^{G^*} only. He would have to take into account that bad agents reject that offer because it violates their (PC).
- This is excluded if we assume

$$q\Pi(e^G, w^G) + (1-q)\Pi(e^B, w^B) \geq q\Pi(e^{G^*}, w^{G^*})$$

  This is basically an assumption that q is not too large.

The optimal menu of contracts
The optimal menu of contracts $\{C^G, C^B\} = \{(e^G, w^G), (e^B, w^B)\}$ is characterized and uniquely determined by the following properties

- (PC^B) is binding, whereas (PC^G) is not
- good agents receive an *information rent*

$$U^G - \underline{U} = (k-1)v(e^B) \geq 0$$

- (FOC^G) is satisfied, there is *no distortion at the top*
- (FOC^B) is distorted by

$$R'(e^B) - \frac{kv'(e^B)}{u'(w^B)} = \frac{q(k-1)}{(1-q)} \frac{v'(e^B)}{u'(w^G)} \geq 0$$

3.5 Signaling

3.5.1 Introduction: A very simple example

Assume we have a labor market with two types of workers

- type G: 'good' productivity ($a = 2$)
- type B: 'bad' productivity ($a = 1$)

Firms compete for workers, but they cannot observe their productivity. Profits are

$$\Pi = a - w$$

If they could observe productivity, they would pay wages equal to productivity.

Before entering the job market agents can spend time on education. Education does not improve on-the-job-productivity! However, it can raise the agent's pay off.

- Let s be the time dedicated to education.
- The costs of s units of education are s/a.
- Education is more costly for agents with lower productivity.
- Firms guess that typically good agents acquire better education.
- Suppose firms believe that a candidate is of type G whenever she has an education level $s \geq s^*$
- With these beliefs they are willing to offer
  $w = 2$ if $s \geq s^*$, and $w = 1$ else.

- Agents invest in education to maximize their surplus

$$U^a = w - s/a$$

- Obviously they either choose $s = 0$ or $s = s^*$.
- The principal's belief is self-affirming if good agents choose $s = s^*$ and bad agents choose $s = 0$.

  This requires

$$\begin{array}{rcll} 2 - s^*/2 & \geq & 1 & \text{for good agents} \\ 1 & \geq & 2 - s^* & \text{for bad agents} \\ \text{or all together} & & & \\ 1 \leq & s^* & \leq 2 & \end{array}$$

- $s^* = 1$ is the least costs signaling equilibrium. I.e. requiring more than s^* leads to an overinvestment in education.

3.5.2 Signaling Principals

In this section we consider an example of indirect or implicit signaling. There are many examples of that kind. Moreover, the example demonstrates that it is not always the principal who suffers from a lack of information. Here the principal holds private information that affects the principal's as well as the agent's pay-off.

- Consider a principal offering two different types of jobs.
- The first job is easy combined with low productivity.
- The second job is arduous with high productivity.
- Profit and utility functions are

$$\begin{aligned} \Pi^h(w,e) &= hR(e) - w \\ U^h(w,e) &= u(w) - hv(e) \end{aligned}$$

  with $h = 1$ in the first job, and $h = k > 1$ in the second job.
- The principal is risk neutral and maximizes profit.
- The agent is risk averse and has reservation utility $\underline{U}$

