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Economic Growth, a Golden Rule of Thumb, and Learning by Doing

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Abstract

A simple rule of thumb which has been successfully used in the basic neoclassical growth model as an alternative to the unstable dynamic optimization solution is shown to be more generally applicable in a non-scale growth model with learning by doing. The model is formulated in accordance with empirical regularities about learning by doing and shown to generate the stylized facts about economic growth reasonably well. The external effects of learning by doing can be internalized by the steady state tax cum subsidy policy implied by a decentralized dynamic optimization approach.
Keywords: Non-Scale Growth – Rules of Thumb – Learning by Doing
JEL-Classification: O41; O30; D90

1 Introduction

While Solow (1956) and other early writers on economic growth used a fixed saving rate as a simple rule of thumb describing economic behavior, the dynamic optimization approach is by now the unquestioned standard in the literature. At least since the seminal papers on endogenous growth by Romer (1986, 1990) and Lucas (1988) have been published, dynamic optimization methods, which in the 1960's were primarily used in normative applications of growth theory aiming at deriving conditions for optimal economic growth (cf. e.g. Cass, 1965; Arrow, 1968), are usually employed in *positive* theories of economic growth (cf. e.g. Blanchard and Fischer, 1989; Barro and Sala-i-Martin, 1995, and nearly any of the innumerable recent publications on economic growth). This paper rejects the paradigm of dynamic optimization and shows how a simple and reasonable rule of thumb can be used as an alternative in (semi-) endogenous growth theory.

Starting with Simon (1955, 1957), the hypothesis of unbounded rationality has been criticized by various authors for several good reasons. Moreover, even if agents behavior was fully rational, in macroeconomic equilibrium a positive fraction of these *rational* agents would use a simple rule of thumb (e.g. a fixed saving rate) unless effort costs of using a sophisticated optimum solution are extremely low (cf. Krusell and Smith, 1996).

The question in this respect is, which degree of sophistication is really rational? An important point to add is, as Buhr and Christiaans (2001) have emphasized, that unbounded rationality is incompatible with modern brain research results according to which the functions of large part of the brain are structured in the form of programs which may be interpreted as a set of rules. Thus, rules of thumb should play an important role in modeling human behavior. Even if one does not believe that agents do not try to calculate optimum solutions, real life decisions are made in an uncertain environment; reliance on neoclassical economics therefore requires that adjustment processes exist which lead to the true optimum if an exact calculation is not feasible. Examples of such an approach are the discussion of the Baumol and Quandt (1964) analysis of a rule of thumb for a monopolistic firm in Buhr and Christiaans (2001) and the analysis of the gradient method as a stable resource allocation process in Arrow et al. (1958), where the latter is directly concerned with the determination of an optimum and therefore requires more information than the former, however. Reliance on the implications of neoclassical models presupposes the existence of adjustment processes or rules of thumb guiding the way to the vicinity of the true optimizing solutions.

As I have argued in a companion paper (Christiaans, 2001), there is an important difference between adjustment processes for static and dynamic economic models. While in a static model one only has to look for some kind of method to find the roots of a (system of) equation(s) describing the optimum solution, the optimum in dynamic models is described by the solution of a system of differential equations constituting a boundary value problem which is difficult to approximate by a mathematical method even for simple problems. Part of this difficulty stems from the fact that the optimum paths in dynamic economics are usually saddle paths. The slightest error in determining the correct initial values will take the proposed solution further and further away from the optimum. Clearly, these problems are reinforced if the underlying model at hand is not completely known, that is, if the agents do not know the true structure of the economy. Moreover, it is difficult to imagine a *meta-dynamic* adjustment process which could augment the dynamic model in the same way as the gradient process can be used to augment a static optimization problem, e.g. The drastic implications of the instability along saddle paths are illustrated by means of numerical examples in Christiaans (2001). Thus, even if individuals were well trained in Pontryagin's maximum principle and if one disagreed with the arguments of behavioral economists against unbounded rationality, a *positive* theory of economic growth cannot be reasonably based on such unstable dynamic processes. It therefore appears that neoclassical economists have been too frank in transferring the approach of unbounded rationality familiar from static models of economic behavior to dynamic settings without caring about the stability of adjustment processes.

