# Endogenous Growth with a Ceiling on the Stock of Pollution

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**Abstract**

The effects of an agreement such as the Kyoto Protocol, which imposes a ceiling on the stock of pollution, have recently been studied in Hotelling models. We add pollution and a ceiling to the endogenous growth model of Tsur and Zemel (2005) to study the effects of the ceiling on capital accumulation and research investments. The ceiling affects the characteristic lines determining economic development only in the short run, i.e. an economy with a ceiling follows basically the same long run development path as an economy without the ceiling. In the short run, the ceiling imposes an additional scarcity on the exhaustible resource. That boosts backstop resource utilization, which implies the existence of more states - described by the capital stock and technology - where research instead of capital accumulation might be optimal. Thus, depending on its state, the economy may invest more in R&D and less in capital stock in the short run.
*Keywords:* Environmental agreements, Fossil fuels, Nonrenewable resources, Research, Endogenous growth

## 1. Introduction

Climate change has been one of the major issues both in public and academic discussion in recent decades. A wide range of nations agreed in the Kyoto Protocol to limit climate change by meeting a global temperature stabilization target, which allows for long-run global temperature increase of 2°C. According to Graßl et al. (2003) this target translates into a maximum $CO_2$ concentration of 400 - 450ppm, while Hansen et al. (2008) advocate a maximum $CO_2$ concentration of 350ppm. Regardless of who is right, it is necessary to impose a ceiling on the stock of $CO_2$. Those nations that have signed the Kyoto Protocol agreed that the consequences of climate change remain manageable as long as the stabilization target, and therefore the ceiling on $CO_2$ concentration, is not violated. Since other agreements follow a similar approach, e.g. the Montreal Protocol on Substances that Deplete the Ozone Layer, it seems likely that a successor of the Kyoto Protocol will include an implicit ceiling on the stock of $CO_2$. Fossil fuels are one of the main sources of $CO_2$ emissions. Therefore, agreements such as the Kyoto Protocol might have a significant impact on the energy generation of the economy.

In the literature the effects of a ceiling on the stock of $CO_2$, or more generally pollution, have recently been examined by Chakravorty et al. (2006a), Chakravorty et al. (2006b), Chakravorty et al. (2008), Chakravorty et al. (2011) and Lafforgue et al. (2008). This literature analyzes how a ceiling on the stock of pollution changes the optimal resource utilization path. A Hotelling model with polluting exhaustible resources and a renewable non-polluting resource serves as the basic framework, which is augmented in several ways. Abatement activities are considered by Chakravorty et al. (2006a) and Lafforgue et al. (2008). Chakravorty et al. (2008) focus on the consequences of two differently polluting exhaustible resources. Chakravorty et al. (2011) extend the model of Chakravorty et al. (2006a) by technological progress, which is caused by a learning-by-doing effect and decreases the costs of the backstop. It is shown that the optimal resource utilization path depends on the cost structure established by the standard assumption of the Hotelling model and the assumption related to the specific augmentation.

Owning to its Hotelling based structure, the literature fails to consider capital or research activities, which are both determinants of economic growth, structural change and changes of the energy mix as shown by Tsur and Zemel (2005). R&D in particular seems to be a non-negligible factor, as it is the driving force behind a steadily positive growth rate in many endogenous growth models, e.g.

Rivera-Batiz and Romer (1991).[1] Therefore, this paper strives to analyze the effects of a ceiling on the stock of pollution in an economy incorporating a polluting exhaustible resource and a backstop as well as capital and research driven technological progress. For this purpose we augment the suitable endogenous growth model of Tsur and Zemel (2005) with both pollution and a ceiling on the stock of pollution. Utilization of the two resources leads to costs. In contrast to the usual assumption of endogenous growth models, technological progress does not augment the productivity of resources or capital, but reduces the costs associated with the use of the backstop. With regard to fossil fuel based energy generation, the chosen modeling constitutes the more realistic approach. For clarification we refer to Stiglitz (1974). By modifying Solow's neoclassical growth model, Stiglitz shows that sustainable economic development is compatible with an exhaustible resource, if technology, which enhances the resource's productivity, increases sufficiently fast.[2] However, the result rests upon the assumption that a vast amount of goods can be produced by a vanishingly low amount of resources and sufficiently advanced technology. With regards to thermodynamics, such an assumption seems unrealistic if fossil fuels are taken into account.[3] Other features from the Hotelling models, such as abatement or differently polluting exhaustible resources, are left for further research, in order to keep the analysis as simple as possible.

In the present paper we show that the social optimum consists of three phases which appear in the Hotelling models in a similar manner. As in Chakravorty et al. (2006a), the only possible sequence containing all three phases starts with a non-binding ceiling which becomes binding later on. After a phase with a binding ceiling, the ceiling becomes non-binding again and will stay it forever. Thus, neither capital nor research can explain other sequences. However, research reduces the costs of the backstop. As long as the backstop is used, the unit costs

[1] A comprehensive review of the endogenous growth theory is given by Aghion et al. (1998) and Barro and Sala-i Martin (2003).

[2] See Solow (1956).

[3] Compare Meyer et al. (1998), page 171.

of the backstop determine the energy price as well as the marginal costs of the last used unit of exhaustible resources. Technological progress implies therefore a reduction of both. Together with changing energy demand, caused by the variable capital stock, and in contrast to Chakravorty et al. (2006a), the model can explain a decreasing scarcity rent of the exhaustible resource endogenously. By analyzing the development during the three phases and taking the only possible sequence into account, we can describe the optimal path of the economy. The development of the economy depends on its state described by capital stock and technology in relation to two characteristic lines. The ceiling affects one of the lines in the short run, i.e. in the first two phases, resulting in more capital-technology combinations with the optimality of research instead of capital accumulation than in an economy without the ceiling. This could be interpreted as higher incentives for R&D. During the phase with a binding ceiling these excess incentives are eliminated. In the long run, i.e. in the last phase, the lines are identical with those of the unconstrained economy of Tsur and Zemel (2005). Hence, the constrained economy will basically follow the same long run development path as the unconstrained economy. To sum up, the model clarifies how the ceiling affects capital accumulation and research activities and gives an endogenous explanation for decreasing scarcity rents of the exhaustible resource.

To complete the discussion, we decentralize the social optimum in a competitive market. The analysis is based upon a neoclassical framework with price-taking composite product manufacturers and individuals, as well as Cournot competition on the resource market between two resource owning companies. Neither the individuals nor the companies take their influence on the emission stock into account. Therefore, the exhaustible resource has to be taxed in the short run. In the long run, the tax is not needed due to the high scarcity of the resource. To adjust for market power effects resulting from the Cournot competition both resources must be subsidized at all times.

The outline of the paper is as follows. Section 2 gives a description of the model. The social optimum is described in section 3. The market economy and government interventions necessary for the social optimum are discussed in section 4. Section 5 concludes the discussion.

## 2. Model

We augment the endogenous growth model of Tsur and Zemel (2005) with a pollution stock and a ceiling on the stock of pollution. For that purpose we describe the model structure of Tsur and Zemel (2005) briefly.[4] A single composite good $Y$ is produced by using capital $K$ and energy $x$ according to the production function $Y = F(K, x)$, with $F(0, x) = F(K, 0) = 0$, $F_K > 0$, $F_x > 0$, $F_{KK} < 0$, $F_{xx} < 0$, $F_{Kx} = F_{xK} > 0$ and $J = F_{KK}F_{xx} - F_{Kx}^2 > 0$. To avoid a collapse of production, the assumptions $\lim_{K \to 0} F_K = \infty$ and $\lim_{x \to 0} F_x = \infty$ are added. Energy is generated by a one to one transformation of an exhaustible resource $R$ or a backstop $b$, i.e. $x = R + b$. The cost of supplying the resources is $M(R)$ in the case of the exhaustible resource and $M_b B(A)b$ in the case of the backstop. The first cost function is increasing and strictly convex, i.e. $M'(R) > 0$ and $M''(R) > 0$. Furthermore, we assume $M(0) = M'(0) = 0$. The second cost function is composed of a fixed cost parameter $M_b > 0$ and a function $B(A)$.[5] The latter reflects the influence of technology $A$ on the backstop unit costs. Additionally to $B'(A) < 0$ and $\lim_{A \to \infty} B(A) = \bar{B} > 0$ we assume $B(A_0) > 0$, $B''(A) > 0$ and $\lim_{A \to \infty} B'(A) = 0$, where $A_0$ is the initial value of technology. The net income is given at each point in time by $Y^n = F(K, x) - M(R) - M_b B(A)b$ and can be used for consumption $C$, physical capital (dis)investment $\dot{K}$ or research $I$. Then the capital stock evolves as follows:

$$\dot{K} = F(K, x) - C - M(R) - M_b B(A)b - I. \tag{1}$$

Technology $A$ increases in research investment $I$ in compliance with

$$\dot{A} = I. \tag{2}$$

[4] We refer to Tsur and Zemel (2005) for details. Deviations from Tsur and Zemel (2005) are indicated explicitly. For the sake of simplicity time index $t$ is suppressed. It is only added, if needed for understanding.

[5] Tsur and Zemel (2005) assume $M_b = 1$.