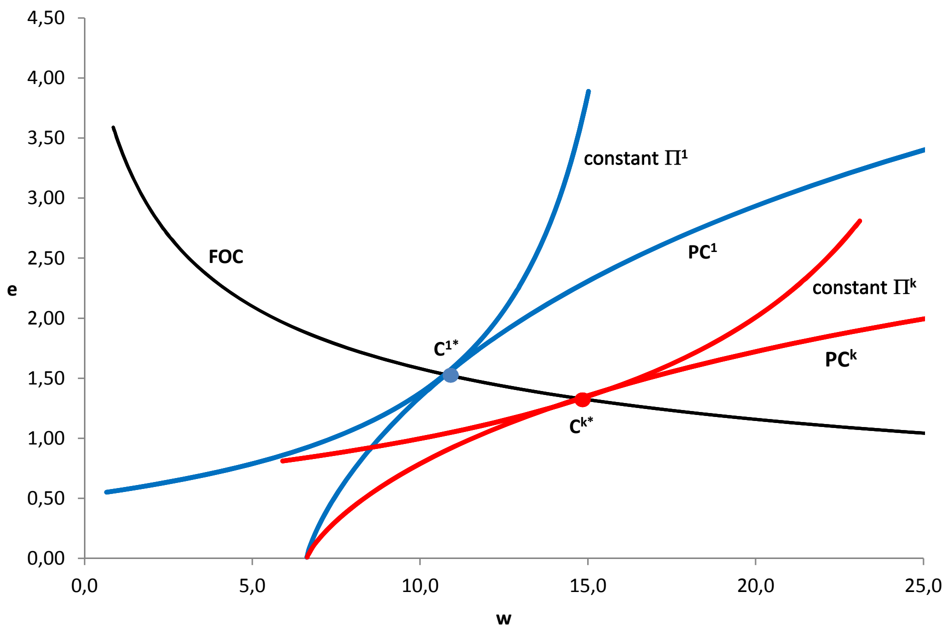
3.5.3 Symmetric full information

The solution of such a problem is characterized by a (binding) participation constraint (**PC**) and a first order condition (**FOC**)

$$u(w^h) - h\,v(e^h) = \underline{U} \qquad (\mathbf{PC})$$

$$\frac{R'(e^h)}{1} = \frac{v'(e^h)}{u'(w^h)} \qquad (\mathbf{FOC})$$

- Notice that the (FOC) is independent of h. This is because the parameter h scales the principals' revenue and the agents' disutility of effort in exactly the same way.
- However, along the (FOC) the slopes of the tangent curves of constant profit or utility in case of type 1 principals are h times the slope of the type h curves.



The menu (C^{1^*}, C^{k^*}) is the equilibrium in case of symmetric information.

3.5.4 Asymmetric information

The menu (C^{1^*}, C^{k^*}) is no longer an equilibrium menu in case of asymmetric information because both types of principals prefer to offer C^{1^*}.

In particular:

- a type k principal has an incentive to pass himself off as type 1, and
- the agent will not sign the contract C^{1^*}, because he is afraid to be tricked into an arduous job.

A separating perfect Bayesian equilibrium[5] consists of

- A menu of contracts $(\tilde{C}^1, \tilde{C}^k)$ designed by the principal.
- A decision of the agent whether to accept the contract or not although he does not know the type of the principal.
- A system of consistent beliefs based on menu offered by the principal.

When a contract $C = (e, w)$ is offered, the agent has to form beliefs about the type of principal who is offering the contract. In the light of the beliefs, he evaluates the offer.

- Let $\mu(C)$ denote the belief that this contract is offered by a type 1 principal. $1-\mu(C)$ is the belief that C is offered by a type k principal.

[5]It is very easy to see that pooling equilibria do not exist in this model.

- In particular the agent checks the participation constraint using the beliefs, i.e. he calculates the expected utility[6] :

$$u(w) - \Big(\mu(C) + (1-\mu(C))\,k\Big)\,v(e) \geq \underline{U} \qquad (\mathbf{PC}^{\mu})$$

- The participation constraint takes the ordinary simple form ($\mathbf{PC}^1$) or ($\mathbf{PC}^k$) if the beliefs are equal to 0 or 1, respectively.

Once more we discuss why (C^{1^*}, C^{k^*}) as not an quilibrium menu. This time we make explicit reference to the role of beliefs:

- The menu can be an equilibrium only if the agent's beliefs are $\mu(C^{1^*}) = 1$, $\mu(C^{k^*}) = 0$.
- I.e., the agent believes to face a principal of type 1 if C^{1^*} is offered to him, and he believes he is of type k if C^{k^*} is offered.
- But these beliefs are not rational in this case because the agent should assume that some firms offering C^{1^*} actually are of type k, i.e. $\mu(C^{1^*}) < 1$.
- Hence, (C^{1^*}, C^{k^*}) cannot occur in a separating perfect Bayesian equilibrium.

