A contest success function for rankings

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Abstract

A contest is a game where several players compete for winning prizes by expending costly efforts. A contest success function determines the probability of winning or losing the contest as a function of these efforts. We assume that the outcome of a contest is an ordered partition of the set of players (a *ranking*) and a contest success function assigns a probability to each possible outcome. We define a contest success function for contests whose outcome is a ranking of any type, i.e., with any number of players at each rank. This approach is new in contest theory since the axiomatic work has exclusively been on contests with single-winner, whose outcome is a ranking with one player in the first rank and all other players in the second rank. The contest success function is characterized by *pair-swap consistency*, which is an axiom of independence of irrelevant alternatives and generalizes the main axiom in Skaperdas (1996).

1 Introduction

A contest is a game where several players compete for winning prizes by expending costly efforts. Contest models are used to study situations of conflict such as war, political lobbying, litigation and R&D races. In this paper we generalize the framework and develop a model for studying a broader class of competitive environments.

In the literature so far, the axiomatic work exclusively focuses on contests with single winner, where one player wins the prize while all others lose. However, it seems particularly natural that a contest may induce different types of outcome. For instance, in the academic job market candidates compete for various research and teaching positions and most of them are recruited for some job. In the competition for tuition waiver rights at a university, some percentage of best performing students may be awarded a full tuition waiver, some others half and the remaining nothing. In both these examples, the outcome of the contest is different than the single-winner type. In this paper we define a model for contests with any type of outcome

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and characterize a class of contests by a property of independence which, loosely speaking, requires the relative performance of any two players in a contest to depend only on their own relative efforts, and not on the efforts of others.

Let us describe the model in more detail. We assume that the outcome of a contest is a *ranking*, which is defined as an ordered partition of the set of players, assigning a rank (or *level*) to each player. The type of a ranking is identified by the exact number of players at each level. When a ranking is *strict*, each player has a different level and no level is shared by more than one player. An outcome of the academic job market is a strict ranking, as each candidate is recruited for a different position. When a ranking is *non-strict*, at least one level is shared by more than one player. There are many types of non-strict ranking. The outcome of a single-winner contest is a non-strict ranking where one player is ranked first and all other players are ranked second. The awarding of tuition waiver rights at a university induces a non-strict ranking of students, dividing them into three categories. The analysis here allows for the study of any type of ranking as the outcome of a contest.[1]

The outcome of a contest is stochastically determined by players' efforts. More precisely, each profile of efforts induces a non-degenerate probability distribution over all possible outcomes of a contest. In the example of the awarding of tuition waiver rights at a university, the more each student has prepared during the academic year the higher its probability of being awarded a tuition waiver. This probability is always positive, as even a completely unprepared student has some chances of passing exams. The function which maps players' efforts into probabilities of outcomes is called *contest success function.*

In the literature so far, the axiomatic characterizations of contest success functions are exclusively for contests with single winner. In this paper we axiomatically characterize a contest success function for contests with any type of outcome. We characterize our success function by a unique property, *pair-swap consistency*. Pair-swap consistency is an axiom of independence, requiring the relative performance of two players in a contest of any type to be exclusively determined by their own efforts, as if they were the only players in the contest. More precisely this axiom requires, for each pair of players, the probability of any ranking where one of them is one level above the other to be proportional to the probability of the same player winning against the other in a two-player contest.

[1] Suppose there are 4 players. In a single-winner contest, a possible outcome is $(2;3,4,1)$, which means Player 2 is ranked the first, and other players share the second place. All other possible outcomes of a single-winner contest share the similar feature. With the approach followed here, apart from this ranking, the following rankings are also possible: $(2;3;4;1)$, a strict ranking where Player 2 is ranked the first, Player 3 the second, Player 4 the third and Player 1 the fourth; $(2;3,4;1)$, a non-strict ranking where Player 2 is ranked the first, Players 3 and 4 share the second rank and Player 4 is ranked the third; $(2,3;4,1)$, a non-strict ranking where Players 2 and 3 share the first rank and Players 4 and 1 share the second rank; $(2,3,4;1)$, a non-strict ranking where Players 2, 3 and 4 share the first rank and Player 1 is ranked the second. Many other types of ranking are possible.

Pair-swap consistent contest success functions are appropriate for modeling certain situations more than others. They are appropriate for modeling competitive situations where each player acts individually and fights each opponent indiscriminately. The academic job market and the awarding of tuition waiver rights at a university are reasonable examples of such cases. On the other hand, these functions are inappropriate for modeling situations where players compete unevenly against their opponents. Examples of such situations are when some players collude, or when there are geographical or cultural barriers to competition between some but not all players.

Related models in contest theory and probabilistic choice

Restricting attention to contests of single-winner type, our contest success function is equivalent to the one characterized in Skaperdas (1996), which is the most popular in the literature. Our axiom is analogous to *sub-contest consistency*, which is the main axiom in Skaperdas (1996) and in most axiomatic characterizations of contest success functions. Loosely speaking, sub-contest consistency is a property of success functions for contests of single-winner type which requires the winning probability of a player to be proportional to its probability of winning in sub-contests restricted to *any subset* of players. For contests of single-winner type, our axiom is equivalent to a weaker version of sub-contest consistency, requiring the property to hold only for sub-contests restricted to *any pair* of players.

Our framework is also meaningful in probabilistic choice and random utility theory. A contest among n players can be interpreted as the probabilistic preferences of a decision maker over a set of n alternatives. Efforts of players correspond to underlying qualities of alternatives and the contest success function gives the probability distribution of the preferences. Pair-swap consistency is analogous to Luce (1959)'s *choice axiom*, also known as *independence of irrelevant alternatives.* This axiom considers probabilistic choices from different subsets of alternatives, and roughly speaking it requires the relative probability of any two alternatives to be independent of other alternatives in the subset. Independence of irrelevant alternatives characterizes an important class of probabilistic choice functions, which includes the *multinomial logit.* The framework developed here generalizes Luce (1959), in that it allows to represent not only probabilistic choices but complete probabilistic preferences.

Applications

In the last section of this paper we apply our success function to contest games with various structures. More specifically, we analyze equilibrium behavior in contests with multiple stages and multiple prizes in order to identify which structures induce higher equilibrium efforts. These topics have been widely studied with success functions of perfectly discriminating type[2] (see Baye et al. (1993), Moldovanu and Sela

[2] A success function is of perfectly discriminating type if the outcome of a contest is non-stochastically determined by players' efforts. For example in an auction players are deterministically

(2001) and Moldovanu and Sela (2006)). However, the study of these subjects with success functions of imperfectly discriminating type[3] is still incomplete (see Clark and Riis (1998b), Wärneryd (2001), Fu and Lu (2006) and Fu and Lu (2009)). Our approach provides a framework for generalizing the analysis.

The paper develops as follows. Section 2 reviews the related literature in contest theory and probabilistic choice. Section 3 describes the model, and section 4 defines the contest success function and pair-swap consistency, and characterizes the success function via this axiom. Section 5 analyzes some properties of the success function and applies the function to the study of equilibrium behavior in contests games with various structures.

2 Related literature

The axiomatic approach in contest theory started with the seminal work of Skaperdas (1996), which defines a contest success function for single-winner contests. The function is characterized by five axioms: probability, monotonicity, anonymity, sub-contest independence and sub-contest consistency[4]. Let us briefly describe these axioms. *Probability* requires the sum of the winning probabilities of all players to be always 1, and *monotonicity* that the winning probability of a player is increasing in its own effort and decreasing in other players' efforts. *Anonymity* imposes that the winning probabilities depend only on the efforts of players and not on their identities, and *sub-contest independence* that the winning probabilities in a sub-contest are independent of the efforts of players which are excluded from the sub-contest. *Sub-contest consistency* requires the winning probability of a player to be proportional to its winning probability in any sub-contest. More specifically, given a set of players N, for any player $i \in N$ and any subset of players $M \subset N$ where $i \in M$, sub-contest consistency requires that player i's winning probability $p(i)$ satisfies $p(i)/\left[\sum_{j\in M} p(j)\right] = p_M(i)$, where $p_M(i)$ is player i's winning probability in the sub-contest restricted to players in M. While the first four axioms are fairly weak, sub-contest consistency is clearly demanding. In this sense we say that sub-contest consistency is the *main* axiom of the characterization. In Skaperdas (1996), a contest success function p fulfills these five axioms if and only if there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that each player $i \in N$'s winning probability can be represented as $p(i) = f(x_i)/\left[\sum_{j\in N} f(x_j)\right]$ if $\sum_{j\in N} f(x_j) > 0$ and $1/n$ otherwise,

ranked according to their bids.