R&D investments are limited by the net income, i.e. $I \in [0, Y^n]$. Hereafter the upper bound is represented by $\bar{I}$. As long as the exhaustible resource is used, the resource stock $S_R$, with the initial value $S_{R_0}$, decreases according to

$$\dot{S}_R = -R. \tag{3}$$

At every point in time the representative household exhibits a strictly concave utility function $U(C)$, which increases in consumption with $\lim\limits_{C\to 0} U'(C) = \infty$. To avoid the optimality of $C = 0$, the assumption $U(0) = -\infty$ is made additionally.[6] Therefore, the utility is given by

$$U(C) \begin{cases} \geq 0, & \text{for } C > 0, \\ = -\infty, & \text{for } C = 0. \end{cases} \tag{4}$$

As in the Hotelling models mentioned above, utilization of exhaustible resources causes pollution $E$. To keep the model simple, it is assumed that utilization of one resource unit generates one unit of pollution, i.e. $R = E$. Thus, $R$ and $E$ are used synonymously. The stock of pollution is $S_E$, while its initial value is denoted by $S_{E_0}$. With $\gamma$ being the natural regeneration rate, $S_E$ develops according to

$$\dot{S}_E = E - \gamma S_E. \tag{5}$$

The ceiling $\bar{S}_E$ is imposed exogenously.[7] Then $\bar{S}_E - S_E \geq 0$ must hold at every point in time. Due to the ceiling, it is possible to divide the complete planning period into three phases depending on the ceiling's status. Phase 1 is characterized by a non-binding ceiling. In phase 2 the ceiling is binding for the time interval $[t_i, t_j[$, with $0 \leq t_i \leq t_j < \infty$. In phase 3 the ceiling is non-binding for $[t_k, \infty]$, with $t_k < \infty$.

[6] Due to $\lim\limits_{K\to 0} F_K = \infty$ and (15) decreasing capital stock is accompanied by increasing consumption. Therefore, $K = 0$ and $C = 0$ could be reached in finite time, if the assumption $U(0) = -\infty$ is not made.

[7] A reason for the ceiling can be prohibitive high costs if the emission stock increases to a level above the ceiling. Since the emission stock does not influence utility or production, the costs of the emission stock are negligible as long as the ceiling is not violated.

## 3. Social Optimum

In the following section we derive the social optimum. Thus, we assume that a social planner maximizes the utility over the complete planning period given the initial state $(K_0, A_0, S_{R_0}, S_{E_0})$ and subject to (1), (2), (3), (5), $\bar{S}_E - S_E \geq 0$, $K \geq 0$, $S_R \geq 0$, $0 \leq I \leq \bar{I}$ and $E, b, C \in [0, \infty[$. The present value of utility is given by $\int_0^\infty U(C)e^{-\rho t}dt$, with $\rho$ as the time preference rate. Thus, with $\lambda$, $\kappa$, $\tau$ and $\theta$ representing the current-value costate variables of $K$, $A$, $S_R$ and $S_E$, and $\mu$ representing the Lagrange multiplier associated with the ceiling, the current-value Lagrangian is

$$L = U(C) + \lambda[F(K, b + R) - C - M(E) - M_b B(A)b - I] + \kappa I - \tau E \\ + \theta[E - \gamma S_E] - \mu[E - \gamma S_E]. \tag{6}$$

Analogous to Tsur and Zemel (2005), an interior optimum is given by the following necessary conditions:[8]

$$\frac{\partial L}{\partial C} = U_C - \lambda = 0, \tag{7}$$

$$\frac{\partial L}{\partial E} = \lambda[F_x - M'] - \tau + \theta - \mu = 0, \tag{8}$$

$$\frac{\partial L}{\partial b} = \lambda[F_x - M_b B(A)] = 0. \tag{9}$$

The total energy supply, as well as the energy mix, can be determined graphically by means of (8) and (9). In Fig. 1 the energy demand function is given by $F_x$, while $M_b B(A)$ and $M' + \frac{\tau - \theta + \mu}{\lambda}$ represent the supply functions of the backstop and the exhaustible resource, respectively. The total energy supply $x$ is given by $F_x = M_b B(A)$ if both resources are used, and by $F_x = M' + \frac{\tau - \theta + \mu}{\lambda}$ if only the exhaustible resource is used.[9] In the latter case, total energy equals $E^{\#}$ and $M_b B(A) > M'(E^{\#}) + \frac{\tau - \theta + \mu}{\lambda}$ must hold. In the first case, the amount

[8] It can be shown that the sufficient conditions hold as long as $B''(A) \geq \frac{M_b}{b}(B'(A))^2 \left[\frac{1}{M''(R)} - \frac{F_{KK}}{J}\right]$. Due to $B''(A) > 0$, $M''(R) > 0$ and $\frac{F_{KK}}{J} < 0$ both sides of the inequality are positive. As long as the backstop is used, which is assumed, the inequality holds if $M_b$ is sufficiently small.

[9] A similar figure with $\theta = \mu = 0$ can be found in Tsur and Zemel (2005), p. 488. Thus, the figure of Tsur and Zemel (2005) is a special case of Fig. 1.

of exhaustible resources is given by $M_b B(A) = M'(E) + \frac{\tau - \theta + \mu}{\lambda}$ and the amount of backstops by the difference $x - E$. Both resources are only used simultaneously as long as $M'(0) + \frac{\tau - \theta + \mu}{\lambda} < M_b B(A) < M'(E^{\#}) + \frac{\tau - \theta + \mu}{\lambda}$ holds. If $M_b B(A) < M'(0) + \frac{\tau - \theta + \mu}{\lambda}$, only the backstop is used.

In the following we assume that both resources are used. The second term of the supply function $M'(E) + \frac{\tau - \theta + \mu}{\lambda}$ compares the shadow prices related to the exhaustible resource with the shadow price of capital, i.e. it reflects the scarcity of the resource in relation to the scarcity of capital. Therefore, $\frac{\tau - \theta + \mu}{\lambda}$ should be called the relative scarcity index. It will be used below to identify the possible sequence of phases. In the following, the index $^*$ denotes optimal values, while

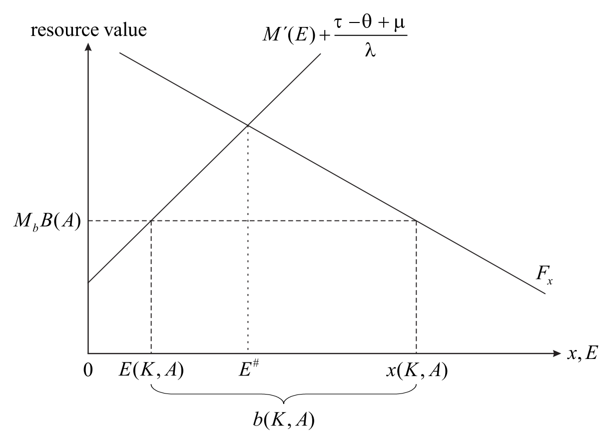


**Figure 1:** Usage of exhaustible resource and backstop

unmarked variables denote values of any possible path. The maximization of (6) with respect to the R&D investments $I$ gives

$$
\begin{aligned}
I^* &= 0, \text{ if } -\lambda + \kappa < 0, \\
0 \leq I^* &\leq \bar{I}, \text{ if } -\lambda + \kappa = 0, \\
I^* &= \bar{I}, \text{ if } -\lambda + \kappa > 0.
\end{aligned}
\tag{10}
$$

Depending on the relation of $\kappa$ to $\lambda$, R&D investments are minimal, singular or maximal. The costate variables grow according to

$$\frac{\partial L}{\partial K} = \lambda F_K = \rho\lambda - \dot{\lambda}, \tag{11}$$
$$\frac{\partial L}{\partial S_E} = -\theta\gamma + \mu\gamma = \rho\theta - \dot{\theta}, \tag{12}$$
$$\frac{\partial L}{\partial S_R} = 0 = \rho\tau - \dot{\tau}, \tag{13}$$
$$\frac{\partial L}{\partial A} = -\lambda M_b b B' = \rho\kappa - \dot{\kappa}. \tag{14}$$

Combining (11) with (7) establishes the well-known Ramsey - rule

$$\hat{C} = \frac{F_K - \rho}{\eta}. \tag{15}$$

This states that the growth rate of consumption $\hat{C}$ is positive as long as the marginal product of capital is higher than the time preference rate. Consumption reacts the stronger to the difference the smaller the positive elasticity of marginal utility ($\eta$) is.
The complementary slackness condition is given by

$$\begin{aligned}
&\frac{\partial L}{\partial \mu} = -E + \gamma S_E \geq 0, \qquad && \mu \geq 0, \qquad && \mu\frac{\partial L}{\partial \mu} = 0, \\
& && \bar{S}_E - S_E \geq 0, && \mu[\bar{S}_E - S_E] = 0, \\
& && \rho\mu - \dot{\mu} \geq 0, && [= 0 \quad \text{if} \quad \bar{S_E} - S_E > 0].
\end{aligned} \tag{16}$$

To complete the equation system the transversality conditions

$$\begin{aligned}
&(a) \lim_{t\to\infty} e^{-\rho t}\lambda(t)\,[K(t) - K^*(t)] \geq 0, \qquad && (b) \lim_{t\to\infty} e^{-\rho t}\tau(t)\,[S_R(t) - S_R^*(t)] \geq 0, \\
&(c) \lim_{t\to\infty} e^{-\rho t}\theta(t)\,[S_E(t) - S_E^*(t)] \geq 0, && (d) \lim_{t\to\infty} e^{-\rho t}\kappa(t)\,[A(t) - A^*(t)] \geq 0
\end{aligned} \tag{17}$$

are needed.