3.5.5 Perfect Bayesian Equilibrium

The Separating Perfect Bayesian Equilibrium
The separating Perfect Bayesian Equilibrium menu $(C^{1^{**}}, C^{k^{**}})$ is the menu determined by:

1. Type 1 principals offer a contract $C^{1^{**}}$ determined by two conditions:
   (i) The participation constraint (PC^1) is satisfied and binding.
   (ii) The self-selection constraint (SSC^k) of firms of type k is satisfied and binding.
2. Type k principals offer $C^{k^{**}} = C^{k^*}$.

Affirmation of condition 2.
Condition (2) can be discussed independent of condition (1). Hence with start with it.

- Assume we have a separating perfect Bayesian equilibrium $(\tilde{C}^1, \tilde{C}^k)$.
- In equilibrium the beliefs must be $\mu(\tilde{C}^1) = 1$, $\mu(\tilde{C}^k) = 0$.
- Furthermore, the participation constraint based on these beliefs must hold, and if $\tilde{C}^k$ is offered, it reduces to $u(\tilde{w}^k) - kv(\tilde{e}^k) \geq \underline{U} \qquad (\mathrm{PC}^k)$
- The profit maximizing contract of a firm of type k satisfying (PC^k) is C^{k^*}.Take a look a the picture with the first best contracts to visualize the argument!

[6]We use the label ($\mathbf{PC}^{\mu}$) to indicate that agent use beliefs because they don't know which type they are facing. Notice that uncertainty only matters for the dis-utility term.

- Whether C^{k^*} is rejected by the agent or accepted may depend on $\mu(C^{k^*})$. But we can show, that C^{k^*} will be accepted independently of the beliefs.

- Indeed, the expected utility of C^{k^*} with beliefs $\mu(C^{k^*})$ is

$$\begin{aligned} U(C^{k^*}) &= \mu(C^{k^*}) \left(u(w^{k^*}) - v(e^{k^*})\right) \\ &\quad + (1 - \mu(C^{k^*})) \left(u(w^{k^*}) - kv(e^{k^*})\right) \\ &= u(w^{k^*}) - \left(\mu(C^{k^*}) + (1 - \mu(C^{k^*}))\, k\right)\; v(e^{k^*}) \end{aligned}$$

  The smaller $\mu(C^{k^*})$, the lower is this utility because $k > 1$.

- But even in case of $\mu(C^{k^*}) = 0$ the agent will accept the contract because then

$$U(C^{k^*}) = u(w^{k^*}) - kv(e^{k^*}) = \underline{U}$$

- In other words, when the agent is offered C^{k^*} he will accept it.

- Summing up: a separating perfect Bayesian equilibrium with $\tilde{C}^k \neq C^{k^*}$ cannot exist because
  - type k principals would prefer to offer C^{k^*}
  - and the agent will accept this offer regardless of his beliefs.

Affirmation of condition 1.
Construction of $\tilde{C}^1$

Knowing $C^{1^{**}}$ we are left with the construction of $C^{k^{**}}$ to complete the equilibrium menu.

- Agents must be willing to accept $\tilde{C}^1$ if $\mu(\tilde{C}^1) = 1$, i.e.

$$U^1(\tilde{C}^1) \geq \underline{U} \qquad (\mathbf{PC}^1)$$

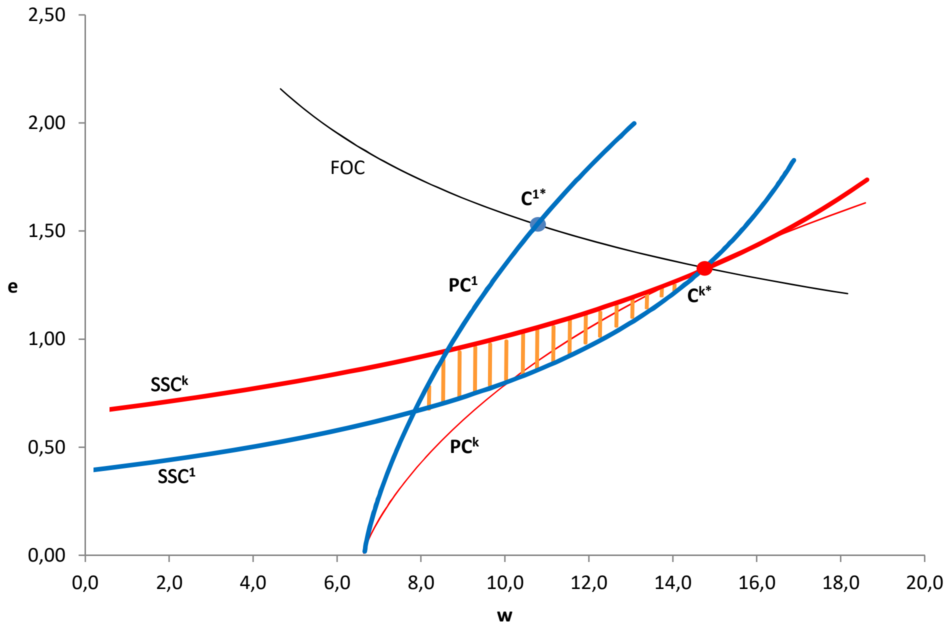
- Principals of type k should weakly prefer C^{k^*} to $\tilde{C}^1$