[3]A contest success function is of imperfectly discriminating type if efforts of players induce a non-degenerate probability distribution over all possible outcomes of a contest. The contest success functions in Skaperdas (1996), Clark and Riis (1996), Fu et al. (2013) and in this paper are of imperfectly discriminating type.

[4]These terms are not used in Skaperdas (1996). We follow the terminologies in Wärneryd (2001) and Münster (2009).

where $x_j \geq 0$ is the effort of player $j \in N$ and $n \geq 2$ is the number of players in N.

The contest success function defined in Skaperdas (1996) has been generalized in several directions by characterizations of different success functions. In all these extensions the outcome of a contest is of single-winner type, and most characterizations relax some of the first four axioms in Skaperdas (1996) and maintain sub-contest consistency. Let us briefly discuss them. Blavatskyy (2010) relaxes probability, introducing a success function for modeling contests with the possibility of a draw. Clark and Riis (1998a) relax anonymity, developing a success function which treats players unfairly. Rai and Sarin (2009) and Arbatskaya and Mialon (2010) define success functions for multi-dimensional efforts of players, and Münster (2009) axiomatizes a success function for contests between groups of players. These last three extensions generalize the framework in Skaperdas (1996) to cases where a player's effort is a vector instead of a scalar[5], and their axioms are generalizations of the ones in Skaperdas (1996) to the multi-dimensional case. Bozbay and Vesperoni (2013) develop a success function for contests where each pair of players can be ally or enemy of each other, defining the function for any possible network of alliances. In their characterization sub-contest independence and sub-contest consistency are violated for any network where some players are allies.

This paper is first in axiomatically characterizing a contest success function for rankings different than the single-winner type. However, there is non-axiomatic work on the subject. Clark and Riis (1996) introduce a success function for contests whose outcome is a strict ranking of players, without providing an axiomatic characterization. Fu and Lu (2011) show that this contest success function is the outcome of a stochastic process, under certain assumptions on the distribution of errors. Fu et al. (2013) propose an alternative success function for contests whose outcome is a strict ranking, deriving it from a different stochastic process. The success function developed here is more general than Clark and Riis (1996) and Fu et al. (2013) in that, besides the possibility of being applied to contests whose outcome is a strict ranking, it proposes a unified framework for modeling contests whose outcome is of any type, including non-strict rankings.

The second body of literature that this paper connects to is probabilistic choice and random utility theory. Luce (1959) axiomatically characterizes a framework for modeling the probabilistic choice of a decision maker from a set of alternatives. The main axiom of the characterization is *independence of irrelevant alternatives* (IIA). McFadden and Zarembka (1974) axiomatizes the most popular functional form of this model, known as the *multinomial logit.* McFadden (1973) provides a non-axiomatic characterization, showing that the choice probabilities can be derived from a stochastic process with errors of the *generalized extreme value* (GEV) family,

[5]In Münster (2009), a group of $n \geq 2$ players can be interpreted as a single player with n dimensional effort.

where the crucial assumption is the independence of these errors. By allowing for correlation of errors, Williams (1977) and McFadden (1978) generalize the framework into a model known as *nested logit*. Beggs et al. (1981) generalize the multinomial logit model in a different direction, providing a representation of probabilities of *strict rankings* of alternatives known as *rank-ordered logit*. Their characterization is non-axiomatic and generalizes the one in McFadden (1973), showing that their model can be derived from a stochastic process with errors of GEV type. While their representation is only for *strict* rankings, here we provide a model for any type of rankings, and our characterization generalizes Luce (1959).

3 Model

There is a set of players $N = \{1, \ldots, n\}$, where $n \geq 2$. A *ranking* r is an ordered partition of N which assigns a level to each player. We write $r(i) = l$ to say that in ranking r player i is ranked at level l[6]. For $N = \{1, 2, 3\}$, if $r(1) = 1$, $r(2) = 1$ and $r(3) = 2$, we write $r = (1, 2; 3)$. We exclude the degenerate case where all players are ranked equally. Then, the set of all rankings R is the set of all ordered partitions of N except (N). We say that player i is *ranked above* player j if $r(i) < r(j)$.

A *type of ranking* describes the exact number of players in each level of a ranking. We write $t(l) = m$ to say that a ranking of type t assigns m players to level l[7]. For instance, any ranking of single-winner type has one player at the first level ($t(1) = 1$) and all other players at the second level ($t(2) = n - 1$). We denote by T the set of all types of ranking.

Given a type of ranking $t \in T$, the *set of possible outcomes* of a contest is the set of all rankings in R of the same type t[8]. For instance, the set of possible outcomes of a single-winner contest is the set of all rankings in R of single-winner type. We denote by $R(t)$ the set of possible outcomes of type $t \in T$, and by $\mathcal{R}$ the set of all sets of possible outcomes.

Each player $i \in N$ is associated with an effort $x_i \in X_i := \mathbb{R}_+$, and $x \in X := \mathbb{R}^n_+$ is an effort profile. Given any type of ranking in T, a *contest success function* defines the probability of each possible outcome in $R(t)$ for any effort profile in X. For any

[6]Formally, a function $r : N \to N$ represents an ordered partition of N if and only if no player is assigned to a level below some empty level (for any i and l in N we have $r(i) = l$ only if for each $l' < l$ there exists $j \in N$ such that $r(j) = l'$). A function r which fulfills this condition is a ranking if and only if there is no level to which all players belong ($r(i) \neq r(j)$ for some $i, j \in N$).

[7]More formally, a function $t : N \to N$ is a type of ranking if and only if (1) each player is assigned to some level ($\sum_{l \in N} t(l) = n$); (2) there is no level to which all players belong ($t(l) \leq n-1$ for all $l \in N$); (3) if a level has some players in it then there are no empty levels above that level ($t(l) \geq 1$ only if $t(l') \geq 1$ for all $l' < l$ and $l \geq 2$).

[8]A set of rankings $A \subseteq R$ is a set of possible outcomes if and only if (1) every $r \in A$ is of same type $t \in T$; (2) for each $r \in A$, there exists $r' \in A$ such that $r'(\psi(i)) = r(i)$ for any player $i \in N$ and permutation ψ of N.

$x \in X$, we write $p(r, x)$ for the probability of $r \in R(t)$ being the outcome. Then, for any type $t \in T$, a contest success function of type t is a mapping $p : R(t) \times X \to [0, 1]$ where $\sum_{r \in R(t)} p(r, x) = 1$ for every $x \in X$.

4 Characterization of the contest success function

Let us define a contest success function for any type of ranking and characterize it via *pair-swap consistency*. To define this axiom, we first define a *pair-swap ranking*. Let $t \in T$ be any type. For each ranking $r \in R(t)$ and each pair of players $i, j \in N$, the pair-swap ranking $r_{i,j}$ is obtained from the ranking r by players i and j swapping their levels and all other players remaining at the same levels[9]. For instance, for $N = \{1, 2, 3, 4\}$, if $r = (2; 3; 4; 1)$ the ranking $r_{2,4}$ is defined as $(4; 3; 2; 1)$, where players 2 and 4 swap their levels and players 3 and 1 remain at the same levels as in r. We can now define pair-swap consistency.

Definition 4.1 *Given a function $f : \mathbb{R}_+ \to \mathbb{R}_{++}$ and a type of ranking $t \in T$, a contest success function satisfies* ***pair-swap consistency*** *if for each effort profile $x \in X$, ranking $r \in R(t)$ and pair of players $i, j \in N$ with $r(j) = r(i) + 1$*

$$\frac{p(r, x)}{p(r, x) + p(r_{i,j}, x)} = \frac{f(x_i)}{f(x_i) + f(x_j)}.$$

For each pair of players $i, j \in N$ and effort profile $\in X$, the RHS of the equation in Definition 4.1 is the winning probability of player i in a two-player contest against player j, where the success function of the two-player contest belongs to the family characterized in Skaperdas (1996)[10]. For the same effort profile $x \in X$ and any type of ranking $t \in T$, the LHS is the probability of a ranking $r \in R(t)$ where player i is one level above player j, *conditional* on all players in $N \setminus \{i, j\}$ being at given levels in the outcome[11]. In other words, the LHS measures how likely it is that i is ranked above j rather than j above i *everything else is equal*, i.e., when any other player is

[9] Given a type $t \in T$, for any ranking $r \in R(t)$ and pair of players $i, j \in N$ the pair-swap ranking $r_{i,j}$ is defined as the ranking in $R(t)$ where (1) $r_{i,j}(j) = r(i)$ and $r_{i,j}(i) = r(j)$; (2) $\forall\ k \in N \setminus \{i, j\}$: $r_{i,j}(k) = r(k)$.