Before analyzing the three phases it is useful to determine the possible sequences of phases. This will also reveal information about the sign and existence of $\theta$ during the three phases. For this purpose, the behavior of the relative scarcity index at the junction points is observed. The jump conditions for the costate variables at a junction point $j$ is[10]

$$\Gamma^{+}(j) = \Gamma^{-}(j) + B\frac{\partial[\bar{S}_E - S_E]}{\partial \Gamma_V},\ B \geq 0, \tag{18}$$

with $\Gamma = \tau, \theta, \lambda$; $\Gamma_V$ being the associated state variable $S_R, S_E, K$ as well as $^+$ and $^-$ denoting the values just after and just before the junction point, respectively. It shows that $\tau$ and $\lambda$ are continuous while $\theta$ may jump. Due to (7), the continuity of $\lambda$ implies a continuous consumption path. Since the indirect approach is used for (6), the jump condition can be written as[11]

$$\theta^{+}(j) = \theta^{-}(j) + \mu^{+}(j) - \mu^{-}(j) + B_\theta,\ B_\theta \geq 0. \tag{19}$$

Due to (16), $\mu = 0$ during phase 1 and 3. In phase 3 the ceiling is non-binding and will never be reached. Since pollution has then no effect on production or utility, it is irrelevant for the social planner. Thus, its shadow price $\theta$ must be zero. Therefore, the relative scarcity index in phase 1, phase 2 and phase 3 is $\frac{\tau-\theta}{\lambda}$, $\frac{\tau-\theta+\mu}{\lambda}$ and $\frac{\tau}{\lambda}$, respectively. At a junction point the used amount of exhaustible resources can exhibit a jump, because $E$ is a control variable. If the ceiling becomes binding at the junction point, a jump upwards is prevented by the natural regeneration rate. If the ceiling becomes non-binding, the ceiling itself prevents an upward jump. However, jumps downward are possible in both cases. The necessary changes can be derived from Fig. 1. It does not matter whether the backstop is used. As the demand function $F_x$ and the marginal extraction costs function $M'(E)$ are not affected by a junction point, a sudden drop in $E$ is only possible if the relative scarcity index increases. Therefore, the following conditions must hold at junction points between phase 1 and 2 as well as between phase 2 and 3.

- At a junction point $t_1$ from phase 1 to phase 2:
  $\frac{\tau^{-}(t_1)-\theta^{-}(t_1)}{\lambda^{-}(t_1)} \leq \frac{\tau^{+}(t_1)-\theta^{+}(t_1)+\mu^{+}(t_1)}{\lambda^{+}(t_1)} \Leftrightarrow \theta^{+}(t_1) \leq \theta^{-}(t_1) + \mu^{+}(t_1)$

[10] Cf. Feichtinger and Hartl (1986), p. 166 et seq.
[11] Cf. Chiang (1992), p. 300 et seq.

- At a junction point $t_2$ from phase 2 to phase 3:
$\frac{\tau^-(t_2)-\theta^-(t_2)+\mu^-(t_2)}{\lambda^-(t_2)} \leq \frac{\tau^+(t_2)}{\lambda^+(t_2)} \Leftrightarrow \mu^-(t_2) \leq \theta^-(t_2)$

- At a junction point $t_3$ from phase 2 to phase 1:
$\frac{\tau^-(t_3)-\theta^-(t_3)+\mu^-(t_3)}{\lambda^-(t_3)} \leq \frac{\tau^+(t_3)-\theta^+(t_3)}{\lambda^+(t_3)} \Leftrightarrow \theta^+(t_3) \leq \theta^-(t_3) - \mu^-(t_3)$

Substituting (19) shows that all three conditions must hold equally. This implies the continuity of $E$, since the state variables capital $K$ and technology $A$ have to be continuous, too. The total energy input depends only on $K$ and $A$, so that its continuity, as well that of $b = x(K, A) - E(K, A)$, follows directly. Thus, both production factors are continuous, which implies the continuity of $Y$. The one of consumption $C$ results from the continuity of $\lambda$ and (7). Therefore, the economy switches smoothly from one phase to the next.

If we denote the variables by the corresponding phase, we can rewrite the conditions at the junction points as $\theta_2(t_1) = \theta_1(t_1) + \mu_2(t_1)$, $\theta_2(t_2) = \mu_2(t_2)$ and $\theta_2(t_3) = \theta_1(t_3) + \mu_2(t_3)$, respectively. Obviously, the first and third conditions are identical. Thus, should there be more than one junction point between phase 1 and 2, the conditions must hold for two or more different points in time. However, by solving (12) and (16) for $\theta_1$, $\theta_2$ and $\mu_2$, i.e. $\theta_1(t) = \theta_{01}e^{(\rho+\gamma)t}$, $\theta_2(t) = \theta_{02}e^{(\rho+\gamma)t} - \gamma\mu_{02}e^{(\rho+\gamma)t}\int e^{-(\rho+\gamma)t+\rho\int\xi(t)dt}dt$ and $\mu_2(t) = \mu_{02}e^{\rho\int\xi(t)dt}$, with $\theta_{01}$, $\theta_{02}$ and $\mu_{02} > 0$ as constants of integration and $\xi(t) \leq 1$, the conditions can be written as

$$\begin{aligned}
&\theta_{02}e^{(\rho+\gamma)t} - \gamma\mu_{02}e^{(\rho+\gamma)t}\int e^{-(\rho+\gamma)t+\rho\int\xi(t)dt}dt = \theta_{01}e^{(\rho+\gamma)t} + \mu_{02}e^{\rho\int\xi(t)dt} \\
\Leftrightarrow\ &\frac{\theta_{02}-\theta_{01}}{\mu_{02}} = e^{-(\rho+\gamma)t+\rho\int\xi(t)dt} + \gamma\int e^{-(\rho+\gamma)t+\rho\int\xi(t)dt}dt, \text{ for } t = t_1, t_3 \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&\theta_{02}e^{(\rho+\gamma)t} - \gamma\mu_{02}e^{(\rho+\gamma)t}\int e^{-(\rho+\gamma)t+\rho\int\xi(t)dt}dt = \mu_{02}e^{\rho\int\xi(t)dt} \\
\Leftrightarrow\ &\frac{\theta_{02}}{\mu_{02}} = e^{-(\rho+\gamma)t+\rho\int\xi(t)dt} + \gamma\int e^{-(\rho+\gamma)t+\rho\int\xi(t)dt}dt, \text{ for } t = t_2. \qquad (21)
\end{aligned}$$

The right hand side of (20) and (21) is called $T_f(t)$. It is continuous in time and $\frac{dT_f}{dt} < 0$ for $\xi(t) < 1$. As long as the growth rate of $\mu$ is lower than the time preference rate, $T_f$ descreases strictly. In this case, (20) as well as (21) holds only for one point in time, which implies just one junction point between both phase 1 and 2 and between phase 2 and 3. Furthermore, $\theta_{01}$ must be negative. Otherwise, the junction point between phase 2 and 3 would be located before the junction point between phase 1 and 2 on the time line, which is impossible due to the definition of phase 3. The only possible sequence containing all phases is 1, 2, 3. In the case of $\xi(t) = 1$ the right hand side of (20) and (21) reduces to zero. It follows $\theta_{02} = \theta_{01} = 0$, i.e. the shadow price of the emission stock equals zero in phase 1 and 2. This implies that the emission path will be only tangent to the ceiling. Therefore, we neglect the second case in the following.

$\theta < 0$ during phase 1 can be easily explained by its interpretation as the shadow price of the emission stock. An external marginal increase of the stock narrows the problem of the social planner. Therefore, the increase has a negative value, which implies $\theta < 0$ in phase 1. The binding ceiling during phase 2 requires a slightly different approach. If only the ceiling is marginally increased, more exhaustible resources can be used. Therefore, the corresponding shadow price $\mu$ is clearly positive. On the other hand, it is possible to increase both the ceiling and the emission stock, so that the ceiling remains binding. The corresponding shadow price is $\mu + \theta > 0$. Because the amount of exhaustible resources which can be used additionally is smaller in the second case, the shadow price must be smaller. This implies $\theta < 0$ in phase 2.

**Proposition 1** *Both consumption and extraction of the exhaustible resource are continuous. The only sequence containing all three phase begins with phase 1, switches over to phase 2 and ends with phase 3.*

### *3.1. The phases*

In this section we turn to the analysis of the three phases. As the ceiling is never reached in phase 3, it is irrelevant and $\mu = \theta = 0$, which implies the identity of phase 3 with an unconstrained economy described by Tsur and Zemel (2005).[12]Therefore, the analysis starts with phase 3. Phase 1 and 2 follow in

[12] $\mu = 0$ follows directly from (16) and $\theta = 0$ from the irrelevance of the emission stock in phase 3.

numerical order.

### *3.1.1. Phase 3 - the long run*

Since phase 3 is identical with the economy of Tsur and Zemel (2005), the following remarks are limited to the extent that is necessary for understanding. For proofs, as well as for more detailed explanations, we refer to Tsur and Zemel (2005). As mentioned above phase 3 is characterized by $\theta = \mu = 0$. If both resources are in use, (8) and (9) gives

$$F_x(K, x(K, A)) = M'(E(A)) + \frac{\tau}{\lambda} = M_b B(A). \tag{22}$$

(22) determines $b(K, A)$, $E(A)$ and $x(K, A)$. The optimal R&D investments are given by (10). Thus, the optimization problem reduces to the task of identifying optimal consumption and capital accumulation for every point in time. Tsur and Zemel (2005) show that this can be done by means of two characteristic lines. Assuming $S_R = \infty$, the first line is given by the no-arbitrage condition $\frac{\partial Y^n}{\partial A} = \frac{\partial Y^n}{\partial K}$ of R&D and capital accumulation:

$$-M_B B'(A) b_\infty(K, A) = F_K(K, x(K, A)). \tag{23}$$

(23) defines a line in the $A, K$ - space given by $K^S(A)$, which describes all points in the $A, K$ - space on which singular R&D is optimal. Therefore, we refer to this line as the singular line (SiL). Assuming monotony, $K^S(A)$ growths in $A$. Above (below) the SiL $\frac{\partial Y^n}{\partial A} > \frac{\partial Y^n}{\partial K}$ ($\frac{\partial Y^n}{\partial A} < \frac{\partial Y^n}{\partial K}$). Since the second characteristic line is defined by the steady state $\hat{C} = \hat{K} = \hat{A} = 0$, we refer to it as the steady state line (SSL). It is given by the derivations of (7) and (11) with respect to time:

$$F_K(K, x(K, A)) - \rho = 0. \tag{24}$$

(24) determines implicitly the function $K^N(A)$, with $\frac{dK^N}{dA} \geq 0$. Above (below) the SSL consumption declines (increases). If $\lim_{A\to\infty} K^S(A) > \lim_{A\to\infty} K^N(A)$ the economy converges to a steady state. On the other hand, if $\lim_{A\to\infty} K^N(A) > \lim_{A\to\infty} K^S(A)$ the economy may grow for ever.