$$\Pi^k(C^{k^*}) \geq \Pi^k(\tilde{C}^1) \qquad (\mathbf{SSC}^k)$$

- Finally, principals of type 1 should weakly prefer $\tilde{C}^1$ to C^{k^*}.

$$\Pi^1(\tilde{C}^1) \geq \Pi^1(C^{k^*}) \qquad (\mathbf{SSC}^1)$$

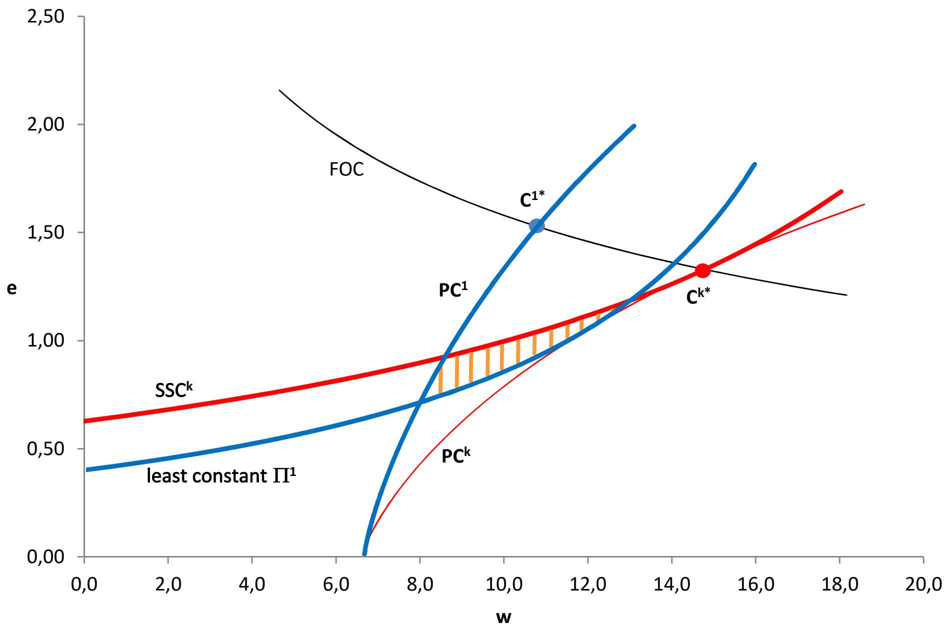
The shaded area in the picture below shows the segment of contracts satisfying the three conditions above.



Beyond the participation constraint (PC^1) and the (SSC) constraints there is a simple implication of (PC^k) reducing the domain of possible candidates for $\tilde{C}^1$:

- All contracts below the (PC^k) curve will be accepted by the agents, regardless of their beliefs.
- Hence $\tilde{C}^1$ should not yield a profit smaller than the maximum of Π^1 along the (PC^k) curve.
- This condition dominates the SSC^1 constraint.

The second picture shows the reduction of the segment by the argument above.



The arguments above and the picture clearly show

Remark

- In any separating equilibrium the contract $\tilde{C}^1$ will offer a lower wage and require a lower level of effort than C^{k^*}.

3.5.6 Equilibrium beliefs

Up to now we only checked the constrains on the menu of contracts. Now we add the (Bayesian) conditions on beliefs.