[10] In Skaperdas (1996) the function f is weakly positive, while here f is required to be *strictly* positive. This is to avoid well known problems in the characterization in Skaperdas (1996) which arise when $f(x_i) = 0$ for some $i \in N$ and $x_i \in X_i$. In short, the contest success function in Skaperdas (1996) is not well defined when this is the case. Moreover in Skaperdas (1996) the function f is *increasing*, while we do not impose such restriction here. We do this only to keep the model more general, as one can imagine competitive situations where a player's performance is not necessarily monotonic in its own effort. However, in the sections of discussion and applications we always assume f to be increasing.

[11] Given an effort profile $x \in X$, the outcome of a contest is a random variable. Let us denote it by ω, where $\omega(i)$ is the level of a player $i \in N$ in the outcome. Then, ω takes value in $R(t)$ and its probability distribution is determined by $\Pr(\omega = r) = p(r, x)$ for any $r \in R(t)$. The LHS of the equation in Definition 4.1 can be written as $\Pr(\omega = r | \omega(h) = r(h) \text{ for any } h \notin \{i, j\})$.

at some level. Then, we can say that pair-swap consistency defines a property of a contest success function $p : R(t) \times X \to [0,1]$ for any type of ranking $t \in T$ which requires, everything else equal, the relative performance of any pair of players to depend only on their relative efforts, as if they were the only players in the contest. Theorem 4.1 defines the contest success function.

Theorem 4.1 *Given a function $f : \mathbb{R}_+ \to \mathbb{R}_{++}$ and a type of ranking $t \in T$, a contest success function is pair-swap consistent if and only if for all $r \in R(t)$ and $x \in X$*

$$p(r,x) = \frac{\prod_{i\in N} f(x_i)^{-r(i)}}{\sum_{r'\in R(t)} \prod_{i\in N} f(x_i)^{-r'(i)}}. \tag{1}$$

Proof: see Appendix.

Restricting our attention to contests of single-winner type, the contest success function in Theorem 4.1 is equivalent to the one in Skaperdas (1996). Moreover, for single-winner contests, pair-swap consistency is equivalent to a *weaker version* of sub-contest consistency, which is the main axiom in Skaperdas (1996). While sub-contest consistency requires a player's winning probability in a contest to be proportional to its probability of winning in every sub-contest restricted to any *subset* of players, pair-swap consistency requires this to hold only for sub-contests restricted to any *pair* of players. Then, our weaker axiom is sufficient for characterizing the success function in Skaperdas (1996).

There is an important difference between our characterization and the one in Skaperdas (1996). While in Definition 4.1 pair-swap consistency is defined for a given function f, the axioms in Skaperdas (1996) describe properties of a success function without assuming any functional form. Loosely speaking, Skaperdas (1996) shows that a success function fulfills some axioms if and only if there exists a function f such that each player's winning probability can be represented as a particular function of $(f(x_i))_{i\in N}$. Then, while we define our success function for a *given* f, Skaperdas (1996) characterizes a *family* of contest success functions parametrized by f. Let us see our methodology in more detail. We start by choosing a function f and taking the success function corresponding to f from the family in Skaperdas (1996). We use this success function to define pair-swap consistency, which characterizes our contest success function for a given f and for all types of ranking. Taking any f, we define a family of contest success functions for any type of ranking which fulfills pair-swap consistency. Formally, in Theorem 4.1 we define a *consistent extension*[12] of each success function in the family of Skaperdas (1996) to the domain of success functions for any type of ranking.

[12]See Thomson (2011) for an introduction to consistent extensions of allocation rules to larger domains. We define our success function as a consistent extension of the one in Skaperdas (1996) only to simplify exposition, as a direct characterization of our function would require heavier notation and repetition of arguments already in Skaperdas (1996).

5 Discussion

In this section we study properties of the contest success function defined in Theorem 4.1. We already know that, given a type $t \in T$, the contest success function is a mapping $p : R(t) \times X \to [0,1]$ such that $\sum_{r \in R(t)} p(r,x) = 1$ for every $x \in X$. Moreover, given a function $f : \mathbb{R}_+ \to \mathbb{R}_{++}$, we know that $p(r,x)$ must fulfill pair-swap consistency for any $r \in R(t)$ and $x \in X$. Can we say anything more about our success function? To do so, we additionally assume f to be increasing, weakly concave and twice differentiable[13]. In the following sections we study properties which hold as a consequence of these.

In section 5.1 we analyze aspects of the success function which are relevant for studying existence and uniqueness of equilibrium in contest games, i.e., monotonicity and concavity of rankings' probabilities in players' efforts. The results show that, as rankings' probabilities are not necessarily monotonic and concave, the characterization of an equilibrium is not always trivial. On the other hand this proposition provides a very intuitive framework to reason on the issue. In section 5.2 we compare probabilities of a player being ranked first in contests with different sets of possible outcomes and same effort profile. These probabilities are generally different from each other and their relative magnitude depends on the effort profile. Our results give insights on the effects of a contest's type on its precision in ranking players proportionally to efforts. In section 5.3 we analyze players' incentives in spending costly efforts in contest games with various structures. More precisely, we analyze equilibrium behavior in contests with multiple stages, multiple prizes and various types of outcome. Our results highlight which structures induce higher equilibrium efforts than others.

In these sections we will often refer to contests of three types: single-winner, multi-winner and strict-ranking. Although their meaning is intuitive, let us define them here. For any set of players N, a *single-winner* contest partitions this set such that one player is ranked first ($t(1) = 1$) and all other players second ($t(2) = n - 1$). A *multi-winner* contest locates $m \geq 2$ players at the first level ($t(1) = m$) and all others at the second ($t(2) = n - m$). In a *strict-ranking* contest the outcome is a strict ranking of players ($t(l) = 1$ for all $l \in N$). Through all the section we will often refer to the following examples with three players. Consider a single winner-contest with $n = 3$. The set of possible outcomes is $R(t) = \{(1; 2, 3), (2; 1, 3), (3; 1, 2)\}$. Given an effort profile $x \in X$ the probability of ranking $r = (1; 2, 3)$ can be written as

$$p(r,x) = \frac{f(x_1)}{f(x_1) + f(x_2) + f(x_3)} \ .$$

In a multiple-winner contest with three players and two winners the set of possible outcomes is $R(t) = \{(1, 2; 3), (1, 3; 2), (2, 3; 1)\}$ and for any $x \in X$ the probability of

[13] This assumption is widely accepted in contest theory. See Skaperdas (1996) for axiomatic characterizations of specific functional forms for f which are positive, increasing and twice differentiable.

ranking $r = (1, 2; 3)$ is

$$p(r, x) = \frac{f(x_1)f(x_2)}{f(x_1)f(x_2) + f(x_1)f(x_3) + f(x_2)f(x_3)} \ .$$

Consider a strict-ranking contest with $n = 3$. The set of possible outcomes is

$$R(t) = \{(1; 2; 3), (1; 3; 2), (2; 1; 3), (2; 3; 1), (3; 1; 2), (3; 2; 1)\} \ .$$

and for any $x \in X$ the probability of $r = (1; 2; 3)$ takes value

$$p(r, x) = \frac{f(x_1)^2 f(x_2)}{f(x_1)^2 f(x_2) + f(x_1)^2 f(x_3) + f(x_2)^2 f(x_1) + f(x_2)^2 f(x_3) + f(x_3)^2 f(x_1) + f(x_3)^2 f(x_2)} \ .$$

5.1 Monotonicity and concavity of rankings' probabilities

In this section we study rankings' probabilities as functions of players' efforts, and find conditions for their monotonicity and concavity. Let us define some concepts to be used in Proposition 5.1. Consider a contest of any type $t \in T$. For any effort profile $x \in X$ and ranking $r \in R(t)$, we say that $p(r, x)$ is *increasing* in the effort of player $i \in N$ if the first derivative $\partial\left[p(r, x)\right]/\partial x_i$ evaluated at x is positive, while we say that $p(r, x)$ is *concave* in player i's effort if the second derivative $\partial^2\left[p(r, x)\right]/\partial^2 x_i$ evaluated at x is negative. Before stating our results, we introduce some more notation. For any effort profile $x \in X$, the level of a player $i \in N$ in the outcome is a random variable and its *expected value* is $\rho(i, x) := \sum_{r \in R(t)} p(r, x) r(i)$. For each ranking $r \in R(t)$, the *distance* of player i's level in r from its expected value is $d_i(r, x) := (r(i) - \rho(i, x))^2$, while its *expected distance*[14] is $\delta_i(x) := \sum_{r \in R(t)} p(r, x)(r(i) - \rho(i, x))^2$.