It is conceivable that the literature on the so-called *(global) asymptotic stability of optimal control* (cf. e.g. Cass and Shell, 1976; Brock and Scheinkman, 1976) has led some economists to regard solutions to optimal control problems as being stable rather than unstable. It is important to recognize that the cited authors call the solutions of dynamic optimization problems *stable* if the optimum solution converges to a steady state along the stable manifold of a saddle point. The equilibrium of the underlying dynamical system, a saddle point, is by no means stable but unstable by definition. This kind of *saddle-path stability* which may be useful in normative applications of optimal control theory, however, provides the basis for the positive interpretation of dynamic optimization growth models mentioned before. It is the author's conviction that it is not possible to make an unstable model stable by redefining the properties of a saddle point.

The general instability of solutions to dynamic economic optimization problems proven by Kurz (1968) concerns their representation in *open loop*. If closed form pure *feedback* solutions providing the optimum values of the control variables as functions of the state variables were available, this problem would cease to exist. Such closed form feedback solutions are difficult if not impossible to obtain even for extremely simplified problems, however. The analysis of reasonable and simple feedback rules of thumb is therefore a promising alternative to dynamic optimization solutions. Such feedback rules may or may not come close to the real optimum. In Christiaans (2001), the following simple rule of thumb for consumption, C, has been proposed:

$$C = wL + \rho K, \tag{1}$$

where w is the wage rate, L population (which equals the labor force), ρ the rate of time preference, and K the capital stock which equals the assets of the households in a closed economy. In per capita terms one gets

$$c = w + \rho k,$$

where c denotes consumption per capita and k the aggregate capital-labor ratio. Equation (1) is a generalization of the classical consumption function taking a positive rate of time preference into account.

According to this rule of thumb, households consume their wage income and a share of their assets equaling their rate of time preference. That is, households consume more than their wage income if they own a positive stock of assets and discount the future. This kind of behavior appears to be intuitively plausible. Moreover, in the special case where ρ equals the interest rate, equation (1) may be interpreted as the optimum solution to a simplified problem of utility maximization with a constant wage and interest rate. It has been shown that (1) performs reasonably well from a normative point of view even if applied to the more general Ramsey-Koopmans-Cass (RKC) problem of economic growth. More precisely, an economy in which households determine their consumption according to equation (1) asymptotically approaches the optimum steady state of the RKC model (the *modified golden rule path*) if there is no technological progress or if the intertemporal elasticity of substitution in consumption is equal to one (Christiaans, 2001).

Even as a positive theory of economic behavior the assumption that the consumption function is given by equation (1) appears to be more plausible than to assume that households solve a dynamical optimization problem with perfect precision. Spending one's wage income plus a share of assets depending on the rate of time preference is a sensible rule of thumb which does not require the solution of any optimization problem nor any knowledge of the true structure of the economy at all. In order to apply the rule, households need only information about the wage rate and their own assets and rate of time preference. This is why I have chosen to call (1) a *golden rule of thumb*. Note that much more information is needed e.g. for calculating the saving rate according to the modified golden rule of accumulation (which also leads to the modified golden rule path). The applicability of the golden rule of thumb in positive economics has been underlined by showing that it is at least as consistent with empirically estimated rates of convergence as the RKC approach.

