Games and Information

MEPS Course

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Contents

In Part II of the course we will look at further ways to classify extensive form games with imperfect information. Moral hazard, adverse selection and signaling will be the elements of structure of games, pooling and separating equilibria the elements of classification of solutions we will discuss.

3 Contract Theory

In Part II of the course we will look at further examples and ways to classify these extensive form games with imperfect information.

Moral hazard, adverse selection and signaling will be the structural elements of these games.

Pooling and separating equilibria we will work out as the categories of classification of the solutions.

3.1 Principal Agent Theory

Consider a framework characterized by the following features

- A principal thinks of entering into an agreement with an agent.
- The agent is supposed to perform a particular task.
- The outcome of the agent's activities depends on the effort the agent applies and is affected by random events.
- The principal receives revenues resulting from the agent's activities.
- The agent receives a (wage) payment from the principal.

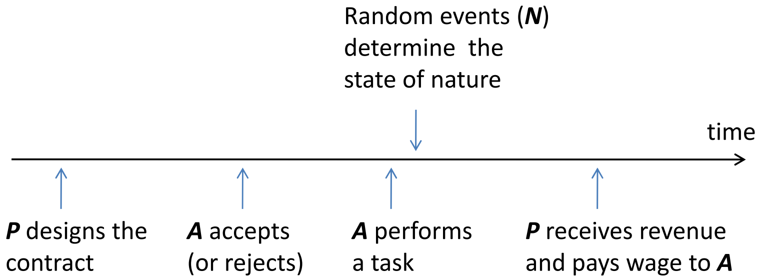
- The principal designs and offers the agent a contract c.
- The agent can reject the contract or accept it.
- If he accepts the offer, he then decides to apply a low or a high level of effort.
- The principal may not know what effort the agent applies. The principal's revenue jointly depends on the unobserved state of nature and the unobserved effort of the agent.
- The agent's pay off depends on revenue.

Representation of a Principal Agent Model

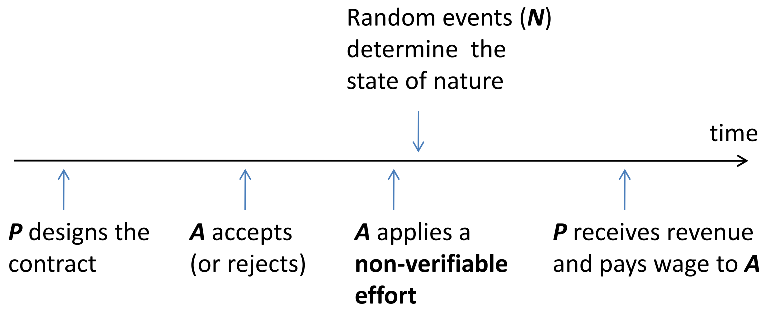
- The elements of the strategic conflict described above fit together in a huge game represented by a game tree.
- However, the nested structure with a continuum of action at each stage and the information structure makes the tree difficult to handle .
- A more appropriate representation sets the standards in the literature.
- It provides a better overview and it helps to classify the different cases of problems.

3.2 Classification of Problems

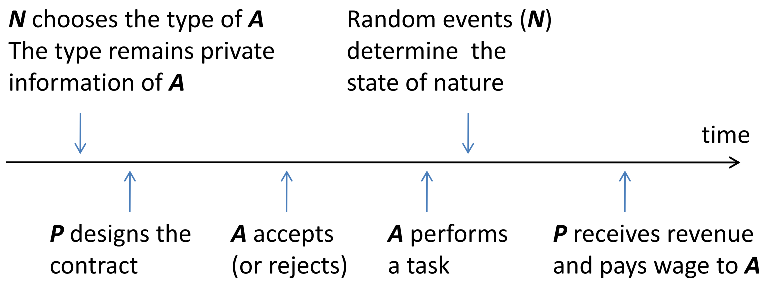
Actions, events and timing of the base model
Let P denote the principal and A the agent.