Proposition 5.1 *For any type $t \in T$ and effort profile $x \in X$*

1. *the probability of a ranking is always increasing (decreasing) in the effort of a player ranked first (last).*
2. *the probability of a ranking is always concave (convex) in the effort of a player ranked first (last) in any contest with only two levels.*
3. *the probability of a ranking $r \in R(t)$ is increasing (decreasing) in the effort of a player $i \in N$ if and only if player i's level in the ranking $r(i)$ is smaller (larger) than its expected value $\rho(i, x)$.*
4. *the probability of a ranking $r \in R(t)$ is concave (convex) in the effort of a player $i \in N$ if player i's level in the ranking $r(i)$ is smaller (larger) than its expected value $\rho(i, x)$ and if the distance $d_i(r, x)$ is smaller (larger) than the expected distance $\delta_i(x)$.*

[14]Notice that the expected distance of player i's level in the outcome is the *variance* of its level in the outcome, hence a measure of dispersion.

Proof: See Appendix.

Let us briefly interpret these results. Proposition 5.1 says that the probability of a ranking is increasing (decreasing) in the effort of a player if and only if the player's level in the outcome is smaller (larger) than its expected level. For example, let $n = 4$ and consider any symmetric effort profile $x' \in X$. Each player $i \in N$'s expected level is $\rho(i, x') = 5/2$, hence the probability of $r = (1; 2; 3; 4)$ at x' is increasing in x_1 and x_2 and decreasing in x_3 and x_4. Intuitively, an increase in a player's effort makes it more likely that the player is ranked at relatively better levels, everything else equal. Proposition 5.1 says also that the probability of a ranking is concave (convex) in the effort of a player if the ranking assigns the player to a level which is smaller (larger) than its expected level and whose distance is smaller (larger) than the expected distance. Let us see an example. For $n = 4$ and any symmetric effort profile $x' \in X$, each player $i \in N$'s expected level is $\rho(i, x') = 5/2$ and expected distance is $\delta(i, x') = 5/4$. Then, the probability of $r = (1; 2; 3; 4)$ at x' is concave in x_2 and convex in x_4. The intuition for this result is less obvious. When the distance is smaller than the expected distance, the ranking assigns the player to a level which is relatively close to the expected level. Then, a ranking's probability is concave in a player's effort when it ranks the player at a level only slightly smaller than its expected level. Intuitively, when this is the case, the ranking's probability is "approximately" the average of the player's winning probabilities in two-player contests against each opponent, for a given effort profile. As a two-player contest has only two levels, each of these winning probabilities are concave in the player's effort, hence their average too.

Let us see how Proposition 5.1 applies to our examples with three players. Consider the single-winner contest. The probability of ranking $r = (1; 2, 3)$ is increasing and concave in the effort of player 1 and decreasing and convex in the efforts of players 2 and 3 for any $x \in X$. For the multi-winner contest, the probability of $r = (1, 2; 3)$ is increasing and concave in the efforts of players 1 and 2 and decreasing and convex in the effort of player 3 for any $x \in X$. Consider the strict-ranking contest. The probability of $r = (1; 2; 3)$ is increasing in the effort of player 1 and decreasing in the effort of player 3 for any $x \in X$. On the other hand, $p(r, x)$ is increasing (decreasing) in the effort of player 2 if and only if player 2's expected level in the outcome $\rho(2, x)$ is larger (smaller) than its level $r(2) = 2$. Notice that the probability of ranking $r = (1; 2; 3)$ is not necessarily concave in the effort of player 1. One can show that a sufficient condition for its concavity in x_1 is $\rho(1, x) \leq 2$.

5.2 Winning probabilities in contests of different types

In this section we compare a player's probability of being ranked first in a contest of type $t \in T$ with its probability of being ranked first in a contest of type $t' \neq t$ with same effort profile. For any $t \in T$, $x \in X$ and $i \in N$, player i's *probability* *of being ranked first* in a contest of type t is the joint probability of all outcomes $r \in R(t)$ where this is the case. Then, player i's probability of being ranked first is $\sum_{r \in R_i(t)} p(r, x)$, where $R_i(t)$ is the set of all rankings $r \in R(t)$ where $r(i) = 1$.

Proposition 5.2 *For any $n \geq 3$ and $x \in X$, a player $i \in N$'s probability of being ranked first in a single-winner contest is*

1. *lower than the probability of being ranked first in a multi-winner contest with same effort profile x.*
2. *higher (lower) than the probability of being ranked first in a contest of a type with one player at first level, one player at second level and $n-2$ players at third level and same effort profile x if and only if $f(x_i)$ is lower (higher) than*

$$\frac{\sum_{j \neq i} f(x_j)^2 (\sum_{k \neq i,j} f(x_k))}{\sum_{j \neq i} f(x_j)(\sum_{k \neq i,j} f(x_k))}.$$

Proof: See Appendix.

Although the results in Proposition 5.2 are far from being a complete analysis of the subject, they give some idea of the forces at play. The intuition of the first result is straightforward. It is easier for any player to be ranked first when there are multiple first positions rather than a single one simply because more chances are available. The second result is not so obvious. Both types of contest rank a single player at the first level, but while one type indiscriminately ranks all other players at the last level, the other type selects also a single player for the second level. A player with a relatively low effort is more likely to be ranked first in the first type of contest, while a player with relatively high effort in the second type. Intuitively, the higher the number of levels in the outcome, the more a contest is "precise". In a contest with three levels, a player with low effort is more likely to be ranked third or second, and a player with high effort to be ranked second or first.

Let us see how the results in Proposition 5.2 apply to our examples with three players. Consider player 1's probability of being ranked first in the single-winner and the multi-winner contests. For the single-winner contest it is the probability of ranking $(1; 2, 3)$, and for the multi-winner contest it is the joint probability of rankings $(1, 2; 3)$ and $(1, 3; 2)$. By Proposition 5.2 player 1's probability of being ranked first is always higher in the multi-winner contest than in the single-winner one. Consider the strict-ranking contest. Player 1's probability of being ranked first in this contest is the joint probability of $(1; 2; 3)$ and $(1; 3; 2)$. For $n = 3$, a strict ranking can be seen as a ranking with one player at first level, one player at second level and $n-2$ players at third level. Then, by Proposition 5.2 player 1's probability of being ranked first is higher (lower) in the single-winner contest than in the strict ranking contest if and only if $f(x_1)$ is lower (higher) than the opponents' average $(f(x_2) + f(x_3))/2$.

5.3 Players' incentives in contests with various structures

In this section we study contest games where players compete for some prizes by spending costly efforts. In Proposition 5.3 we study players' incentives in contests with various types of outcome. More specifically, we compare equilibrium efforts of single-winner, multi-winner and strict-ranking contests. While all these contests are organized in a single stage, in Proposition 5.4 we analyze contests with two stages and compare their equilibrium efforts with the ones of analogous single-stage contests.

Contests with various types of outcome

The outcome of a contest is a ranking of players. For any level $l \in N$ in the outcome, a player ranked at level l receives a prize $v_l \geq 0$. We assume $v_l \geq v_{l+1}$ for all $l \in N$ and $v_l > v_{l+1}$ for some $l \in N$. Players are risk neutral and efforts' costs are linear and sunk. Then, a player $i \in N$'s payoff is

$$\pi_i(x) = \sum_{l=1}^{n} \sum_{r \in R_{i,l}(t)} p(r, x) v_l - x_i$$

where $R_{i,l}(t)$ is the subset of $R(t)$ which includes any ranking r such that $r(i) = l$, for any $l \in N$. We define the *average prize* as $\bar{v} := 1/n \sum_{l \in N} v_l$ and the *average level* as $\bar{l} := 1/n \sum_{l \in N} l$, which is equal to $(n+1)/2$ by Gauss formula.

Proposition 5.3 *Take a single-winner contest awarding a prize to the winner and nothing to others. In a symmetric interior equilibrium its total efforts are*

1. *higher than the ones in the symmetric equilibrium of any multi-winner contest which equally shares the same prize among $m \geq 2$ winners and gives nothing to others. The equilibrium efforts of this multi-winner contest are equal to the ones of multiple separated single-winner contests, each with n/m players and each awarding one of these prize's shares to its winner.*

2. *lower than the ones in the symmetric equilibrium of a strict-ranking contest awarding the same prize to the player ranked first and nothing to others. For any prize profile $(v_l)_{l \in N}$ the equilibrium efforts of this strict-ranking contest are such that*

$$\frac{f(x_i)}{f'(x_i)} = \bar{l}\bar{v} - \frac{1}{n} \sum_{l=1}^{n} l v_l$$

for any $i \in N$. For any level $l \in N$, a larger prize v_l induces higher (lower) equilibrium efforts if and only if l is smaller (larger) than the average level.