The SiL has been defined for $S_R = \infty$. Taking the limited resource stock into

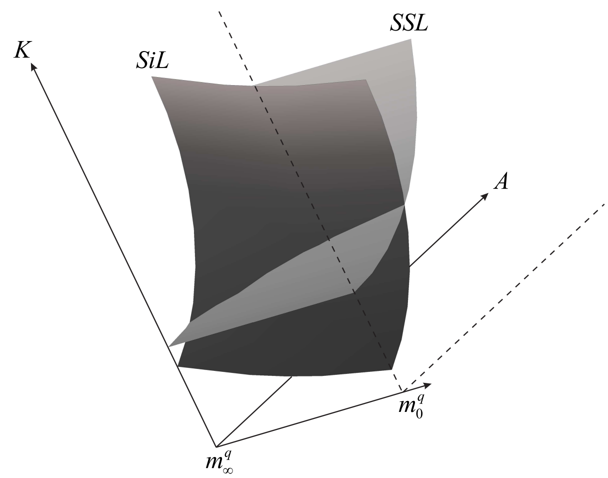


**Figure 2:** SiL and SSL in the $A, K, m^q$ space

account, and therefore the relative scarcity $m^q$, changes the position of the SiL. As can be seen from Fig. 1, the amount of backstop resource utilization increases under ceteris paribus conditions if $m^q$ is higher. Thus, the higher $m^q$ the greater is the left hand side of $-M_B B'(A) b(K, A, m^q) = F_K(K, x(K, A))$, the equation that establishes the SiL. $K$ has to be smaller for all $A$ to guarantee equality, implying a lower position of the SiL in the $A, K$ space. Therefore, the SiL depends as well on $m^q$ or $S_R$, respectively, as $S_R$ declines in time and $m^q$ increases, with $\frac{dK^S}{dm^q} < 0$. The SiL is then a two-dimensional manifold in the three-dimensional $(A, K, m^q)$ or $(A, K, t)$ space, respectively. On the other hand, the SSL is not affected by the scarcity of $R$, which implies $\frac{dK^N}{dm^q} = 0$. The situation is illustrated by Fig. 2. To describe the optimal development path we use the formulation "the economy lies above (on, below) the SiL or SSL, repectively", if the point in the $A, K, m^q$ space describing the economy's $A, K, m^q$ combination is located above (on, below) the SiL (SSL). Assuming maximal one intersection of SiL and SSL, Tsur and Zemel (2005) show that the economy develops according to the following program. Above the SiL the economy approaches the SiL with maximal R&D (i. standard case) or a steady state on the SSL (i. exception), which lies below the SiL in this case, with minimal R&D. Both cases establish a most rapid approach path (MRAP). Once reached, the SiL cannot be departed from, i.e. the economy conducts singular R&D for ever, or switches into a steady state at the intersection of SSL and SiL for $S_R = 0$, if the SiL lies below the SSL for huge $A$. Below the SiL the economy follows an MRAP with minimal R&D investments and approaches either the SiL (ii. standard case) or the SSL (ii. exception) by means of capital accumulation or reduction. Thus, positive R&D investments are only feasible above or on the SiL, while capital can only accumulate on or below the SiL.

**Proposition 2** *(Property 3.1 of Tsur and Zemel (2005)) The optimal path converges either to the SSL or the SiL. In the first case, the economy switches into a steady state. In the latter, capital and technology grow forever.*

The interpretation of $\frac{dK^S}{dm^q} < 0$ is straightforward. Since the scarcity of $R$ implies greater utilization of the backstop, a reduction of its supply costs has a greater effect on net production $Y^n$. On the other hand, the effect of capital accumulation remains unchanged. Thus, R&D becomes feasible for more $A, K$ combinations as scarcity of $R$ increases, while capital accumulation becomes unfeasible for these $A, K$ combinations. This process could be interpreted as an increase of R&D incentives, and will play a major role in the following discussion of the other two phases and economic development over the whole time.

*3.1.2. Phase 1*

Phase 1 is characterized by a non-binding ceiling that becomes binding in the future. Thus, from (16) we get $\mu = 0$. Since the ceiling becomes binding later on, changes of the emission stock are valued by the social planner by $\theta < 0$, as shown before. By using $\theta_1$ to indicate the phase the shadow price belongs to, we get from (8) the variant of phase 1 for (22):

$$F_x(K, x(K,A)) = M'(E(A)) + \frac{\tau - \theta_1}{\lambda} = M_b B(A). \tag{25}$$

The relative scarcity index $m_1^q$ is now given by $\frac{\tau + |\theta_1|}{\lambda}$. Its growth rate is

$$\hat{m}_1^q = F_K + \gamma \frac{|\theta_1|}{\chi}, \text{ with } \chi := \tau + |\theta_1|. \tag{26}$$

For all capital - technology - combinations both the relative scarcity index and its growth rate are higher than in an economy in the same situation but without the ceiling, i.e. with the same $A, K$ combination and the same costate variables but without $\theta_1$. In the following, such a economy will be called unbounded. The effect of the higher relative scarcity index are shown in Fig. 3. As long as both

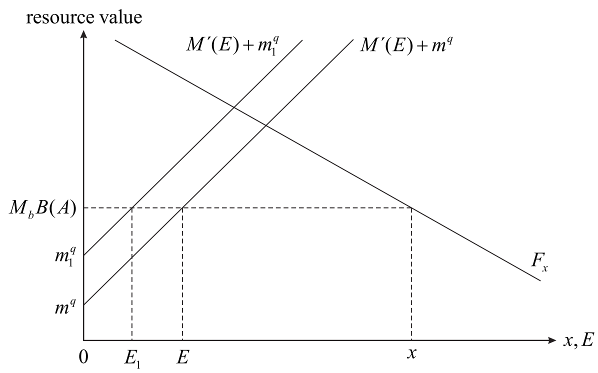


**Figure 3:** Usage of exhaustible resource and backstop in phase 1

resources are used, $x$ remains unchanged. However, a greater scarcity index increases the marginal costs of the exhaustible resource, implying a reduction of its utilization. Since the marginal costs of the backstop are not affected, the gap in energy supply $(E-E_1)$ is closed by an increase of backstop utilization. As shown in chapter 3.1.1, a higher $b$ affects the position of the SiL in the $A, K$ space. With a greater left hand side of $-M_bB'(A)b(K, A, m_1^q) = F_K(K, x(K, A))$, a lower capital stock is needed for all $A$ to guarantee equality. Thus, the SiL of the bounded economy lies below the SiL of an unbounded one. Following the known argumentation, the higher scarcity of $R$ causes a higher utilization of $b$, implying a greater effect on net production of decreasing backstop supply costs. Therefore, R&D instead of capital accumulation becomes feasible for more $A, K$ combinations. Since $m_1^q$ grows steadily in time, the SiL is shifting downwards in the $A, K$ space, i.e. $\frac{dK^S}{dm_1^q} < 0$.

As shown in chapter 3.1.1, the SSL is not affected by the scarcity. Therefore, the artificial scarcity has no effect on the SSL.

The development program is not affected by the ceiling. However, the economy cannot be in phase 1 for ever. If it were, the emission stock converges to the ceiling for $t \to \infty$. This implies $\lim_{t\to\infty} E(t) = \gamma \bar{S}_E$ and therefore the exhaustion of $S_R$ in finite time. But with $S_R = 0$ the emission stock decreases to zero, contradicting $\lim_{t\to\infty} S_E(t) = \bar{S}_E$. Thus, the economy cannot reach the steady state at the intersection of SSL and SiL for $S_R = 0$ during phase 1. Two possibilities for reaching a steady state remain. In the first one, the capital stock is too low to conduct R&D and the economy converges to the SSL by capital reduction. In the other case, the capital stock is high and the economy conducts maximal R&D to switch to minimal R&D and reach the SSL by capital reduction.

**Proposition 3** *During phase 1 the SiL is shifted downward in the $A, K$ space by the prospectively binding ceiling, which increases the scarcity of the exhaustible resource. The relative scarcity index increases monotonously, with $\frac{dK^S}{dm_1^q} < 0$. Compared with a economy with the same capital stock, technology and costate variables, but without the ceiling, the bounded economy exhibits higher and increasing research incentives. The SSL is not affected by the ceiling during phase 1.*

*3.1.3. Phase 2*

During phase 2 the ceiling is binding. (16) implies $\mu_2 > 0$. Since the emission stock is still of relevance, $\theta < 0$ remains. To indicate the phase we use the notation $\theta_2$ and $\mu_2$. (8) can be rewritten to form the variant of phase 2 for (22):

$$F_x(K, x(K, A)) = M'(\bar{E}) + \frac{\tau - \theta_2 + \mu_2}{\lambda} = M_b B(A). \tag{27}$$

The relative scarcity index is given by $m_2^q := \frac{\tau - \theta_2 + \mu_2}{\lambda}$. The economy cannot stay in phase 2 forever, since $E(t) = \gamma \bar{S}_E$ implying the exhaustion of $S_R$ in finite time and therefore the violation of $S_E(t) = \bar{S}_E$, $\forall t > t_1$, with $t_1$ being the point in time the ceiling becomes binding. Thus, the economy has to switch from phase 2 to phase 3 at $t = t_2$. It was shown that $\mu_2(t) - \theta_2(t) \geq 0,\ t \in [t_1, t_2[$ holds. Therefore, the relative scarcity index $m_2^q$ is higher than in an unbounded economy. The growth rate $\hat{m}_2^q$ is unknown, because no exact information is given about $\hat{\mu}_2$ in (16):

$$\hat{m}_2^q = F_K + \gamma - \frac{\mu_2}{\psi}[\rho - \hat{\mu}_2] - \gamma \frac{\tau}{\psi}, \text{ with } \psi := \tau - \theta_2 + \mu_2. \tag{28}$$