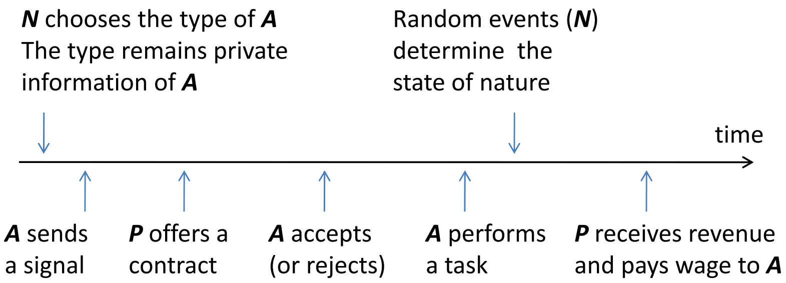
Moral Hazard
The agent's action cannot be observed by the principal (hidden action)



Adverse selection
The agent holds private information from the beginning



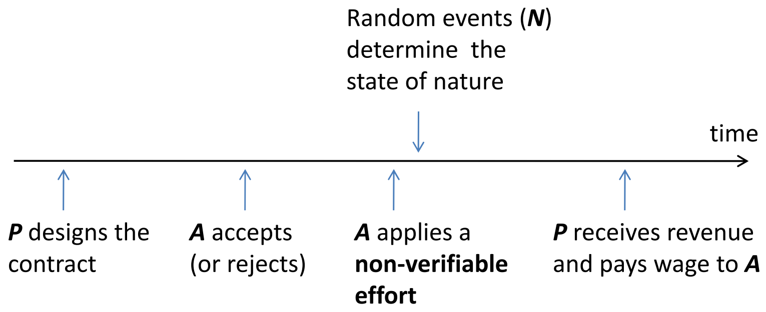
Signalling
The agent can send a signal to the principal



3.3 The Moral Hazard Problem

Hidden action

The agent's action is not observable



Searching for an optimal contract

- There is s strategic conflict between principal and agent.
- The agent's action cannot be observed by the principal.
- The principal infers a rational prediction about the agent's action.
- The principal solves the problem by backward induction.
- The solution is a subgame perfect Nash-equilibrium.

The framework

- The agent is supposed to perform a task yielding outcome x in monetary value.
- Consider n different possible outcomes x_i, $i = 1, \dots, n$.
- They occur randomly with probalities p_i and will be observed by the principal and the agent. We refer to i as the state of nature.
- The probabilities may depend on the level of effort e applied by the agent, $p_i(e)$.
- The contract offers the agent a wage w_i in state i. The wage is not specified conditional on effort because e is private information of the agent.
- The agent has utility $u(w_i)$ of wage and dis-utility $v(e)$ of effort. He prefers high wage and low effort.

Backward induction

- Final stage:
  (i) The principal wants the agent to accept the contract,
  (ii) and the agent chooses a utility maximizing effort level.

  (i) imposes a constraint on the contract called
  participation constraint (**PC**)
  $\sum_{i=1}^{n} p_i(e)u(w_i) - v(e) \geq \underline{U}$
  The agent's expected utility should not fall short of the level he can achieve somewhere else.

  (ii) imposes a further constraint on the contract called
  incentive compatibility constraint (**ICC**)
  e solves: $\max_e \sum_{i=1}^{n} p_i(e)u(w_i) - v(e)$
  Without observing the effort the principal should expect the agent to apply this level of effort.

- First stage: The principal proposes a contract[1] $(w_1, \ldots, w_n)$ based on the possible revenues $x_1 \leq \cdots \leq x_n$ which maximizes the principal's profit (net revenue) subject to the constraints of the second stage [2]:

$$\max_{\{w_1,\ldots,w_n\}} \quad \sum_{i=1}^{n} p_i(e)\,(x_i - w_i)$$

$$\text{s.t. (PC) and (ICC)}$$

- The moral hazard problem of this form can be considered as a sequential game with the principal as a first player, the agent as second player, and nature as third player. (Nature does not pursue its own goal.)
- The strategy set of the principal is the set of all wage schedules $\{(w_1, \ldots, w_n)\}$.
- The strategy set of the agent is the set of all feasible effort levels $\{e\}$.
- The solution is the subgame perfect equilibrium of the corresponding game given by the strategy profile $((w_1, \ldots, w_n), e)$ of principal and agent.

[1] One may include the desired level of effort in the contract. However, it is determined endogenously by the incentive compatibility constraint and is not observable, anyway.

[2] Notice that we consider the case of a risk neutral principal. He considers the expected value of profit, whereas the agent uses utility and dis-utility functions reflecting risk aversion.