Given a budget constraint $\sum_{l \in N} v_l \leq V$ equilibrium efforts are maximized by giving the whole budget to the player ranked first and nothing to others.

Proof: See Appendix.

Let us briefly discuss these results. Consider the first point in Proposition 5.3. Firstly, it says that a single-winner contest induces higher efforts than any multi-winner contest which shares the same prize among its multiple winners. This result is intuitive and related to the first point in Proposition 5.2, which argues that it is always easier to be ranked first in a multi-winner contest than in a single-winner one. Secondly, it says that the multi-winner contest induces same efforts as separate single-winner contests each with n/m players and each awarding one of its multiple prizes. Intuitively, this result follows from pair-swap consistency, which says that the relative performance of any pair of players should depend only on their relative efforts, as if they were competing alone in a single-winner contest[15]. Consider the second point in Proposition 5.3. Firstly, we see that a single-winner contest with a unique prize induces lower efforts than a strict-ranking contest which gives the same unique prize to the player ranked first. This result is not immediate, but it can be understood by referring to the second point in Proposition 5.2, which hints that contests with more levels are more competitive. Secondly, in the second point of the proposition we have a simple formula for equilibrium efforts of strict-ranking contests, from where we derive some conclusions on the effects of prizes on efforts[16]. Intuitively, giving higher prizes to players ranked at best levels and lower prizes to players ranked at worst levels should increase competition.

Two-stage contests

A *two-stage contest* is a game where players compete for a single prize by spending costly efforts and the competition is organized in two stages. In the first stage all players engage in a multi-winner contest, which selects a set $M \subset N$ of winners[17]. The set of all possible sets of winners is denoted by $\mathcal{M}$. Only players in M access the second stage, where they engage in a single-winner contest where the winner receives a prize $v_1 > 0$ and the others nothing. Notice that there is a different second-stage contest for each possible set of winners $M \in \mathcal{M}$ of the first stage. A two-stage contest has *contingent expenditures* when each player $i \in N$ spends a different effort in each stage. Denote by x_i^1 player $i \in N$'s effort in the first stage, and by x_i^M its effort in the second stage for each $M \in \mathcal{M}$. For any $M \in \mathcal{M}$, if $i \notin M$ then $x_i^M = 0$. Call $x^1 \in X$ a first stage effort profile, and $x^M \in X$ a second stage effort profile for any $M \in \mathcal{M}$. For each set of second stage participants $M \in \mathcal{M}$, a player $i \in M$'s

[15] Wärneryd (2001) already shows that a single-winner contest awarding a unique prize v_1 induces higher efforts than m symmetric separated single-winner contests each with n/m players and each awarding a prize v_1/m. The results in the first point of Proposition 5.3 complete this analysis.

[16] On the other hand, the study of incentives in strict-ranking contests with different success functions has often been analytically demanding. See Clark and Riis (1996), Clark and Riis (1998b), Fu and Lu (2006) and Fu and Lu (2009).

[17] The cardinality $m \geq 2$ of the set of winners fully determines the type of the first stage's contest.

second stage payoff is

$$\pi_i^M(x^M) = p^M(i, x^M)v_1 - x_i^M$$

where $p^M(i, x^M)$ denotes the probability of the ranking of M where player i is ranked first. Given a second stage effort profile $x^M \in X$ for each $M \in \mathcal{M}$, a player $i \in N$'s first stage payoff is

$$\pi_i^1(x^1) = \sum_{M \in \mathcal{M}_i} p(M, x^1)\pi_i^M(x^M) - x_i^1$$

where $\mathcal{M}_i$ is the set of all sets of winners $M \in \mathcal{M}$ where $i \in M$ and $p(M, x^1)$ denotes the probability of the ranking of N where players in M are ranked first.

A two-stage contest has *generic expenditures* when a player $i \in N$ makes a unique expenditure y_i in the first stage, which determines the first stage's effort $x_i^1 := y_i$ and the second stage's effort $x_i^M := y_i$ for any $M \in \mathcal{M}$. A player $i \in N$'s payoff is

$$\pi_i(y) = \sum_{M \in \mathcal{M}_i} p(M, y)p^M(i, y)v_1 - y_i \ .$$

Proposition 5.4 *Take a single-winner contest awarding a prize to the winner and nothing to others. In a symmetric interior equilibrium its total efforts are*

1. *lower than the ones in the symmetric equilibrium of a two-stage contest with generic expenditures awarding the same prize to the winner and nothing to others. The equilibrium efforts of this two-stage contest are maximized by eliminating (the closest natural number to) $n - \sqrt{n}$ players in the first stage.*
2. *higher than the ones in the symmetric subgame perfect equilibrium of a two-stage contest with contingent expenditures awarding the same prize to the winner and nothing to others, given f is a power function*[18].

Proof: See Appendix.

Let us comment these results. The first point of Proposition 5.4 says that a contest which awards a prize in two stages induces higher efforts than the one which awards it in a single stage if its expenditures are generic. The second point of Proposition 5.4 says that the opposite holds if the contest has contingent expenditures. Notice that a two-stage contest implicitly ranks players in three levels: first stage losers, second stage losers and the second stage winner. If efforts are generic, a two-stage

[18] A function f is a *power function* if it takes the form $f(x_i) = (x_i + \beta)^\alpha$ for some $\alpha, \beta > 0$. The power function is the most popular functional form in the contests' literature and the case with $\beta = 0$ is axiomatically characterized in Skaperdas (1996).

contest is analogous to a single-stage contest with three levels. As hinted by the second point of Proposition 5.3, adding levels in the outcome tends to induce higher efforts. This explains the first point. The intuition for the second point is that with contingent efforts a player spends in the second stage only if it has passed the first stage, hence can wait and see if the contest is worth its efforts[19].

6 Appendix

Proof of Theorem 4.1

It is easy to verify that (1) satisfies pair-swap consistency. Moreover, for $n = 2$ the only type of ranking in T is single-winner, and pair-swap consistency directly defines p for any $r \in R(t)$ and $x \in X$. Let us show that pair-swap consistency requires p to take the functional form in (1) for all $n \geq 3$, $t \in T$, $r \in R(t)$ and $x \in X$. Given any $n \geq 3$, $t \in T$ and $x \in X$, pair-swap consistency requires

$$\frac{p(r, x)}{p(r_{i,j}, x)} = \frac{f(x_i)}{f(x_j)} \qquad (2)$$

for any $r \in R(t)$ and pair of players $i, j \in N$ ranked at consecutive levels ($r(j) = r(i) + 1$). Such players always exist since $n \geq 3$ and there are at least two levels in a ranking. Given any $n \geq 3$, consider types $t \in T$ with only *two levels*. Among these types, let us focus first on the ones with at least two players at the second level. For any of these types, without loss of generality consider a ranking $r \in R(t)$ such that $r = (1, \ldots; 2, 3, \ldots)$. Take any $x \in X$. By (2) $p(r, x)/p(r_{1,2}, x) = f(x_1)/f(x_2)$, hence there exists a function $F(x) \neq 0$ such that $p(r, x) = f(x_1)F(x)$ and $p(r_{1,2}, x) = f(x_2)F(x)$. By the same argument, as $p(r, x)/p(r_{1,3}, x) = f(x_1)/f(x_3)$, there exists $F'(x) \neq 0$ such that $p(r, x) = f(x_1)F'(x)$ and $p(r_{1,3}, x) = f(x_3)F'(x)$. As $p(r, x) = f(x_1)F(x) = f(x_1)F'(x)$, we necessarily have $F(x) = F'(x)$, hence

$$p(r, x) = f(x_1)F(x)\ ,\ p(r_{1,2}, x) = f(x_2)F(x) \text{ and } p(r_{1,3}, x) = f(x_3)F(x)\ . \qquad (3)$$

Among the types with only two levels, let us now focus on the ones with at least two players at the first level. For any of these types, without loss of generality

[19] This result relates to previous studies of multiple-stage contests with contingent expenditures and different success functions. Clark and Riis (1998b) and Fu and Lu (2006) show that adding a stage to a contest with contingent expenditures always induces higher efforts. As argued by the second point in Proposition 5.4 the opposite holds with our success function. The reason is not obvious. Here an interpretation. As they focus on success functions of strict-ranking type only, every time they add a stage they also add levels in the outcome. From the second point of Proposition 5.3 we know that adding levels in the outcome tends to induce higher efforts. Then, their result may be driven by implicitly adding levels in the outcome.