However, the binding ceiling implies $E(t) = \bar{E},\ t \in [t_1, t_2[$. Figure 4 shows how the binding ceiling restricts the possible changes of $m_2^q$. As long as both

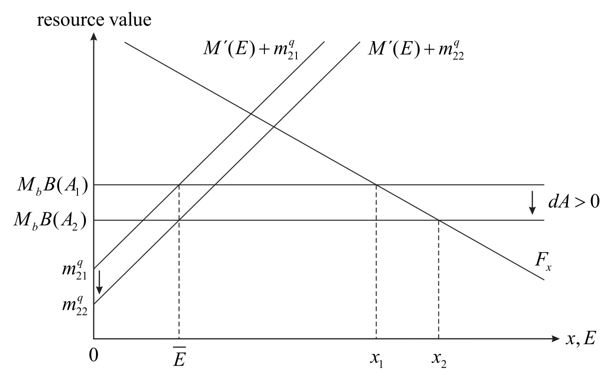


**Figure 4:** Usage of exhaustible resource and backstop in phase 2

resources are used, an increase of $m_2^q$ implies a decrease of $R$ and is therefore not possible. $m_2^q$ remains constant if the supply costs of the backstop are constant, i.e. if no R&D is conducted. If R&D investments are either maximal or singular, $A$ increases ($A_1 \to A_2$) and the backstop supply costs decline. To keep the utilization of the exhaustible resource constant $m_2^q$ must decline, too ($m_{21}^q \to m_{22}^q$). Thus, the binding ceiling establishes a link between R&D investments and the relative scarcity index during phase 2. Together with $\mu_2(t) - \theta_2(t) \geq 0,\ t \in [t_1, t_2[$ the consequences for the SiL and SSL are as follows. The SSL is not affected, because it is independent of the scarcity of the exhaustible resource. Since the relative scarcity index is higher than in an unbounded economy, backstop utilization is higher. As in phase 1 this implies a lower position of the SiL in the $A, K$ space. The development of the SiL depends on $m_2^q$. As discussed in chapter 3.1.1 a higher (lower) relative scarcity index increases (decreases) backstop utilization under ceteris paribus conditions, implying a higher (lower) left hand side of $-M_b B'(A) b(K, A, m_2^q) = F_K(K, x(K, A))$ and therefore a lower (higher) capital stock, which guarantees equality, i.e. $\frac{dK^s}{dm_2^q} < 0$. Thus, as long as R&D is minimal and the relative scarcity index constant, the position of the SiL in the $A, K$ space is unchanged. With $I > 0$ and a decreasing relative scarcity the SiL shifts upwards in the $A, K$ space. Since a development path needs to be located on or above the SiL to allow R&D investments, R&D reduces/increases the number of $A, K$ combinations at which R&D/capital accumulation can be optimal. The reason will be discussed as a part of the following chapter.

As in phase 1 the development program is not affected by the ceiling, just as the steady state at the intersection of SSL and SiL for $S_R = 0$ is ruled out by the fact that the economy cannot be in phase 2 for ever. Therefore, only the two possibilities mentioned in chapter 3.1.2 are left to reach a steady state during phase 2.

**Proposition 4** *Due to the constant resource input $\bar{R}$, the ceiling establishes a link between research activities and the development of the SiL during phase 2. The SiL remains unchanged if R&D investments are minimal and shifts upwards in the $A, K$ space if R&D investments are either singular or maximal, reducing the number of economy states with feasibility of research. Compared with a economy with the same capital stock, technology and costate variables but without the ceiling, the bounded economy exhibits higher research incentives. The SSL is not affected by the ceiling during phase 2.*

### *3.2. Optimal development*

To analyze the development over the whole planning period $[0, \infty]$ it is necessary to join the analysis of the three phases. For this purpose we use the relative scarcity indices $m_1^q$, $m_2^q$ and $m^q$. Furthermore, the smooth transition from one phase to the next must be noticed. Therefore, the SiL and SSL of the three phases can be attached to each other. Since the SSL is independent of scarcity and time $t$, respectively, it is qualitatively identical to the one described in chapter 3.1.1. The SiL shifts steadily downwards in the $A, K$ space during phase 1 and phase 3. During phase 2 it can shift upward as well as keep unchanged. Since the upward shift requires singular or maximal R&D investments, there are two more, mixed possibilities. The first one appears if the development path approaches the SiL from below, i.e. the economy is accumulating capital to realize the research option. In this case, the SiL remains unchanged until the path reaches it and then shifts upwards afterwards. In the second case, maximal R&D investments are reduced to minimal investments before the SiL is reached, i.e. the economy converges to a steady state. Then the SiL shifts upwards at the beginning of the phase and remains unchanged at the end. Figure 5 illustrates the two border cases of a completely unchanged and a steadily upward shifting SiL in the $A, K, t$ space. The development of the SiL is closely related to the

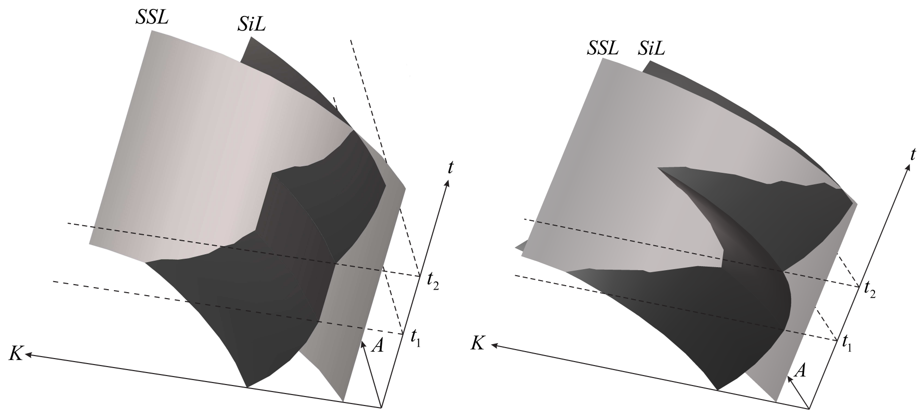


**Figure 5:** SSL and SiL with no or maximal R&D in Phase 2

relative scarcity indices, since a higher index corresponds with a lower SiL, and therefore with a higher research incentive. Figure 6 shows how the development of the index in time must look like to generate the both SiL variants of Fig. 5. $T_R$ denotes the point in time at which the resource stock $S_R$ is exhausted. As stated in chapter 3.1.2, the relative scarcity index of phase 1 ($m_1^q$) both lies above and grows faster than its equivalent of an economy that ignores the ceiling ($m^q$), implying higher and faster growing R&D incentives. The driving force behind the increase of $m^q = \frac{\tau}{\lambda}$ is the exhaustibility of $R$, which is referred to below as the natural scarcity. In phase 1, the relative scarcity index also entails the term $\frac{|\theta_1|}{\lambda} > 0$, which represents the prospectively binding ceiling. Therefore, the ceiling enhances the scarcity of $R$ during phase 1 by adding an artificial scarcity to the natural one. As shown above, a higher relative scarcity index implies higher R&D incentives, which corresponds with a decline in the number of $A, K$ combinations with feasibility of capital accumulation. The additional R&D incentives increase during phase 1, because the artificial scarcity $\frac{|\theta_1|}{\lambda}$ grows at the rate $\widehat{\frac{|\theta_1|}{\lambda}} = \gamma + F_K > 0$, explaining $\hat{m_1^q} > \hat{m^q}$. In Fig. 6, the artificial scarcity equals the gap between $m^q$ and $m_1^q$. The gap also indicates the amount of $R$ which would be used, if the ceiling were ignored. During phase

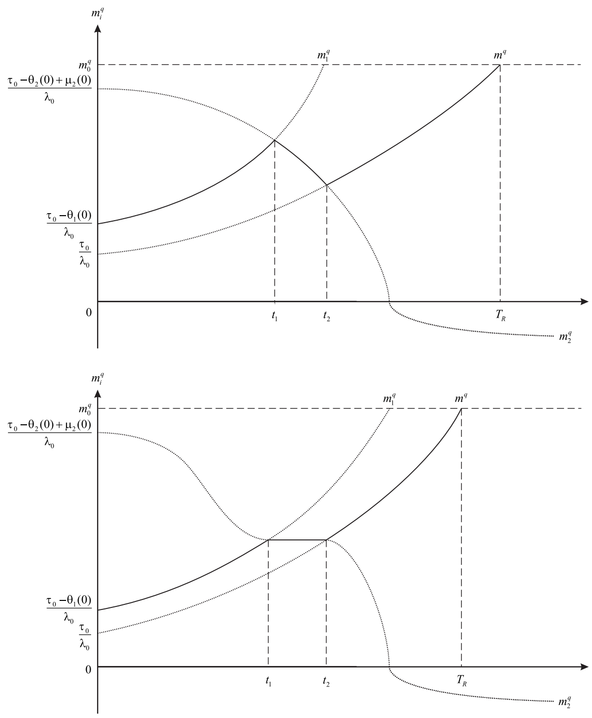


**Figure 6:** Relative scarcity index

2 the artificial scarcity is reduced passively and possibly actively. In the first case, R&D investments are minimal and the relative scarcity index $m_2^q$ remains constant. Since the utilization of the exhaustible resource is constant, the natural scarcity increases, reducing the gap between $m_2^q$ and $m^q$, i.e. the artificial scarcity. In the second case, R&D investments are either singular or maximal. Therefore, utilization of the backstop increases, implying a decreasing relation of $R$ to total production $Y$. Since the exhaustible resource is less important, its relative scarcity declines. Hence, singular or maximal R&D reduces the artificial scarcity, actively establishing a second driving force in addition to the passive reduction of artificial scarcity. As backstop utilization declines ceteris paribus with a lower scarcity index, the incentives for R&D (capital accumulation) decrease (increase). To switch over to phase 3, the artificial scarcity must be eliminated completely. The phase itself is equivalent to an economy without

a ceiling.