The case of two effort levels

- Assume $e \in \{e_L, e_H\}$, x_i ordered from worst to best: $x_1 \leq \cdots \leq x_n$.
- Notation $p_i(e_H) = p_i^H$ and $p_i(e_L) = p_i^L$.
- It seems natural that the principal prefers high effort to low effort. A first assumption in this direction addresses expected revenues. High effort should yield a gain in expected revenues

$$\sum_{i=1}^{n} p_i^H x_i \geq \sum_{i=1}^{n} p_i^L x_i \quad \text{or equivalently} \quad \sum_{i=1}^{n} (p_i^H - p_i^L) x_i \geq 0$$

- However, it is profit that counts for the principal

$$\sum_{i=1}^{n} p_i^H (x_i - w_i) \geq \sum_{i=1}^{n} p_i^L (x_i - w_i)$$

- We rearrange terms and get

$$\sum_{i=1}^{n} (p_i^H - p_i^L) x_i \geq \sum_{i=1}^{n} (p_i^H - p_i^L) w_i$$

  - Obviously, the (weak) inequality holds if wages are equal to revenues wiping out all profits.
  - Furthermore, it holds if high effort yields a gain in expected revenues, and if all wages are identical. In this case the left hand side is positive, and the right hand side is equal to zero.

$$\sum_{i=1}^{n} (p_i^H - p_i^L) w = w \left(\sum_{i=1}^{n} p_i^H - \sum_{i=1}^{n} p_i^L \right) = 0$$

- However, the latter never provides an incentive to supply high effort! Indeed, in this case the expected utility of wage income is independent of effort

$$\sum_{i=1}^{n} p_i(e) u(w) = u(w) \sum_{i=1}^{n} p_i(e) = u(w)$$

  whereas dis-utility is smaller with low effort.
- It remains to find conditions for the probabilities such that the gain in revenues due to high effort can outweigh the gain in expected wage by more than the dis-utility of higher effort.

Summing up we have the following two conditions which must hold for revenues, wages and probabilities of the solution

$$\sum_{i=1}^{n}(p_i^H - p_i^L)x_i \geq \sum_{i=1}^{n}(p_i^H - p_i^L)w_i$$

and

$$\sum_{i=1}^{n}(p_i^H - p_i^L)u(w_i) \geq v(e^H) - v(e^L)$$

We can evaluate the ranking of revenues to get two kinds of conditions such that at least expected revenues raise with higher effort. It then remains to check whether there is enough scope for appropriate wage setting.

- An intuitive assumption is that of monotonous likelihood

$$\frac{p_1^L}{p_1^H} > \frac{p_1^L + p_2^L}{p_1^H + p_2^H} > \dots$$

  It is not necessary but sufficient for the preference for high effort.

  In particular it implies

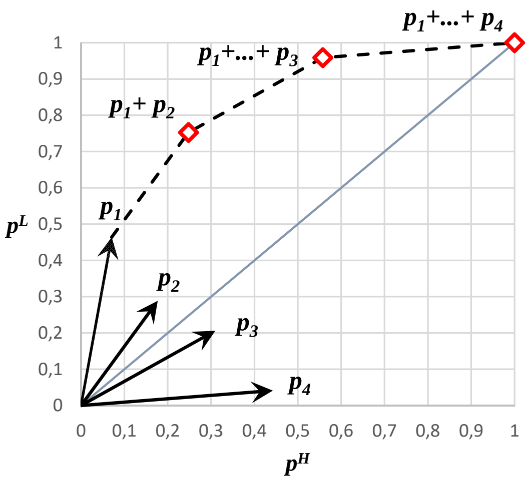
$$\frac{p_1^L}{p_1^H} > \frac{p_n^L}{p_n^H}$$

The case of two effort levels

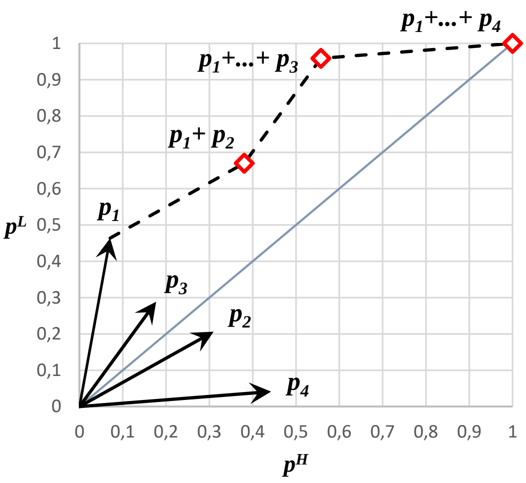
- A weaker but still sufficient assumption is, that p^H stochastically dominates p^L of first order.

$$p_1^H < p_1^L \text{ , and } p_1^H + p_2^H < p_1^L + p_2^L, \dots$$

Monotonous Likelihood in Case of Two Effort Levels



Non-monotonous Likelihood in Case of Two Effort Levels



The incentive compatibility constraint is supposed to implement an incentive to supply high effort.