consider a ranking $r \in R(t)$ such that $r = (1, 2, \ldots; 3, \ldots)$. Take any $x \in X$. By (2) $p(r,x)/p(r_{1,3},x) = f(x_1)/f(x_3)$, hence there exists $F(x) \neq 0$ such that $p(r,x) = f(x_1)F(x)$ and $p(r_{1,3},x) = f(x_3)F(x)$. Moreover, $p(r,x)/p(r_{2,3},x) = f(x_2)/f(x_3)$, hence there exists $F'(x) \neq 0$ such that $p(r,x) = f(x_2)F'(x)$ and $p(r_{2,3},x) = f(x_3)F'(x)$. We now have obtained $p(r,x) = f(x_1)F(x) = f(x_2)F'(x)$. Then, there exists $F''(x) \neq 0$ such that $p(r,x) = f(x_1)f(x_2)F''(x)$, and we necessarily have

$$p(r,x) = f(x_1)f(x_2)F''(x)\ ,\ p(r_{1,3},x) = f(x_2)f(x_3)F''(x)$$

$$\text{and } p(r_{2,3},x) = f(x_1)f(x_3)F''(x)\ . \tag{4}$$

Given any $x \in X$, by iteratively applying the arguments which brought to (3) and (4) one can show that for every type of ranking t with only two levels there exists a function $F_t(x) \neq 0$ such that $p(r,x) = \prod_{i \in A_r} f(x_i)F_t(x)$ for each $r \in R(t)$, where $A_r \subset N$ is the set of all payers ranked at first level in r. Notice that this expression can also be written as $p(r,x) = \prod_{i \in N} f(x_i)^{-r(i)}G_t(x)$, where $G_t(x) = F_t(x)\prod_{i \in N} f(x_i)^2$. By the definition of contest success function, the sum of the probabilities of all rankings in $R(t)$ is 1, hence

$$G_t(x) = \sum_{r \in R(t)} \prod_{i \in N} f(x_i)^{-r(i)}.$$

It follows that $p(r,x)$ must be as in (1) for any $n \geq 3$, $x \in X$, type t with only two levels and ranking $r \in R(t)$.

Given any $n \geq 3$ and type $t \in T$ with *at least three levels*, consider any ranking $r \in R(t)$. Take any triple of players $i, j, h \in N$ ranked $r(h) = r(j) + 1 = r(i) + 2$. Given r, consider the ranking $r' \in R(t)$ where i, j, h are ranked $r'(h) = r(i)$, $r'(j) = r(h)$ and $r'(i) = r(j)$ and all other players are ranked as in r. With some abuse of notation, these two rankings are represented by $r = (i \ldots; j \ldots; h \ldots)$ and $r' = (h \ldots; i \ldots; j \ldots)$. Take any $x \in X$. As the pair-swap ranking $r_{j,h} = (h \ldots; j \ldots; i \ldots)$ is equal to the pair-swap ranking $r'_{i,h}$, we have

$$\frac{p(r,x)}{p(r',x)} = \frac{p(r,x)}{p(r_{j,h},x)} \frac{p(r'_{i,h},x)}{p(r',x)}.$$

By (2), $p(r,x)/p(r_{j,h},x) = f(x_j)/f(x_h)$ and $p(r',x)/p(r'_{i,h},x) = f(x_i)/f(x_h)$, hence

$$\frac{p(r,x)}{p(r',x)} = f(x_i)f(x_j)f(x_h)^{-2}. \tag{5}$$

Notice that $r'(i) - r(i) = 1$, $r'(j) - r(j) = 1$ and $r'(h) - r(h) = -2$. Then, the RHS of (5) can be written as $\prod_{k \in \{i,j,h\}} f(x_k)^{r'(k)-r(k)}$. As the pair-swap ranking $r_{i,h}$ is equal to the pair-swap ranking $r'_{i,j}$, we have

$$\frac{p(r,x)}{p(r_{i,h},x)} = \frac{p(r,x)}{p(r',x)} \frac{p(r',x)}{p(r'_{i,j},x)}.$$

By (2) and (5), we have

$$\frac{p(r,x)}{p(r_{i,h},x)} = f(x_i)^2 f(x_h)^{-2} \tag{6}$$

Again, notice that $r_{i,h}(i) - r(i) = 2$, $r_{i,h}(j) - r(j) = 0$ and $r_{i,h}(h) - r(h) = -2$, and the RHS of (6) can be written as $\prod_{k \in \{i,j,h\}} f(x_k)^{r_{i,h}(k)-r(k)}$. By iteratively applying the arguments which led to (5) and (6), we obtain

$$\frac{p(r,x)}{p(r',x)} = \prod_{i \in N} f(x_i)^{r'(i)-r(i)} \text{ for each pair of rankings } r, r' \in R(t). \tag{7}$$

By (7), there must exist a function $G_t(x) \neq 0$ such that

$$p(r,x) = \prod_{i \in N} f(x_i)^{-r(i)} G_t(x) \text{ for each } r \in R(t).$$

By the definition of contest success function the sum of the probabilities of all rankings in $R(t)$ is 1, hence $G_t(x) = \sum_{r \in R(t)} \prod_{i \in N} f(x_i)^{-r(i)}$. It follows that $p(r,x)$ must be as in (1) for any $n \geq 3$, $x \in X$, $t \in T$ and $r \in R(t)$.

□

Proof of Proposition 5.1

Given a function $f : \mathbb{R}_+ \to \mathbb{R}_{++}$, for any $t \in T$ and $x \in X$, the probability of a ranking $r' \in R(t)$ is

$$p(r',x) = \frac{\prod_{i \in N} f(x_i)^{-r'(i)}}{\sum_{r \in R(t)} \prod_{i \in N} f(x_i)^{-r(i)}}.$$

Assume f twice differentiable, increasing ($f' > 0$) and weakly concave ($f'' \leq 0$). The first derivative of $p(r',x)$ with respect to x_i is

$$\frac{\partial}{\partial x_i} \left[p(r',x) \right] = \frac{f'(x_i)}{f(x_i)} p(r',x) \left[\rho(i,x) - r'(i) \right] \ .$$

For any $x \in X$, $\partial [p(r', x)]/\partial x_i \geq 0$ if and only if $r'(i) < \rho(i, x)$, hence $p(r', x)$ is increasing in x_i if r' ranks player i above its expected level in the outcome, while $p(r', x)$ is decreasing otherwise. Consider any type of ranking $t \in T$ with only two levels. As $r(i) \in \{1, 2\}$ for any $r \in R(t)$, then $\rho(i, x) \in [1, 2]$ for any $x \in X$. Then, as $\rho(i, x) \geq 1$ for any $x \in X$, the probability $p(r', x)$ is always increasing in x_i for any $r' \in R(t)$ where $r'(i) = 1$. Similarly, $p(r', x)$ is decreasing in x_i for any $r' \in R(t)$ where $r'(i) = 2$.

Let us consider concavity. The second derivative of $p(r', x)$ with respect to x_i is

$$\frac{\partial^2}{\partial^2 x_i} \left[p(r', x)\right] = H(x_i) p(r', x) \left[r'(i) - \rho(i, x)\right] + \\ + \left[\frac{f'(x_i)}{f(x_i)}\right]^2 p(r', x) \left[d_i(r', x) - \delta_i(x)\right] \tag{8}$$

where $H(x_i) = (f'(x_i)/f(x_i))^2 - f''(x_i)/f(x_i)$. The RHS of (8) is the sum of two parts. As $H(x_i) > 0$ for any $x_i \geq 0$, the first part is negative if and only if $r'(i) < \rho(i, x)$, hence $p(r', x)$ tends to be concave when increasing, and convex when decreasing. The second part is negative if and only if $d_i(r', x) < \delta_i(x)$. Then, $r'(i) < \rho(i, x)$ and $d_i(r', x) < \delta_i(x)$ are sufficient conditions for the concavity of $p(r', x)$ in x_i, while $r'(i) > \rho(i, x)$ and $d_i(r, x) > \delta_i(x)$ are sufficient conditions for its convexity.

Consider any type of ranking t with only two levels. For each player $i \in N$, take any $r \in R(t)$ where i is ranked first ($r(i) = 1$). By (8) $\rho(i, x)^2 \leq 3\rho(i, x) - 2$ is a sufficient condition for $p(r, x)$ being concave in x_i for any $x \in X$. As rankings have only two levels, we have $\rho(i, x) \in [1, 2]$. The LHS and RHS of the condition are equal to each other at $\rho(i, x) = 1$ and $\rho(i, x) = 2$. Moreover the LHS is convex in $\rho(i, x)$, while the RHS is linear. It follows that the LHS is smaller than the RHS for any $\rho(i, x)$ and $p(r, x)$ is concave in x_i for any $x \in X$. By similar arguments, one can show that for any type $t \in T$ with only two levels $p(r, x)$ is convex in x_i for any $x \in X$ and $r \in R(t)$ where $r(i) = 2$.