Thus, the ceiling causes an increase in the scarcity of the exhaustible resource and thereby a reduction in its usage during phase 1 and 2. Since the ceiling would be violated without the additional or artificial scarcity, the result is quite intuitive. The artificial scarcity increases as the pollution concentration approaches the ceiling, indicating a smaller growth rate of $R$. At the ceiling, the artificial scarcity decreases due to the declining resource stock $S_R$ and possibly increasing utilization of the backstop. The scarcity induced reduction of $R$ causes an increase in backstop utilization. This connotes that the reduction of backstop unit costs has a stronger effect on net income $Y^n$. Therefore, more possible $A, K$ states exist at which the effect is stronger than that of capital accumulation, implying stronger R&D incentives and weaker capital accumulation incentives.

The analysis of Chakravorty et al. (2006a) is based essentially on the scarcity of the exhaustible resource, which translates directly into the price of the resource in a Hotelling model. Not every feature related to abatement activities appears here, since abatement and other possible extensions are ignored. However, Chakravorty et al. (2006a) needed the assumption of decreasing global energy demand to explain a decreasing scarcity of the exhaustible resource at the ceiling. In the current model a rising (declining) capital stock increases (decreases) energy demand.[13]However, the backstop absorbs all changes in total energy demand caused by a variation of the capital stock at the ceiling.[14]A decreasing scarcity is caused here by R&D, which increases the utilization of the backstop. Consequently, the importance of the exhaustible resource for total production declines, i.e. its share of total energy output is shrinking.[15]The opposite effect of increasing scarcity at the ceiling cannot be explained, since there is no possibility of increasing the (potential) importance of the exhaustible

[13]See Fig. 1.

[14]If $b = 0$, a decreasing (increasing) capital stock and therefore a lower (higher) energy demand implies a lower (higher) scarcity to ensure $\bar{E}$.

[15]Chakravorty et al. (2011) get a similar result due to a learning - by - doing effect which decreases the costs of the backstop.

resource. For this purpose, either depreciation with respect to technology or a second technology that is related to the exhaustible resource must be taken into account.

The development of the relative scarcity index together with Fig. 1 allows a qualitative statement about the extraction path of the exhaustible resource.

**Proposition 5** *During phase 1 (3) $R$ decreases monotonically due to the increasing $m_1^q$ ($m^q$) and the constant or decreasing unit costs of the backstop resource. On the other hand, phase 2 is characterized by constant utilization of the exhaustible resource.*

Figure 7 illustrates a corresponding path. The path denoted with $E_b$ shows

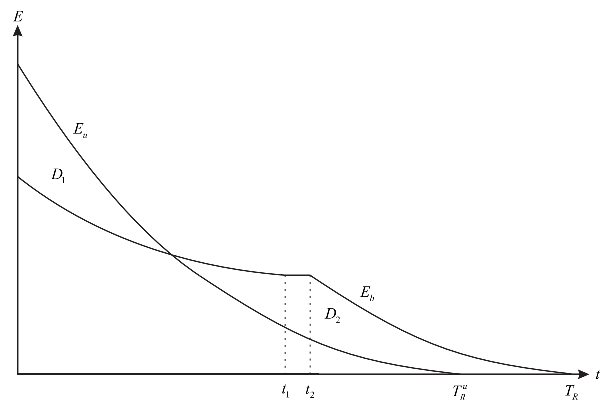


**Figure 7:** extraction path of the exhaustible resource

how the resource is extracted in the bounded economy. The other path, $E_u$, illustrates the path of an economy that is identical with the bounded one at $t = 0$ but without the ceiling. Since the complete resource stock $S_R$ must be used, the area under $E_b$, and under $E_u$, equals $S_R$. Thus, the area marked with $D_1$ represents the amount of exhaustible resource that is not used at early points of time in the bounded economy. Therefore, this amount must be used later on. The corresponding area is denoted by $D_2$. Because the areas under both paths equal $S_R$, $D_1 = D_2$ must hold. Note that $E_b$ must not lie below $E_U$ at $t = 0$. Since both economies are identical at the starting time, this only happens if the relative scarcity index $m_1^q(0) = \frac{\tau(0)+|\theta_1(0)|}{\lambda(0)}$ is greater than its equivalent of the economy without the ceiling $m_u^q(0) = \frac{\tau_u(0)}{\lambda_u(0)}$. Even if $\tau_u(0) = \tau(0)$ holds, there are two possible effects left. On the one hand $|\theta_1(0)| > 0$ increases the numerator of $m_1^q(0)$, indicating less usage of the exhaustible resource. On the other hand $\lambda(0)$ can be greater than $\lambda_u(0)$. Due to (7) this implies lower consumption in the bounded economy. If the reduction of consumption is large enough, the second effect outweighs the first one and $E_b(0) > E_u(0)$ holds. In this case, the economy uses more of the exhaustible resource and consumes less. Both imply a higher net income $Y^n$, which can be used for either capital accumulation or research. Thus, the economy tends to adjust to the ceiling with strong measures than by gaining time to implement the necessary measures. The latter happens rather if $E_b(0) < E_u(0)$, as shown in Fig. 7. This solution is the one intuition suggests, since it would be expected that a ceiling on the stock of pollution, as an environmentally friendly measure, should decrease utilization of the polluting resource. Therefore, if $E_U(0) < E_b(0)$ holds, we have a kind of a green paradox.[16]However, in contrast to Sinn (2008a) and Sinn (2008b) the greater usage of the polluting resource is part of an optimal path for the entire economy and does not violate the environmental protection measure, i.e. the ceiling. If the intuitive solution holds, the economy will use more backstop, which implies a lower net income. But the lower resource utilization $R$ extends the time period until the ceiling binds.

It is important to notice that both economies will exhibit the same SiL (and SSL) after the resource stock $S_R$ is exhausted, because all the energy is then generated by the backstop. Therefore, (23) allocates all capital stock to the same technology level whether or not the economy was restricted by a ceiling. This implies for the two standard cases that the position of the long run development path is not affected by the ceiling. However, the position of a bounded economy on the path may be different from that of an economy without

[16]The concept of the green paradox was introduced by Sinn (2008a) and Sinn (2008b) to describe a situation where a tighter environmental policy on the demand side of the economy induces a higher supply of polluting goods, therefore harming the environment instead of protecting it. The idea can be applied here in a more general sense.

the ceiling at some specific point in time. In case of exception ii., the economy does not conduct R&D and converges to the SSL. Positive R&D would be only possible if the relative scarcity reaches a level higher than that of an exhausted resource stock ($m_0^q$). However, in this case the exhaustible resource is no longer used. Thus, if the artificial scarcity had increased the relative scarcity index to this level, it would have been impossible to reach or stay at the ceiling, which contradicts the fact that the economy must be in phase 1 or 2 to justify the artificial scarcity. However, the long run development may be affected by the ceiling in case of exception i. This economy follows an MRAP with minimal R&D investments above the SiL. If the MRAP with minimal R&D investments is only a part of the development path, and was preceded by a part with maximal R&D investments, higher (lower) net production in early periods may cause a higher (lower) technology level and capital stock in the steady state.

The development of the emission stock follows directly from Fig. 7. It increases during phase 1, is constant during phase 2 and converges to 0 during phase 3. Figure 8 shows an appropriate path.

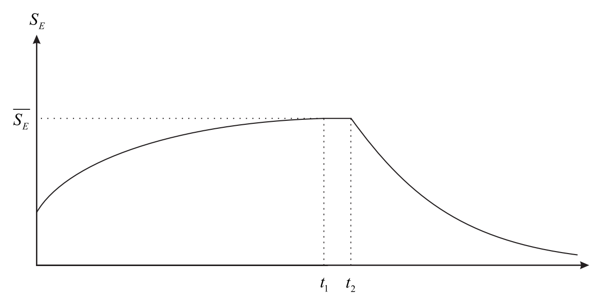


**Figure 8:** emission stock development

## 4. Market Economy

After having analyzed the social optimum we turn to a market economy in this section. This economy consists of a great number of identical individuals and composite good producers as well as two resource owning companies.[17] The individuals own all companies in the $Y$ - and resource sector, as well as the capital stock. They maximize their intertemporal utility with respect to their budget constraint. The companies in the composite good sector rent capital and buy resources to generate energy. Since they do not face an intertemporal problem, they maximize their profit at every point in time. The two resource owners sell the resources and, in the case of the backstop resource owning company, conduct research. Therefore, they maximize their intertemporal profit with respect to either the resource stock or the technology. We assume a Cournot competition on the resource market and perfect competition on all other markets. The government has the possibility of taxing the exhaustible resource, with $\phi$ denoting the corresponding quantity tax rate. Additionally, both resources can be subsidized with $s_R$ or $s_b$, respectively. To balance the budget, the government can levy a lump-sum tax or grant a lump-sum transfer, both denoted with $T \lesseqgtr 0$.