In other words, the expected utility of high effort must be larger than or at least equal to that of low effort.

$$\textbf{(ICC)} \qquad \sum_{i=1}^{n} p_i^H u(w_i) - v(e^H) \geq \sum_{i=1}^{n} p_i^L u(w_i) - v(e^L)$$

$$\text{or equivalently} \qquad \sum_{i=1}^{n} (p_i^H - p_i^L)\, u(w_i) \geq v(e^H) - v(e^L)$$

The latter version of (ICC) means that an increase in expected utility of income due to higher effort has to outweigh the increase of dis-utility.

Searching for an optimal contract $(w_1, \ldots, w_n)$

$$\max_{\{w_1,\ldots,w_n\}} \quad \sum_{i=1}^{n} p_i^H\,(x_i - w_i) \qquad \text{s.t. (PC) and (ICC)}$$

Lagrangean function[3]

$$\begin{aligned}\mathcal{L}(w_i, \lambda, \mu) \;&=\; \sum_{i=1}^{n} p_i^H\,(x_i - w_i) + \lambda\left[\sum_{i=1}^{n} p_i^H u(w_i) - v(e^H) - \underline{U}\right] \\ &\quad + \mu\left[\sum_{i=1}^{n} (p_i^H - p_i^L)\,u(w_i) - v(e^H) + v(e^L)\right]\end{aligned}$$

First order conditions and shadow prices
The first order conditions with respect to w_i are

$$-p_i^H + \lambda p_i^H u'(w_i) + \mu(p_i^H - p_i^L)u'(w_i) = 0 \quad \text{for } i = 1, \ldots, n$$

Rearrange terms to check whether the constraints are binding

$$\frac{p_i^H}{u'(w_i)} = \lambda\, p_i^H + \mu\,(p_i^H - p_i^L) \quad \text{for } i = 1, \ldots, n$$

Summation over i yields $0 < \sum_{i=1}^{n} \frac{p_i^H}{u'(w_i)} \;=\; \lambda$

Notice, that all other terms vanish, because the probabilities p_i^H and p_i^L resp. sum up to unity.

We conclude $\lambda > 0$, and therefore (PC) is binding.

The interpretation of the shadow price equation for λ is straight forward.

$$\lambda = \sum_{i=1}^{n} \frac{p_i^H}{u'(w_i)} = \sum_{i=1}^{n} p_i^H \cdot \frac{dw_i}{du}$$

dw_i/du is the marginal wage reduction caused by a shrinking of utility. This marginal effect is small if w_i is small; it is large if w_i is large.

- The sum of these wage reductions weighted by probabilities amounts to an expected profit gain of the principal.
- The relaxation of the binding effect of (PC) as it may be caused by a lower value of $\underline{U}$ allows the principal to reduce the expected utility of the agent.
- The cost reduction raises the principal's expected profit, and this is what λ is supposed to measure!

[3] The arguments of the Lagrangean function always are the elements of the contract together with all shadow prices. In order to simplify notation we drop the arguments in what follows.

In order to check whether ICC is binding we assume the opposite, i.e. μ is equal to zero. Then the first order conditions reduce to $\lambda u'(w_i) = 1$ and hence the wage schedule w has to be flat.

But then (ICC)

$$\sum_{i=1}^{n} (p_i^H - p_i^L)\, u(w_i) \geq v(e^H) - v(e^L)$$

reduces to $0 \geq v(e^H) - v(e^L)$ which is wrong by assumption.

We conclude $\mu > 0$, and therefore (ICC) is binding as well.

- Although both constraints are binding as before, the optimal contract creates the incentives to make the agent supply high effort.
- The principal does not have to pay higher wages on average, he only has to offer a differentiated wage schedule instead of full insurance.