□

Proof of Proposition 5.2

Without loss of generality, let us focus on the probabilities of player 1 being ranked first. The probability of player 1 being ranked first in a single-winner contest is $f(x_1)/\left[\sum_{i \in N} f(x_i)\right]$ for some f positive and increasing. Consider a multi-winner contest with $m \geq 2$ winners. Call $\mathcal{M}$ the set of all $M \subset N$ of cardinality m. $\mathcal{M}$ is the set of all sets of winners. For any player $i \in N$, $\mathcal{M}_i$ is the set of all sets of winners $M \in \mathcal{M}$ where $i \in M$. With some abuse of notation, for each $M \in \mathcal{M}$ denote by $p(M, x)$ the probability of the ranking $r \in R(t)$ where $r(i) = 1$ for any $i \in M$. Then, the probability of 1 being ranked first is $\sum_{M \in \mathcal{M}_1} p(M, x)$. For any

$x \in X$, player 1 is more likely to be ranked first in the multi-winner contest than in the single-winner contest if and only if

$$\frac{\sum_{M\in\mathcal{M}_1}\prod_{i\in M} f(x_i)}{\sum_{M\in\mathcal{M}}\prod_{j\in M} f(x_j)} > \frac{f(x_1)}{\sum_{j\in N} f(x_j)}$$

After few manipulations this condition becomes

$$\sum_{M\in\mathcal{M}_1}\prod_{i\in M\setminus\{1\}} f(x_i)\left[\sum_{j\in N} f(x_j)\right] > \sum_{M\in\mathcal{M}_1}\prod_{i\in M\setminus\{1\}} f(x_i)\left[\sum_{j\notin M} f(x_j) + f(x_1)\right]$$

which is always true.

Consider a contest of a type with one player at the first level, one player at the second level and $n-2$ players at the third level. For any $x \in X$, player 1's probability of being ranked first in this contest is

$$\frac{\sum_{j\neq 1} f(x_1)^2 f(x_j)}{\sum_{i\in N}\sum_{j\neq i} f(x_i)^2 f(x_j)} \;.$$

Then, comparing this probability with $f(x_1)/\left[\sum_{i\in N} f(x_i)\right]$, one can show that player 1 is more likely to be ranked first in the single winner contest if and only if

$$f(x_1) < \frac{\sum_{j\neq 1} f(x_j)^2(\sum_{k\neq 1,j} f(x_k))}{\sum_{j\neq 1} f(x_j)(\sum_{k\neq 1,j} f(x_k))} \text{ for any } x \in X.$$

□

Proof of the first point of Proposition 5.3

Consider a single-winner contest. By Proposition 5.1 the probability of a ranking where player $i \in N$ is ranked first is increasing and concave in x_i for any $x \in X$, hence player i's payoff is concave in x_i. One can show that for any f where $f(0)/f'(0) < (n-1)v/n^2$ there exists a unique equilibrium where each player $i \in N$ exerts effort x_i^* such that

$$\frac{f(x_i^*)}{f'(x_i^*)} = \frac{(n-1)v_1}{n^2} \;. \tag{9}$$

Consider $k \geq 2$ single-winner contests, each equivalent to the one above. There are k groups of n players, hence $m = n \cdot k$ players in total. The set of all players is

$M = \{1, \ldots, m\}$. As the n players of each group compete only within their group, equilibrium efforts are as in (9).

Suppose now that the players in M do not compete only within their group, but contend the k prizes in a grand contest all against all. Consider a grand contest of single-winner type, where the winner gets all the k prizes and others nothing. One can show that for any f where $f(0)/f'(0) < (m-1)kv_1/m^2$ there exists a unique equilibrium where each player $i \in M$ exerts effort x_i^* such that

$$\frac{f(x_i^*)}{f'(x_i^*)} = \frac{(m-1)kv_1}{m^2} \ . \tag{10}$$

Suppose now that in the merged contest each player $i \in M$ can obtain *at most one* of the k prizes. The merged contest is a multi-winner contest with m players and k winners ($t(1) = k$ and $t(2) = m - k$). Let $\mathcal{W}$ be the set of all subsets of M of cardinality k, representing the set of all possible sets of winners $W \in \mathcal{W}$. For any $x \in X$ and $W \in \mathcal{W}$, the probability that W is the set of winners is

$$p(W, x) = \frac{\prod_{i \in W} f(x_i)}{\sum_{W' \in \mathcal{W}} \prod_{i \in W'} f(x_i)}$$

where with some abuse of notation $p(W, x)$ represents the probability of the ranking $r \in R(t)$ where $r(i) = 1$ for each $i \in W$. The payoff of a player $i \in M$ is

$$\pi_i(x) = \sum_{W \in \mathcal{W}_i} p(W, x)v_1 - x_i$$

where $\mathcal{W}_i$ is the set of all $W \in \mathcal{W}$ where $i \in W$. Assume f increasing, concave and twice differentiable. By Proposition 5.1 the probability of a ranking where a player $i \in M$ is ranked first is increasing and concave in x_i. Then, player i's payoff is concave in x_i and an interior best reply is identified by the first order condition

$$\sum_{W \in \mathcal{W}_i} \frac{\partial}{\partial x_i} p(W, x)v_1 = 1 \ .$$

It can be shown that for any f where $f(0)/f'(0) < k(m-k)v_1/m^2$ there exists a symmetric interior equilibrium where the effort of each player $i \in M$ is such that

$$\frac{f(x_i^*)}{f'(x_i^*)} = \frac{k(m-k)v_1}{m^2} \tag{11}$$

It is easy to verify that the RHS of (10) is always larger than the RHS of (11), hence the single-winner grand contest always induces higher efforts than the multi-winner grand contest. Moreover, as $m = n \cdot k$, the RHS of (11) is equal to the RHS of (9), hence the multi-winner grand contest induces the same efforts as the separated single-winner contests. Corollary 6.1 summarizes the results.

Corollary 6.1 *Suppose f is increasing, concave and twice differentiable. For some f, there exists an equilibrium of the multi-winner grand contest such that the efforts in the multi-winner grand contest are equal to the ones in the unique equilibrium of the single-winner contests. Moreover, in the unique equilibrium of the single-winner grand contest efforts are larger than in the equilibrium of the multi-winner grand contest.*

□

Proof of the second point of Proposition 5.3

Consider a strict-ranking contest. If there exists an interior equilibrium, the best reply of each player $i \in N$ is identified by the first order condition

$$\sum_{l=1}^{n} \sum_{r \in R_{i,l}(t)} \frac{\partial}{\partial x_i} [p(r,x)]\, v_l = 1 \ .$$

If the equilibrium is symmetric, the equilibrium effort x_i^* of each player $i \in N$ is such that

$$\frac{f(x_i^*)}{f'(x_i^*)} = \frac{1}{n} \sum_{l=1}^{n} \left[\bar{l} - l\right] v_l \tag{12}$$

where $\bar{l} = 1/n \sum_{l=1}^{n} l$ is the average level. Notice that, by Gauss formula, $\bar{l}$ is equal to $(n+1)/2$, hence (12) can also be written as

$$\frac{f(x_i^*)}{f'(x_i^*)} = \frac{(n+1)}{2}\bar{v} - \frac{1}{n}\sum_{l=1}^{n} l v_l$$

where $\bar{v} = 1/n \sum_{l=1}^{n} v_l$ is the average prize. As f is increasing and concave, the LHS of (12) is always increasing in x_i^*. The RHS of (12) is a weighted average of the prizes. Notice that the weight of prize v_l is positive if and only if l is smaller than the average level $\bar{l}$. Then, given a prize profile $(v_l)_{l \in N}$, increasing a prize v_l induces higher efforts if and only if $l < \bar{l}$.

Notice that in the RHS of (12) the weight of a prize v_l is always larger that the weight of v_{l+1} for any $l < n$. Then, given a limited budget $\sum_{l \in N} v_l \leq V$, equilibrium efforts are maximized by giving a unique large first prize $v_1 = V$. By (12), if $v_1 = V$, each player exerts effort such that $f(x_i^*)/f'(x_i^*) = (n-1)V/(2n)$. Consider the single-winner contest which gives a unique prize V to the winner. It is easy to verify that in a symmetric interior equilibrium this contest induces efforts such that $f(x_i^*)/f'(x_i^*) = (n-1)V/n^2$. It follows that, in a symmetric interior equilibrium, the strict ranking contest with $v_1 = V$ always induces higher efforts than the single-winner contest with prize V. Corollary 6.2 summarizes the results.