It is the purpose of the present paper to show that the golden rule of thumb (1) is more generally applicable by using it as a consumption hypothesis in a non-scale growth model of the learning by doing type with endogenous technological change. This model is formulated in accordance with empirical regularities about learning by doing and it turns out to be consistent with the so-called *stylized facts* about economic growth. This result can be interpreted as a further preliminary test of the golden rule of thumb since it shows that it is consistent with stylized empirical data. It is therefore a reasonable working hypothesis about actual economic behavior. The non-scale growth learning by doing approach has also been chosen for similar reasons. There is ample empirical evidence about the importance of learning by doing in production and it will be shown that endogenous non-scale growth is possible with learning elasticities smaller than one. While Lucas (1988) in his model of learning by doing assumed learning elasticities equal to one in order to obtain endogenous growth, it will be shown that this assumption – which is at odds with empirical evidence – is not necessary unless learning by doing is the only engine of growth. It can be shown in a standard model of non-scale growth that in the presence of capital accumulation and growth of the labor force output per capita grows in the long run at a positive rate if the learning elasticity corresponds to the empirically estimated values around 0.32 (the *80%-curve*, cf. e.g. Hirsch, 1956). This result can already be found in Arrow (1962) or in Sheshinski (1967a), who at that time did not emphasize endogenous long-run per capita growth, however. In contrast to these early papers, cumulated production instead of cumulated investment is used as the learning index. This approach is more naturally associated with learning *by doing* and has the advantage of being compatible with a rising saving rate during the transition to the long-run equilibrium if the golden rule of thumb is used as the consumption hypothesis.

While this paper is rather critical about the standard dynamic optimization approach to economic growth, it nevertheless includes some good news for neoclassical growth economics. The proposed rule of thumb often has implications similar to those of the standard optimization model. E.g., it will be shown that the tax cum subsidy policy used to internalize the external effects of learning by doing in the steady state of a decentralized dynamic optimization economy can be used in a rule of thumb economy in order to lead households to the optimum long-run equilibrium. This raises the reliance on the implications of neoclassical growth economics. It should be noted, however, that much empirical work needs to be done in order to decide whether households actually use rules of thumb similar to the one proposed here. As a matter of fact, the number of possible consumption hypotheses is infinite, and of course the most of them will lead to different conclusions than the hypothesis of unbounded rationality.

The organization of the paper is as follows. In section 2 the basic non-scale growth model with learning by doing is introduced. While the model is extremely simplified, its formulation is nevertheless in line with empirical regularities about learning by doing. Various consumption hypotheses (the golden rule of thumb and the centralized and decentralized dynamic optimization solution) are analyzed in section 3, where it is also shown how the external effects of learning by doing can be internalized in the decentralized dynamic optimization economy and in the rule of thumb economy, respectively. It is shown in section 4 that the model generates the stylized facts about economic growth reasonably well. The final section offers concluding remarks and directions for further research. Some of the mathematical derivations are relegated to appendices.

2 The Basic Model

An important problem of the basic neoclassical growth model is its failure to endogenously explain long-run growth of per capita income. Of course, this criticism does also apply to the basic neoclassical model with rule of thumb consumers analyzed in Christiaans (2001). The present paper is devoted to the analysis of the golden rule of thumb in the context of an endogenous non-scale growth model of the learning by doing type which is consistent with most of the so-called *stylized facts* about economic growth.

While the new growth theory – cf. e.g. Romer (1990) and Grossman and Helpman (1991) – succeeded in explaining endogenous per capita growth, Jones (1995) has pointed out a serious problem of these models. His criticism pertains to the *scale effects*, according to which an increase in the size or scale of the economy permanently increases its long-run growth rate. For example, if the number of scientists engaged in R&D in the Romer (1990) model is doubled, the per capita growth rate of output will also double. Since this number of scientists is proportional to the size of the labor force, the growth rate of per capita output is itself proportional to the labor force. This prediction implies that the per capita growth rate itself grows at the rate of population growth and is clearly at odds with empirical evidence. Jones (1995) has shown that a plausible model can be constructed which maintains some of the features of the R&D models but does not involve such scale effects. Such a model, however, does also alter some of the main implications of the new growth theory as regards the possibility of the government to influence the long-run growth rate, which turns out to depend on parameters usually taken to be invariant to government policy. This is a return to Solow-like implications for long-run growth. Nevertheless, such *non-scale* growth models are able to endogenously explain technological progress and are therefore sometimes denoted as *semi-endogenous* growth models. Note that the approach of Sheshinski (1967a) may be considered as an early contribution to this theory; only the emphasis on long-run growth has changed. It should also be noted that the term *non-scale growth* is a bit misleading since the scale effect of the *level* of population is eliminated in these models but shows up with respect to the *growth rate* of population. That is, an increase in the rate of population growth increases the steady state growth rate of per capita income in non-scale models of endogenous growth, whereas in scale models such an increase raises the growth rate of the growth rate of per capita income.