1. The principal chooses $\tilde{C}^T$ if he is of type T.
2. The agent's beliefs are:
   - $\mu(\tilde{C}^1) = 1$
   - $\mu(C) = 0$ for all other contracts C (including $\tilde{C}^k$).
3. The agent accepts both contracts $\tilde{C}^1$ and $\tilde{C}^k$ on the basis of the corresponding beliefs. Moreover he accepts any other contract C such that $U^k(C) \geq \underline{U}$. However, he rejects all other remaining contracts.

Reasonable beliefs

- The set of feasible contracts described and illustrated above obviously is the set of all perfect Bayesian equilibria.

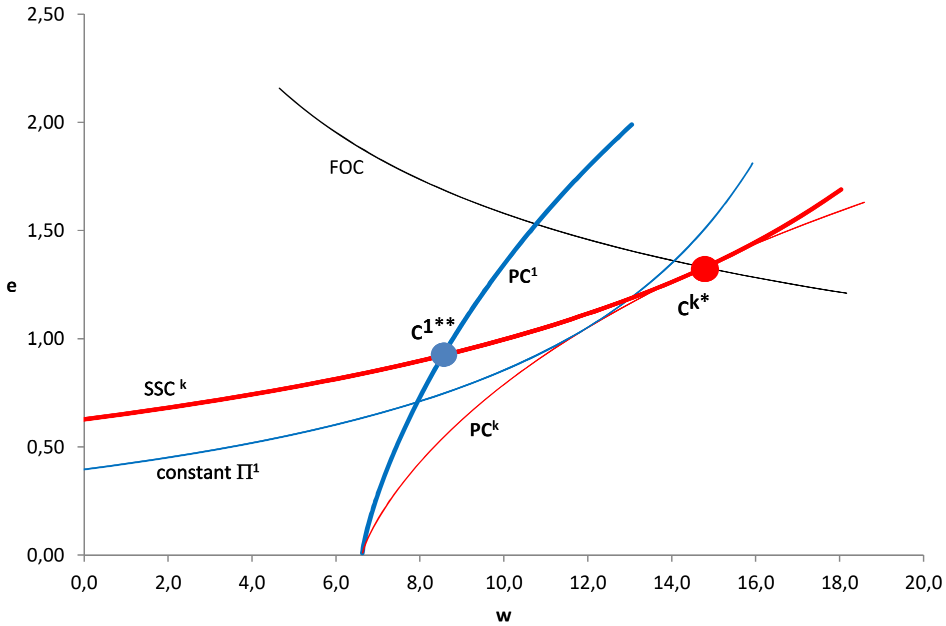
- The intuitive criterion of Cho and Kreps eliminates unreasonable beliefs[7] and reduces the set of perfect Bayesian equilibria to a single menu of contracts. Result 3 characterizes the solution.

Condition 2
$\tilde{C}^1$ is the contract $C^{1^{**}}$ as depicted in the graph below and it is characterized by

$$U^1(C^{1^{**}}) = \underline{U}$$

$$\Pi^k(C^{1^{**}}) = \Pi^k(C^{k^*})$$

$C^{1^{**}}$ maximizes profits of type 1 principals within the set of perfect Bayesian equilibria.



Uniqueness

- Recall that $C^{1^{**}}$ maximizes profits of type 1 principals within the set of perfect Bayesian equilibria.
- Remember that type k principals are not interested in offering $C^{1^{**}}$ instead of C^{k^*} independent of the agents's beliefs.
- Therefore, agents attribute any contract different from C^{k^*} to a type 1 principal.
- Type 1 principals on the other hand have a clear reason to offer $C^{1^{**}}$ rather than any other feasible contract.
- Therefore, agents[8] resist to believe that type 1 agents would not choose $C^{1^{**}}$.

[7]Cho, I. K., and D. Kreps 1987, Quarterly Journal of Economics
[8]according to Cho and Kreps!

3.5.7 Final remark on signaling

We have seen that principals do not have to "buy" a signal at extra costs. They can use the contracts they offer in order transmit the information and reveal their type. The signaling costs are implicit, they come in form of a loss of expected profits of the separating menu.

If agents understand the situation they will trust the revealing character of the signal.