Corollary 6.2 *Suppose f is increasing, concave and twice differentiable. If a symmetric interior equilibrium exists, the equilibrium efforts are as in (12) and*

1. *given a prize profile $(v_l)_{l\in N}$, increasing v_l induces higher efforts if and only if $l < (n+1)/2$ for any $l \in N$;*
2. *given a budget $\sum_{l\in N} v_l \leq V$, equilibrium efforts are maximized by the prize profile $v_1 = V$ and $v_l = 0$ for all $l \geq 2$. These efforts are higher than the ones induced in the symmetric interior equilibrium of the single-winner contest with prize V.*

Let us see an example where the effort profile in Corollary 6.2 is an equilibrium. Let $N = \{1, 2, 3\}$. The set of possible outcomes is

$$R(t) = \{(1;2;3), (1;3;2), (2;1;3), (2;3;1), (3;1;2), (3;2;1)\}.$$

Without loss of generality, suppose $v_3 = 0$. Player 1's payoff is

$$\pi_1(x) = [p((1;2;3), x) + p(1;3;2), x)]\, v_1 + [p((2;1;3), x) + p(3;1;2), x)]\, v_2 - x_1$$

and similarly defined for other players. By (12) we know that, if a symmetric interior equilibrium exists for some f, equilibrium efforts are such that $f(x_i^*)/f'(x_i^*) = v_1/3$ for all $i \in N$, rankings' probabilities are $p(r, x^*) = 1/6$ for all $r \in R(t)$ and each player $i \in N$'s expected level in the outcome is $\rho(i, x^*) = 2$. Take any f where $f(0)/f'(0) < v_1/3$. By Proposition 5.1, for any $x \in X$, $r \in R(t)$ and $i \in N$, the probability $p(r, x)$ is increasing in x_i if and only if $r(i)$ is smaller than $\rho(i, x)$. Then, for any $r \in R(t)$ and $i \in N$, the probability $p(r, x)$ valued at x^* is increasing in x_i if $r(i) > 2$ and constant if $r(i) = 2$. Moreover, for any $r \in R(t)$ and $i \in N$ one can show that the probability $p(r, x)$ valuated at at x^* is concave in x_i if $r(i) \leq 2$. Then, for any $r \in R(t)$ and $i \in N$, the probability $p(r, x)$ valued at at x^* is increasing and concave in x_i if $r(i) = 1$ and constant and concave in x_i if $r(i) = 2$. It follows that player i's payoff at x^* is concave in x_i, and x^* is an equilibrium for some f. Corollary 6.3 summarizes the results.

Corollary 6.3 *For $n = 3$ the effort profile in Corollary 6.2 is an equilibrium for some f.*

□

Proof of the first point of Proposition 5.4

Consider a two-stage contest with generic expenditures. If an interior equilibrium exists, the best reply of each player is identified by a first order condition. In a symmetric equilibrium, the efforts are such that

$$\frac{f(y_i^*)}{f'(y_i^*)} = \frac{v_1}{n}\left(2 - \frac{m}{n} - \frac{1}{m}\right) . \tag{13}$$

By (15), a one-stage contest with prize v induces efforts such that

$$\frac{f(y_i^*)}{f'(y_i^*)} = \frac{v_1}{n}\left(1 - \frac{1}{n}\right) . \tag{14}$$

For any n and m the RHS of (14) is smaller than the RHS of (13). As the LHS of (13) is increasing in y_i^*, any two-stage contest induces higher efforts than the analogous one-stage contest. Let us find m which maximizes the efforts of a two-stage contest. It is easy to verify that the RHS of (13) is maximized at (the closest natural number to) $m^* = \sqrt{n}$, taking value (approximately)

$$\frac{f(y_i^*)}{f'(y_i^*)} = \frac{2v_1}{n}\left(1 - \frac{1}{\sqrt{n}}\right) .$$

Corollary 6.4 summarizes the results.

Corollary 6.4 *Suppose f is increasing, concave and twice differentiable. If a symmetric interior equilibrium exists, it is such that*

1. *any two-stage contest induces higher efforts than the analogous one-stage contest.*
2. *the equilibrium efforts are maximized when the contest eliminates (the closest natural number to) $n - \sqrt{n}$ players in stage-*1*;*

Let us show an example where the effort profile described in Corollary 6.4 is an equilibrium. Consider $n = 4$ and $m = \sqrt{n} = 2$. Without loss of generality, let us focus on player 1. The set of all sets of stage-1 winners to which 1 belongs is $\mathcal{M}_i = \{\{1,2\},\{1,3\},\{1,4\}\}$. The payoff of player 1 is $\pi(y) =$

$$= p(\{1,2\},y)\, p^{\{1,2\}}(1,y)v_1 + p(\{1,3\},y)\, p^{\{1,3\}}(1,y)v_1 +$$

$$+ p(\{1,4\},y)\, p^{\{1,4\}}(1,y)v_1 - y_1 .$$

The payoff of 1 is concave in y_1 if $p(M,y)\, p^M(1,y)$ is concave for any $M \in \mathcal{M}_i$. It is easy to verify that the second derivative of $p(M,y)\, p^M(1,y)$ with respect to y_1, when valuated at a symmetric effort profile, is always negative. Then, player $i \in N$'s payoff $\pi_i(y)$ is concave in y_i when valuated at y^*. Moreover, one can show that for any f where $f(0)/f'(0) < v_1/4$ the effort profile y^* is an equilibrium.

Corollary 6.5 *For $n = 4$ and $m = 2$ the effort profile in Corollary 6.4 is an equilibrium for some f.*

□

Proof of the second point of Proposition 5.4

Consider a two-stage contest with contingent expenditures. By results in Proposition 5.1, for any $M \in \mathcal{M}$ a player $i \in M$'s stage-2 payoff is concave in x_i^M. Then, one can show that for any f where $f(0)/f'(0) < (m-1)v/m^2$ there exists an equilibrium of stage-2 contest where each player $i \in M$ exerts effort x_i^{M*} such that

$$\frac{f(x_i^{M*})}{f'(x_i^{M*})} = \frac{(m-1)v_1}{m^2} \tag{15}$$

and its equilibrium stage-2 payoff is $\pi_i^M(x^{M*}) = v_1/m^2$. By backward induction, a player $i \in N$'s stage-1 payoff is

$$\pi_i^1(x^1) = \sum_{M \in \mathcal{M}_i} \frac{\prod_{i \in M} f(x_i^1)}{\sum_{M' \in \mathcal{M}} \prod_{i \in M'} f(x_i^1)} \pi_i^M(x^{M*}) - x_i^1$$

where $\mathcal{M}_i$ is the set of all $M \in \mathcal{M}$ where $i \in M$. It can be shown that, by results in Proposition 5.1, a player $i \in N$'s stage-1 payoff is concave in x_i^1. For any f where $f(0)/f'(0) < (n-m)v_1/(n^2 m)$ there exists an equilibrium where each player $i \in N$ exerts effort x_i^{1*} such that

$$\frac{f(x_i^{1*})}{f'(x_i^{1*})} = \frac{(n-m)v_1}{n^2 m} \ . \tag{16}$$

The sum of all efforts spent by all players in the two stages is $nx_i^{1*} + mx_i^{M*}$. As this sum cannot be computed from (15) and (16) for every function f, consider f such that $f(x_i^1) = (x_i^1 + \beta)^\alpha$, where $\alpha \in (0,1]$ and $\beta > 0$. With this functional form the equilibrium efforts are

$$x_i^{1*} = \frac{\alpha(n-m)v_1}{n^2 m} - \beta \ , \ x_i^{M*} = \frac{\alpha(m-1)v_1}{m^2} - \beta \tag{17}$$

for any $i \in N$ and $M \in \mathcal{M}$ and the sum of all efforts is

$$\frac{\alpha(n-1)v_1}{n} - (n+m)\beta \ . \tag{18}$$

Consider a one-stage contest where the n players contend all against all the same prize $v_1 > 0$. It is easy to verify that the total expenditures in the one-stage contest are $\alpha(n-1)v/n - n\beta$, hence always larger than the ones in (18) by the amount $m \cdot \beta$. Corollary 6.6 summarizes the results.

Corollary 6.6 *Suppose $f(x_{i,1}) = (x_i^1 + \beta)^\alpha$, where $\alpha \in (0,1]$ and $\beta > 0$. For some α and β, there exists an equilibrium of the two-stage contest where*

1. *total efforts are always lower than the ones of the one-stage contest;*
2. *the limit of total efforts for $\beta \to 0$ is equal to total efforts of the one-stage contest.*

□

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