Characterization of the differentiated wage schedule
Recall the first order conditions and divide by p_i^H

$$\frac{1}{u'(w_i)} = \lambda + \mu\, \frac{p_i^H - p_i^L}{p_i^H} \quad \text{for } i = 1, \ldots, n$$

- The term
$$\frac{p_i^H - p_i^L}{p_i^H} = 1 - \frac{p_i^L}{p_i^H}$$
measures the relative change of probabilities due to an increase in effort from low to high.
- p_i^L / p_i^H is called the likelihood ratio due to a decrease of effort.
- Due to first order stochastic dominance the term one minus likelihood ratio is negative for $i = 1$, changes sign for some i, and is positive for $i = n$.
- The shadow prices λ and μ measure the value the principal can assign to a relaxation of the corresponding constraint.

Recall the first order conditions devided by p_i^H again

- If $p_i^H = p_i^L$ for some i, the wage rate is independent of probabilities and solves

$$u'(w_i) = 1/\lambda$$

 It depends on the shadow price of the participation constraint (PC).

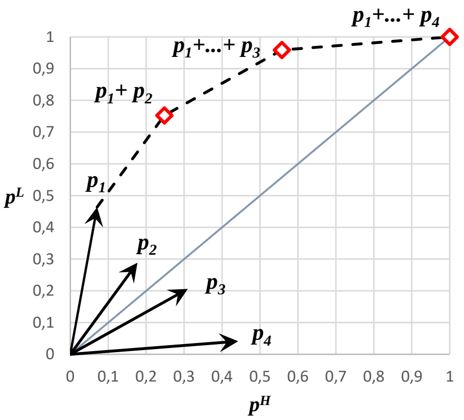
- The smaller the likelihood ratio (i.e. the less revealing a high outcome w.r.t. the effort devoted to the job), the larger the right hand side of the FOC above

 $\hookrightarrow$ the larger w_i must be.If the right hand side is larger, the **marginal utility** must be smaller, and therefore the wage must be larger!

- Remember: $x_1 \leq x_2 \leq x_3 \ldots$
- Use notation $p_i = (p_i^H, p_i^L)$.
- Stochastic dominance: p_1, $p_1 + p_2$, $p_1 + p_2 + p_3$ are located above the diagonal in a p_i^H, p_i^L-diagram.
- The likelihood ratios are the slopes of the vectors p_i.
- w_i increases with i, if and only if the likelihood ratio decreases with i.

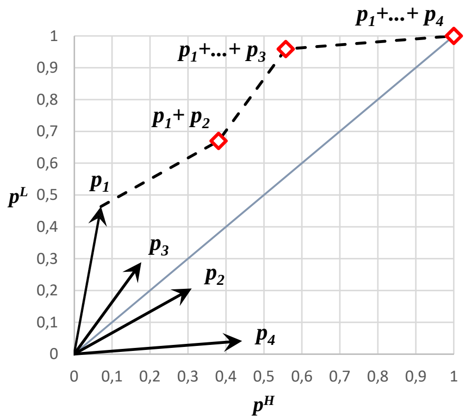
The pictures below illustrate different cases of monotonous and non-monotonous relationships between revenues and wages.

The case of monotonous likelihood ratio



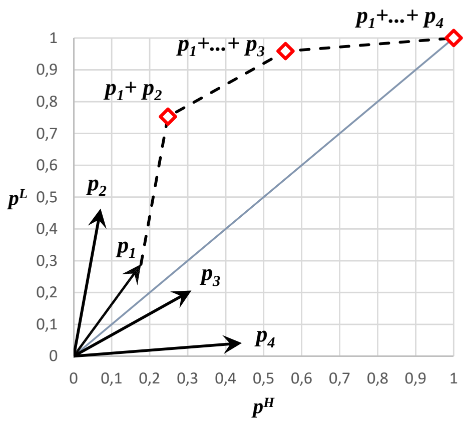
$\boxed{w_1 < w_2 < w_3 < w_4}$

The case of a non-monotonous likelihood ratio I



$\boxed{w_1 < w_3 < w_2 < w_4}$

The case of a non-monotonous likelihood ratio II



$\boxed{w_2 < w_1 < w_3 < w_4}$

Final remarks on moral hazard

- The analysis can be extended to the case of a continuous effort variable in a straight forward manner (cf. Macho-Stadler and Pérez-Castrillo, 2001).
- We considered the case of hidden action and not that of hidden information (cf. Hart and Holmström, 1987).
- There are interesting recent empirical investigations on moral hazard (e.g. Hoppe and Schmitz, 2018).
- The main conclusion is that risk can be shared through a contract and at the same time incentives can be implemented to reduce the risk.

- Nevertheless, although the starting point of reasoning is the the assumption of a lack of information, designing "good" contracts requires a lot of basic information about the environment, preferences ect.