The theory proposed here belongs to this group of non-scale growth models but instead of R&D uses learning by doing as the source of technological progress. The production side is a special case of the general two-sector non-scale growth model outlined in Eicher and Turnovsky (1999), where it is shown that the equilibrium growth rate is completely determined by the technology and population growth, independent of demand. It follows that if a steady state exists, the equilibrium growth rate in a rule of thumb model is the same as in a dynamic optimization framework. In contrast to the formulation of Eicher and Turnovsky (1999), the stock of technology, A, grows as a by-product of production and does not require exclusively allocated resources. This is the essential characteristic of learning *by doing*.

As in the seminal paper on the economic implications of learning by doing by Arrow (1962), it is usual in growth models to measure the *learning index* (technological progress) by the integral of past gross investment (cf. e.g. Sheshinski, 1967a; Romer, 1986). This approach has the obvious advantage that no new state variable besides the capital stock must be introduced. The drawback of using gross investment as the source of learning, however, is that the notion of learning is more naturally associated with production than with investment. Accordingly, most of the empirical studies on learning by doing employ cumulated production as the learning index (cf. e.g Wright, 1936; Hirsch, 1956). While even cumulated gross investment plays an important role for learning (cf. Sheshinski, 1967b; Lieberman, 1984), it seems nevertheless to be important to distinguish between learning by *doing* as measured by production and learning by *investment*.[1] Some of the empirical results, essentially summarized from Lieberman (1984), are of particular importance for the subsequent theoretical analysis:[2]

(1) R&D serves as an accelerator of the learning process, and learning may lead to productivity increases several orders of magnitude greater than those of the original inventions.

(2) There is a high degree of externalities associated with learning by doing (knowledge spillovers).

(3) The *learning elasticity* (the parameter β in equation (2) below) takes on values near $\beta = 0.32$ (corresponding to the so-called 80%-curve). Although different estimates are available, they are in general positive and much smaller than unity.

(4) All quoted papers on empirical evidence assume that technological progress is Hicks-neutral.

(5) The learning elasticity rises if capital is used more intensively.

These results substantiate the following assumptions of the model to be presented in the sequel: (1) Learning by doing is the source of technological progress. This is not to say that R&D – the accelerator of the learning process – does not matter, it is just interesting enough to consider the implications of learning by doing. (2) As a simplified approximation to a high degree of externalities, learning by doing is treated as being purely external at the firm level. This assumption enables a framework of perfect competition. (3) The learning elasticity β is assumed to be much smaller than unity. While Lucas (1988) in his Ricardian model of learning by doing assumed $\beta = 1$ in order to obtain endogenous long-run growth, it will be shown that this assumption, which without further qualifications is at odds with empirical evidence, is not necessary for positive per capita growth in the long-run if capital is accumulated and the growth rate of population is positive. (4) The learning function enters the production function in a multiplicative way, that is, technological progress is Hicks-neutral. While this assumption has the disadvantage that a steady state does not exist unless the production function is of the Cobb-Douglas type (which implies the equivalence of Hicks-neutrality and Harrod-neutrality), it appears to

[1] It is interesting to note that usually cumulative production enters the production function in a multiplicative way in models of learning by doing which neglect the accumulation of physical capital. Cf. e.g. Bardhan (1971) and Christiaans (1997, in German).

[2] A more detailed summary of empirical results on learning by doing as well as an assessment of the alternative approaches of estimating the *learning curve* instead of the *learning function* can be found in Christiaans (1997, ch. II.2).

be more in agreement with the empirical evidence than the usual Harrod-neutral form assumed in Sheshinski (1967a), e.g. Due to the mentioned equivalence one could of course argue that the model is actually one with Harrod-neutrality. I consider the model as a simple approximation to other functional forms in which this equivalence does not hold, however. (5) Since an aggregate model with a single commodity is considered, there is no room for introducing different learning elasticities. This empirical result, however, will help to shed light on the results of the model and it may be useful with respect to a possible two-sector extension of the theory.