Given the described market structure, a representative individual faces the following intertemporal optimization problem:

$$\max_{C} \int_0^\infty \left[U(C)e^{-\rho t}dt\right],$$

$$\text{subject to } \dot{K} = \frac{r}{p_Y}K + \frac{\pi}{p_Y} + \frac{\pi_b}{p_Y} + \frac{\pi_R}{p_Y} + \frac{T}{p_Y} - C. \tag{29}$$

The interest rate and the price of the composite good are represented by $r$ and $p_Y$ respectively. $\pi$, $\pi_b$ and $\pi_R$ denote the profits of the composite good producers and the two resource owners. From the necessary conditions for an optimum

$$\lambda_H = U'(C), \tag{30}$$

$$\hat{\lambda}_H = \rho - \frac{r}{p_Y} \tag{31}$$

[17] The assumption of two resource owners and a Cournot competition is made in order to have some degree of private research expenditure. Otherwise, R&D would be totally government driven, due to the linear backstop production function.

we get[18]

$$\hat{C} = \frac{\frac{r}{p_Y} - \rho}{\eta}. \tag{32}$$

$\lambda_H$ represents the costate variable associated with the capital stock and $\eta$ the (positive) elasticity of $U'(C)$. The Ramsey rule (32) states that consumption will increase as long as the real interest rate is greater than the time preference rate. According to (30), the marginal utility equals $\lambda_H$, i.e. the price the individual would pay for an increase of his capital stock. For an equilibrium on the composite good market $\lambda_H = p_Y$ is necessary. Otherwise, the individual would buy more (less) $Y$ for investing in the capital stock and consumption, if $\lambda_H < p_Y$ ($\lambda_H > p_Y$).

As mentioned above, the composite good producers do not have to solve an intertemporal optimization problem. Instead they maximize their profit at every point in time. Omitting a firm index and with $p_b$ and $p_R$ denoting the resource prices, the representative producer's profit is given by

$$\pi = p_Y F(K, b + R) - rK - p_b b - (p_R + \phi)R. \tag{33}$$

The first order conditions $\frac{\partial F}{\partial K} = \frac{r}{p_Y}$ and $\frac{\partial F}{\partial x} = \frac{p_b}{p_Y} = \frac{p_R + \phi}{p_Y}$ state that the marginal product of any input has to equal its real price. Since the composite good producers have no market power, these conditions hold, if the capital and resource market are cleared. By substituting $\frac{r}{p_Y}$ into (32) we get the socially optimal Ramsey rule (15).

The resource owners know the profit maximization problem of the composite good producers and therefore the price-demand functions for both resources:[19]

$$p_Y F_x(K, b + R) = p_b = p_R + \phi. \tag{34}$$

[18]The current value Lagrangian as well as the necessary conditions are presented in Appendix AppendixA.1

[19]For the resource owners the capital stock is a known but exogenous factor.

The profits of the resource owners for each point in time are then given by

$$\pi_R = [p_Y F_x(K, b+R) - \phi]R - p_Y M(R) + s_R R, \tag{35}$$

$$\pi_b = p_Y F_x(K, b+R)b - M_b B(A)b + s_b b - p_Y I. \tag{36}$$

The owner of the exhaustible resource maximizes its discounted flow of profits with respect to $S_R \geq 0$ and $\dot{S}_R = -R \leq 0$. Appendix AppendixA.2 shows the current value Lagrangian as well as the derivation of the first order conditions. As long as the exhaustible resource is used, the first order condition with respect to $R$ can be written as

$$F_x(K, b+R) = M'(R) - Fxx(K, b+R)R + \frac{\phi}{p_Y} - \frac{s_R}{p_Y} + \frac{\tau_M}{p_Y}. \tag{37}$$

$\tau_M$ denotes the costate variable of the resource stock, which grows with the constant rate $\rho$ and is therefore determinate by its initial value $\tau_{0M}$. Since the capital stock, the tax, the subsidy and the price $p_Y$ are exogenous to both resource owner and $\tau_M$ determinate by $\tau_{0M}$, equation (37) defines implicitly the optimal resource supply $R^*$ subject to the amount of supplied backstop. Thus, the reaction function is given by $R^* = R^*(b)$.
Using the same approach for the backstop owner we get for $b > 0$:[20]

$$F_x(K, b+R) = M_b B(A) - F_{xx}(K, b+R)b - \frac{s_b}{p_Y}. \tag{38}$$

(38) defines implicitly the optimal backstop supply subject to $R$ and the technology level $A$, which increases with the resource owner's R&D expenditures. We get $b^* = b^*(R, A)$. The optimal R&D expenditures are given by the maximization of the Lagrangian with respect to $I$, with $\kappa_M$ denoting the costate variable of technology:

$$\begin{aligned} I^* &= 0, \text{ if } -p_Y + \kappa_M < 0, \\ 0 \leq I^* &\leq \bar{I}, \text{ if } -p_Y + \kappa_M = 0, \\ I^* &= \bar{I}, \text{ if } -p_Y + \kappa_M > 0. \end{aligned} \tag{39}$$

[20]The Lagrangian and the derivation of the first order conditions can be found in Appendix AppendixA.2.

$\kappa_M$ evolves according to

$$\hat{\kappa}_M = \rho + \frac{p_Y}{\kappa_M} M_b B'(A) b. \tag{40}$$

By substituting $R^*(b)$ and $b^*(R, A)$ in (37) and (38) respectively, the Nash - Cournot equilibrium is implicitly given:

$$\begin{aligned} F_x(K, b^*(R^*, A) + R^*) =& M'(R^*) - F_{xx}(K, b^*(R^*, A) + R^*)R^* + \frac{\phi}{p_Y} - \\ & \frac{s_R}{p_Y} + \frac{\tau_M}{p_Y}, \qquad (41) \\ F_x(K, b^* + R^*(b^*)) =& M_b B(A) - F_{xx}(K, b^* + R^*(b^*))b^* - \frac{s_b}{p_Y}. \qquad (42) \end{aligned}$$

Table 1 summarizes and compares the results of the market economy with the socially optimal solution. The Ramsey rule of the social optimum is identical with that of market equilibrium. The same holds for the capital accumulation equations. To reveal this, we substitute (33), (35), (36) and the government's budget constraint $T = \phi R - s_b b - s_R R$ into $\dot{K} = \frac{r}{p_Y} K + \frac{\pi}{p_Y} + \frac{\pi_b}{p_Y} + \frac{\pi_R}{p_Y} + \frac{T}{p_Y} - C$. For the further analysis we assume that the social planner values capital, the resource stock $S_R$, and knowledge $A$ in the same way as the subjects of the economy, which implies $\lambda = \lambda_H = p_Y$, $\tau = \tau_M$ and $\kappa = \kappa_M$. In this case the equations related to R&D are identical. Since the ceiling has no effect on the economy in phase 3, the optimal subsidies will be calculated by comparing the marginal products of $R$ and $b$ respectively. The irrelevance of the ceiling suggests $\phi = 0$. We then get for the subsidies

$$s_b = -p_Y F_{xx} b > 0, \tag{43}$$

$$s_R = -p_Y F_{xx} R > 0. \tag{44}$$

Using (44) we get $\phi_1 = -\theta$ for the optimal tax during phase 1, and $\phi_2 = \mu - \theta$ during phase 2. Using the index $i = 1, 2$ as in section 2 the tax $\phi(t)$ is given by

$$\phi(t) = \begin{cases} |\theta_1(t)|, & \text{if } t \in [0, t_1), \\ \mu_2(t) - \theta_2(t), & \text{if } t \in [t_1, t_2), \\ 0, & \text{if } t \geq t_2. \end{cases} \tag{45}$$

| | Phase One | Phase Two | Phase Three | Market Equilibrium |
|---|---|---|---|---|
| Ramsey - rule | $\hat{C} = \frac{F_K - \rho}{\eta}$ | | | $\hat{C} = \frac{F_K - \rho}{\eta}$ |
| marginal product of $R$ | $F_x = M'(R) + \frac{\tau - \theta}{\lambda}$ | $F_x = M'(R) + \frac{\tau - \theta + \mu}{\lambda}$ | $F_x = M'(R) + \frac{\tau}{\lambda}$ | $F_x = M'(R) - F_{xx} R + \frac{\phi}{p_Y} - \frac{s_R}{p_Y} + \frac{\tau_M}{p_Y}$ |
| marginal product of $b$ | $F_x = M_b B(A)$ | | | $F_x = M_b B(A) - F_{xx} b - \frac{s_b}{p_Y}$ |
| capital accumulation | $\dot{K} = F(K, x) - M_b B(A) b - M(R) - I - C$ | | | $\dot{K} = \frac{r}{p_Y} K + \frac{\pi}{p_Y} + \frac{\pi_b}{p_Y} + \frac{\pi_R}{p_Y} + \frac{T}{p_Y} - C$ |
| R&D | $\hat{\kappa} = \rho + \frac{\lambda}{\kappa} M_b B'(A) b$<br>$I^* = 0, \text{ if } -\lambda + \kappa < 0$<br>$0 \le I^* \le \bar{I}, \text{ if } -\lambda + \kappa = 0$<br>$I^* = \bar{I}, \text{ if } -\lambda + \kappa > 0$ | | | $\hat{\kappa}_M = \rho + \frac{p_Y}{\kappa_M} M_b B'(A) b$<br>$I^* = 0, \text{ if } -p_Y + \kappa_M < 0$<br>$0 \le I^* \le \bar{I}, \text{ if } -p_Y + \kappa_M = 0$<br>$I^* = \bar{I}, \text{ if } -p_Y + \kappa_M > 0$ |

**Table 1:** Comparison of the market equilibrium and the socially optimal solution

**Proposition 6** *The market equilibrium replicates the social optimum, if* $\lambda = \lambda_H$, $\tau = \tau_M$ *and* $\kappa = \kappa_M$ *holds, the usage of both resources is subsidized according to* (43) *and* (44), *and the exhaustible resource is taxed according to* (45).

Since $\theta_1(t) = \theta_{01}e^{(\rho+\gamma)t}$, the tax increases during phase 1 at the rate $\rho + \gamma$, reflecting the increasing emission stock, and therefore the tightening ceiling. In other words, as the emission stock increases, the amount of possible new emissions decreases, implying a higher tax. During phase 2, this amount is fixed at $\bar{E} = \gamma \bar{S}_E$. However, this does not imply a constant tax but, (in the long run) a decreasing one, because the natural scarcity of the exhaustible resource increases. Thus, the tax at the second junction point at $t = t_2$ equals $\phi(t_2) = \mu_2(t_2) - \theta_2(t_2) = 0$. Using the equations for $\theta_2(t)$ and $\mu_2(t)$ the tax during phase 2 is given by

$$\phi_2(t) = \mu_{02}e^{(\rho+\gamma)t}\left[T_f(t) - \frac{\theta_{02}}{\mu_{02}}\right]. \tag{46}$$

Since the growth rate of $T_f(t)$ is not known exactly, $\phi_2(t)$ may increase as well as decrease. Nevertheless, Fig. 9 shows that the tax can only increase, if the composite good price $p_Y$ growths fast, i.e. if the inflation is sufficiently high. In this case the growth rate of the natural scarcity $\frac{\tau}{p_Y}$ is small, while the

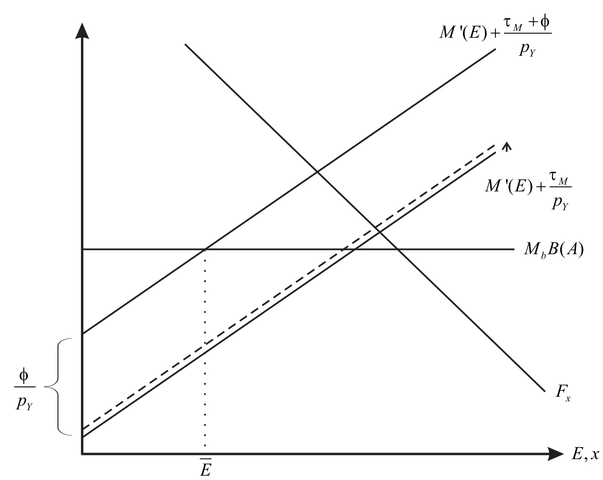


**Figure 9:** Condition for an increasing tax during phase 2

denominator of $\frac{\phi}{p_Y}$ increases. To comply with $E(t) = \bar{E}$ the tax must increase.

Since $\hat{p}_Y = \rho - F_K$, a high capital stock is necessary for a sufficiently high inflation rate. Due to the Ramsey rule (32), a low marginal product of capital implies decreasing consumption. By rewriting the growth rate of $\phi_2$ we can show that consumption must indeed decline if the tax increases. Fur this purpose we write $\hat{\phi}_2$ as

$$\hat{\phi}_2 = \rho + \gamma - \frac{\mu_2}{\phi_2}[\rho - \hat{\mu}_2]. \tag{47}$$

The tax increases if, and only if, $F_K + \gamma - \frac{\mu_2}{\psi}[\rho - \hat{\mu}_2] - \gamma\frac{\tau}{\psi} > F_K - \rho\frac{\phi_2}{\psi}$ holds. The left hand side equals $\hat{m}_2^q$, which can only decrease or stay unchanged. Therefore, $0 > F_K - \rho\frac{\phi_2}{\psi}$ and because of $\phi_2 < \psi$ it follows $F_K - \rho < 0$. According to the Ramsey rule (32) $F_K - \rho < 0$ implies decreasing consumption, which is only possible if the capital - technology - combination describing the status of the economy lies above the SSL.

**Proposition 7** *The optimal tax increases during phase 1 due to the increasing emission stock. Since the natural scarcity increases monotonically, the tax decreases to zero during phase 2. If inflation is sufficiently high, and consumption declines, the tax increases in the short run.*

## 5. Conclusion

This paper analyzes the effects on R&D and the capital stock of a ceiling on the pollution stock. For this purpose we augment the endogenous growth model of Tsur and Zemel (2005) with a polluting resource and a ceiling on the stock of pollution as known from e.g. Chakravorty et al. (2006a). We show that the ceiling mainly affects the short run development of the economy by imposing an artificial scarcity on the exhaustible resource. Since the costs of the backstop can be reduced by R&D, the artificial scarcity increases the advantageousness of R&D, such that R&D is optimal at more possible states of the economy. Inversely, the number of possible states allowing capital accumulation declines. As long as the economy belongs to one of the two standard cases, which both require singular R&D investments, the long run development is hardly affected by the ceiling. In other words, the development path remains unchanged, whereas the position of the economy on the path to one specific moment in time may be altered. If R&D is omitted, the ceiling has no effect. The ceiling may alter the steady state of the economy if and only if maximal R&D is canceled in favor of minimal R&D.

As in Chakravorty et al. (2006a), Chakravorty et al. (2006b), Chakravorty et al. (2008), Chakravorty et al. (2011) and Lafforgue et al. (2008), we are able to distinguish three time phases. Phases 1 and 3 are characterized by a non-binding ceiling. However, in phase 1 it will become binding later on, while it will never be binding in phase 3. During phase 2, the pollution stock is at the ceiling. Analogous to Chakravorty et al. (2006a), the only sequence containing all three phases starts with a non-binding ceiling that will bind later on to become non-binding afterwards. In contrast to the Hotelling models we can explain changes of total energy demand endogenously, since capital is taken into account. Similar to Chakravorty et al. (2011) a declining resource scarcity at the ceiling is caused by an increasing technology level. However, the necessary R&D is an explicit decision and R&D can be abandoned, while Chakravorty et al. (2011) assumes a cost reducing learning-by-doing effect. In both cases, the importance of the exhaustible resource for production vanishes as the utilization of the backstop is intensified.

The optimal resource extraction path is affected by the ceiling, since it exhibits a plateau during phase 2. While intuition suggests a reduction of resource utilization at the starting time to delay the moment the ceiling becomes binding, the results also permit some kind of green paradox. If the natural scarcity of the exhaustible resource is not affected by the ceiling, both resource utilization and non-consumed income are higher in this case, implying greater investments in the capital stock and/or research to adjust to the ceiling.

We show that the social optimum is implemented by a market economy, if the government subsidizes both resources to counter market power effects resulting from the Cournot competition on the resource market. Additionally, the exhaustible resource has to be taxed during phases 1 and 2 to comply with the ceiling. During phase 1, the tax increases monotonically, reflecting the rising emission stock and the increasing artificial scarcity. During phase 2, the emission stock remains unchanged, while the natural scarcity of the resource increases, resulting in the tax being abolished at the end of phase 2. If inflation is sufficiently high, the tax can increases in phase 2 temporary. In this case, the capital stock must be high, and consumption decreases. It is noteworthy that the model does not support subsidies for the backstop that are granted for pollution control reasons.

To keep the model as simple as possible we have omitted several augmentations of the Hotelling models, such as abatement activities or differently polluting exhaustible resources. These are left for further research as are a second technology, or depreciations of technology that may explain an increasing scarcity of the exhaustible resource at the ceiling.


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## AppendixA. Appendix

### *AppendixA.1. Individual*

The current value Hamiltonian of the representative individual is:

$$H = U(C) + \lambda_H \left[ \frac{r}{p_Y} K + \frac{\pi}{p_Y} + \frac{\pi_b}{p_Y} + \frac{\pi_R}{p_Y} + \frac{T}{p_Y} - C \right] \tag{A.1}$$

The first order conditions and the transversality condition are given by:

$$\frac{\partial H}{\partial C} = U'(C) - \lambda_H = 0 \tag{A.2}$$

$$\frac{\partial H}{\partial K} = \lambda_H \frac{r}{p_Y} = \rho \lambda_H - \dot{\lambda}_H \tag{A.3}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda_H(t) [K(t) - K^*(t)] \geq 0 \tag{A.4}$$

### *AppendixA.2. Resource Owners*

The current value Hamiltonian of the firm owning the exhaustible resource is:

$$H = [p_Y F_x(K, b + R) - \phi] R - p_Y M(R) + s_R R - \tau_M R \tag{A.5}$$

The first order condition as well as the transversality condition are given by:

$$\frac{\partial H}{\partial R} = p_Y F_{xx}(K, x) R + p_Y F_x(K, x) - \phi - p_Y M'(R) + s_R - \tau_M = 0 \tag{A.6}$$

$$\frac{\partial H}{\partial S_R} = 0 = \rho \tau_M - \dot{\tau}_M \tag{A.7}$$

$$\tau_M(T_R) = \gamma_{S_R} \tag{A.8}$$

$$\gamma_{S_R} \geq 0, \ \ \gamma_{S_R} S_R(T_R) = 0 \tag{A.9}$$

$$H(T_R) = \begin{cases} \leq 0, \ \text{ if } T_R = 0 \\ = 0, \ \text{ if } 0 < T_R < \infty \\ \geq 0, \ \text{ if } T_R = \infty \end{cases} \tag{A.10}$$

$T_R$ denotes the point in time the resource stock $S_R$ becomes exhausted.

The current value Hamiltonian of the firm owning the backstop is:

$$H = p_Y F_x(K, b + R) b - p_Y M_b B(A) b + s_b b - p_Y I + \kappa_M I \tag{A.11}$$

The first order condition as well as the transversality condition are given by:

$$\frac{\partial H}{\partial b} = p_Y F_{xx}(K,x)b + p_Y F_x(K,x) - p_Y M_b B(A) + s_b = 0 \tag{A.12}$$

$$\frac{\partial H}{\partial A} = -p_Y M_b B'(A) b = \rho \kappa_M - \dot{\kappa}_M \tag{A.13}$$

$$\lim_{t\to\infty} e^{-\rho t} \kappa_M(t)[A(t) - A^*(t)] \geq 0 \tag{A.14}$$

The maximization of the Hamiltonian with respect to $I$ gives:

$$I^* = 0, \text{ if } -p_Y + \kappa_M < 0 \tag{A.15}$$

$$0 \leq I^* \leq \bar{I}, \text{ if } -p_Y + \kappa_M = 0 \tag{A.16}$$

$$I^* = \bar{I}, \text{ if } -p_Y + \kappa_M > 0 \tag{A.17}$$

The index $^*$ marks the optimal value of the variable in question.