In accordance with the preceding discussion, an aggregate production function of the Cobb-Douglas type is assumed:

$$Y = A^{\beta} K^{\alpha} L^{1-\alpha}, \quad 0 < \alpha < 1,\ \beta > 0,\ \alpha + \beta < 1, \tag{2}$$

where Y denotes the flow of output produced. There are two inputs, physical capital, K, and labor, L. Of course, Y, K, and L as well as A depend on time, but for the sake of notational convenience the time index t has been dropped. Note that the assumptions about the parameters accord well with empirical evidence; as has been noted before, β should be expected to be positive but much smaller than unity (about 0.32), and the same applies to α, which equals the capital share of gross output and is about $1/3$ empirically. The population equals the labor force and grows at an exogenous and constant rate n, $0 < n < 1$:

$$w_L := \frac{\dot{L}}{L} = n.$$

The variable A is the learning index and defined as cumulative production,

$$A(t) = A_0 + \int_0^t Y(\tau)\, d\tau.$$

Thus, the time derivative of A is

$$\dot{A} = A^{\beta} K^{\alpha} L^{1-\alpha}. \tag{3}$$

Output Y can both be consumed and invested. With short-run equilibrium prevailing at every point in time, aggregate gross investment, I, is given by the difference between output and consumption, C:

$$I = Y - C.$$

Assuming a constant rate of depreciation δ, the time-derivative of the capital stock, which equals net investment, is therefore

$$\dot{K} = I - \delta K = Y - C - \delta K. \tag{4}$$

A steady state growth path is defined as a path along which the capital-output-ratio is constant ($w_Y = w_K$) and all variables grow at constant rates (as in Eicher and Turnovsky, 1999, p. 399). Imposing the conditions of steady state growth and using $w_L = n$, logarithmic differentiation of equations (2) and (3) yields the following growth rates (cf. Appendix A):

$$w_Y = w_K = w_A = \underbrace{\frac{1-\alpha}{1-\alpha-\beta}}_{=:\gamma} n = \gamma n. \tag{5}$$

Since it has been assumed that $1 - \alpha - \beta > 0$, it follows that $\gamma > 1$ and $w_Y > w_L = n$, that is, the model generates (semi-) endogenous non-scale per capita growth:[3]

$$w_{Y/L} = (\gamma - 1)n > 0. \tag{6}$$

In a steady state, $w_Y - \gamma w_L = 0$, which implies that the following *scale adjusted* per capita variables are constant in long-run equilibrium:

$$y := \frac{Y}{L^\gamma},\ k := \frac{K}{L^\gamma},\ a := \frac{A}{L^\gamma}. \tag{7}$$

Dividing the production function (2) by L^γ yields the scale adjusted per capita production function (cf. Appendix B)

$$y = a^\beta k^\alpha. \tag{8}$$

A similar procedure applied to equation (3) implies that the scale adjusted per capita learning index a follows the differential equation (cf. Appendix C)

$$\dot{a} = a^\beta k^\alpha - \gamma n a. \tag{9}$$

In order to obtain the dynamic equation for the scale adjusted capital-labor ratio k, equation (4) is divided by K. Taking $w_k = w_K - \gamma n$ according to (7) into account, one gets

$$\dot{k} = \frac{Y}{K}\frac{K}{L^\gamma} - \frac{C}{K}\frac{K}{L^\gamma} - \delta\frac{K}{L^\gamma} - \gamma n\frac{K}{L^\gamma}.$$

Rearranging and using (8) and the definition

$$c := \frac{C}{L^\gamma}$$

implies

$$\dot{k} = a^\beta k^\alpha - (\gamma n + \delta)k - c. \tag{10}$$

Equations (9) and (10) form a system of two differential equations describing the dynamic behavior of the model. The initial values $a(0) = a_0$ and $k(0) = k_0$ are given historically. In order to close the model, a hypothesis on the consumption decision and therefore the time path of $c(t)$ is required. In view of the foregoing analysis it will come as no surprise that emphasis is placed on the golden rule of thumb. The dynamic optimization solution will be analyzed briefly in order to evaluate the normative performance of the golden rule of thumb by comparing the respective long-run equilibria and to derive the optimum policy for internalizing the external effects of learning by doing.

3 Analysis of Various Consumption Hypotheses

3.1 Golden Rule of Thumb

According to the golden rule of thumb introduced in Christiaans (2001), aggregate consumption is given by

$$C = wL + \rho K, \tag{1}$$

[3]It is also shown in Appendix A that no steady state with positive growth rates exists if $1 - \alpha - \beta < 0$. If a steady state with positive growth rates exists, $1 - \alpha - \beta = 0$ if and only if $n = 0$. Cf. also Jones (1995, footnote 9) for a similar result.

which is repeated here for convenience. The wage rate is designated by w and ρ is the representative household's rate of time preference. In competitive equilibrium, the wage rate is

$$w = \frac{\partial Y}{\partial L} = (1-\alpha)A^{\beta}K^{\alpha}L^{-\alpha},$$

which by substitution in (1) implies

$$C = (1-\alpha)Y + \rho K. \tag{11}$$

Dividing by L^{γ} and using (8), the scale adjusted per capita consumption becomes

$$c = (1-\alpha)y + \rho k = (1-\alpha)a^{\beta}k^{\alpha} + \rho k,$$

which can be inserted into equation (10) to get

$$\dot{k} = \alpha a^{\beta}k^{\alpha} - (\gamma n + \delta + \rho)k. \tag{12}$$

Figure 1 shows the phase portrait of the dynamical system (9), (12). The shape of the isoclines $\dot{a} = 0$ and $\dot{k} = 0$ and the local stability are analyzed in Appendix D. The phase diagram reveals that the equilibrium $E = (\bar{k}, \bar{a})$ is globally asymptotically stable for all strictly positive initial values.[4] Although the transition dynamics has interesting implications e.g. with respect to the behavior of the saving rate, stability justifies centering the analysis around the properties of the steady state. At this stage, only one result of special importance shall be mentioned. As can be seen from figure 1, in contrast to the monotonicity of the capital-labor ratio in the RKC model, the paths of a and k need not be monotonous in the golden rule of thumb model with learning by doing (cf. e.g. the trajectory O).

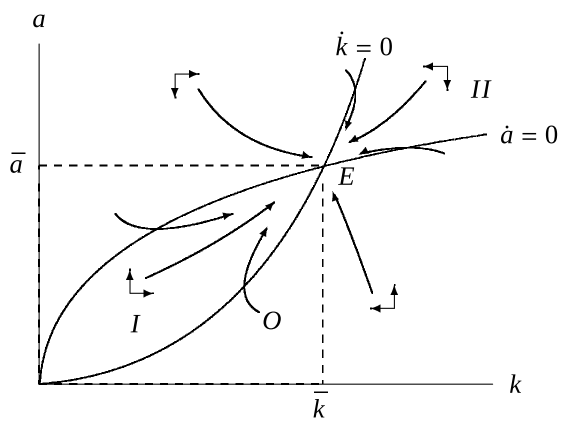

Figure 1: Phase Portrait of the Golden Rule of Thumb Non-Scale Growth Model

[4]The global asymptotic stability can be proven more formally by noting that the curves $\dot{k} = 0$, $\dot{a} = 0$, and the coordinate axes bound four open sets in the upper right quadrant, two of which are positive invariant regions (I and II in figure 1). Since in any of these open sets the trajectories are bounded and monotone, they either stay in one region for all $t \geqq 0$, or they enter one of the positive invariant regions. In any case, they converge to the equilibrium E. A detailed analysis of this kind can be found in Hirsch and Smale (1974, ch. 12).

For the purposes of the comparative statics of the long-run equilibrium, the steady state $E = (\bar{k}, \bar{a})$ can be calculated explicitly by setting the equations (9) and (12) equal to zero:

$$\begin{aligned} \bar{k} &= \left[\frac{\alpha}{(\gamma n + \delta + \rho)(\gamma n)^{\beta/(1-\beta)}}\right]^{\frac{1-\beta}{1-\alpha-\beta}} \\ \bar{a} &= \left[\frac{\alpha}{(\gamma n + \delta + \rho)(\gamma n)^{(1-\alpha)/\alpha}}\right]^{\frac{\alpha}{1-\alpha-\beta}} \end{aligned} \tag{13}$$

Although it would be possible to calculate the derivatives with respect to all parameters of the model, I refrain from presenting the extremely complicated expressions (note that γ is itself a function of α and β). The effect on the equilibrium values of k and a of those parameters which also influence the growth rate $(\gamma - 1)n$ of per capita output are not very useful anyway, as may be illustrated by the following example. If the growth rate of population, n, increases, it follows from (13) that both $\bar{k}$ and $\bar{a}$ decrease. This implies that a country with a higher rate of population growth has ceteris paribus a lower scale adjusted per capita output $y = Y/L^{\gamma}$ in the steady state [cf. (8)]. What matters, however, is not Y/L^{γ}, but per capita output Y/L. Since a higher rate of population growth implies a higher growth rate $(\gamma - 1)n$ of per capita output, a country with a higher value of n will, although scale adjusted per capita output in long-run equilibrium decreases with n, ceteris paribus enjoy a higher value of per capita output in the long run.

It follows that only the parameters δ and ρ in equations (13) are reasonable candidates for long-run comparative statics. It is clear at first sight that $\bar{k}$ and $\bar{a}$ and therefore also the steady state value of y decrease in δ and in ρ. Since both parameters do not affect the growth rate $(\gamma - 1)n$, even per capita output is ceteris paribus decreasing in δ and ρ at every point in time. A higher value of ρ corresponds to a lower saving rate. Thus, the classical result of Solow (1956) that the saving rate has no growth but only level effects continues to hold in this model.

The effects of the model's parameters on the equilibrium growth rate of output, γn, respectively output per capita, $(\gamma - 1)n$, are more important and straightforward to calculate:

$$\frac{\partial \gamma n}{\partial n} = \gamma > 0, \tag{14}$$

$$\frac{\partial \gamma n}{\partial \alpha} = \frac{\beta}{(1-\alpha-\beta)^2} n > 0, \tag{15}$$

$$\frac{\partial \gamma n}{\partial \beta} = \frac{1-\alpha}{(1-\alpha-\beta)^2} n > 0. \tag{16}$$

Since $\gamma - 1 > 0$, the derivatives of $(\gamma - 1)n$ with respect to these parameters have the same signs. These expressions show that a country with a higher growth rate of population, n, a production function using capital relatively more intensively (higher α), or a higher learning elasticity, β, respectively, enjoys ceteris paribus a higher growth rate of per capita output.

3.2 Dynamic Optimization

a) The Centrally Planned Economy Because of the learning by doing externality in this model, it is necessary to consider two different versions of dynamic optimization solutions. While in the centralized version the government is supposed to take the externality generated by learning by doing into account, households and firms neglect this externality in the decentralized version. It will be shown that the steady state of the decentralized optimization solution coincides with the steady state of the golden rule of thumb model and that the tax cum subsidy policy used to ensure a first best solution in the steady state of the decentralized optimization model can be used to lead golden rule of thumb consumers asymptotically to the optimum long-run equilibrium.

As in the Ramsey-Koopmans-Cass (RKC) model, the planner's objective is the maximization of the discounted integral of utility derived from per capita consumption. The results of this section serve mainly as a benchmark for the performance of the golden rule of thumb. It has been shown that the steady state is generally not independent of the instantaneous utility function in the presence of technological change (cf. e.g. Blanchard and Fischer, 1989). Since there is nothing to be gained from deriving this result again and to keep matters as simple as possible, the logarithmic instantaneous utility function with an intertemporal elasticity of substitution, σ, equal to one will be used. It follows for the RKC model with labor-augmenting technological change that the golden rule of thumb approaches the optimum long-run equilibrium in this case (Christiaans, 2001). An analogous result applies in the present setting with respect to the steady state of the decentralized optimization solution. At the same time, it is straightforward to show that the assumption $\sigma = 1$ in connection with a positive rate of time preference, $\rho > 0$, suffices to ensure the convergence of the objective functional. Thus, there is no need to resort to more sophisticated optimality criteria than simple maximization.

The social planner's optimization problem is described as follows:

$$\begin{aligned}
&\max_{c\in\bar{C}[0,\infty)} \int_0^\infty \ln(cL^{\gamma-1})e^{-\rho t}\,dt,\\
&\quad\text{subject to}\\
&\dot{k} = a^\beta k^\alpha - (\gamma n + \delta)k - c, \quad k(0) = k_0,\ \liminf_{t\to\infty} k(t) \geqq 0,\\
&\dot{a} = a^\beta k^\alpha - \gamma n a, \quad a(0) = a_0,\\
&c(t) \geqq 0,
\end{aligned} \tag{17}$$

where, since $c = C/L^\gamma$, consumption per capita is given by $cL^{\gamma-1} = C/L$. From the social planner's point of view, $L = L(t)$ is an exogenous function of time. The symbol $\bar{C}[0,\infty)$ denotes the set of piecewise continuous real valued functions defined on $t \in [0,\infty)$.

In order to derive sufficient conditions for an optimum solution, the current value Hamiltonian to this maximization problem is defined as

$$H = \ln(cL^{\gamma-1}) + \lambda[a^\beta k^\alpha - (\gamma n + \delta)k - c] + \mu[a^\beta k^\alpha - \gamma n a].$$

The optimum conditions derived from H are

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0,$$
$$\dot{\lambda} = \rho\lambda - \lambda(\alpha a^{\beta} k^{\alpha-1} - \gamma n - \delta) - \mu\alpha a^{\beta} k^{\alpha-1},$$
$$\dot{\mu} = \rho\mu - \lambda\beta a^{\beta-1} k^{\alpha} - \mu(\beta a^{\beta-1} k^{\alpha} - \gamma n).$$

The first of these conditions implies $\dot{c}/c = -\dot{\lambda}/\lambda$, which along with $\mu/\lambda = \mu c$ by substitution into the second equation yields

$$w_c := \frac{\dot{c}}{c} = (1 + \mu c)\alpha a^{\beta} k^{\alpha-1} - (\gamma n + \delta + \rho).$$

Replacing λ in the third condition by $1/c$, one arrives at a system of four differential equations in k, a, c, and μ. This set of equations consists of those for $\dot{k}$ and $\dot{a}$ in (17) and the equations just derived for $\dot{c}$ and $\dot{\mu}$.

It is sufficient for the present purposes to analyze the long-run equilibrium and to prove that it is an optimum solution for appropriate initial values of the state variables k and a. Setting $\dot{k} = \dot{a} = \dot{c} = \dot{\mu} = 0$ and imposing the condition that all four variables be positive, one gets the system of equations defining the optimum steady state (where the first two equations have been divided by k respectively a):

$$a^{\beta} k^{\alpha-1} = (\gamma n + \delta) + c/k, \tag{18}$$
$$a^{\beta-1} k^{\alpha} = \gamma n, \tag{19}$$
$$(1 + \mu c)\alpha a^{\beta} k^{\alpha-1} = \gamma n + \delta + \rho, \tag{20}$$
$$(1 + 1/(\mu c))\beta a^{\beta-1} k^{\alpha} = \gamma n + \rho. \tag{21}$$

Using (19) to replace $a^{\beta-1}k^{\alpha}$ in (21) by γn yields

$$\hat{\mu}\hat{c} = \beta\gamma n/[(1-\beta)\gamma n + \rho] > 0, \tag{22}$$

where $\hat{\mu}$ and $\hat{c}$ denote the long-run equilibrium values of μ and c, respectively. By substitution into (20) one gets

$$\alpha a^{\beta} k^{\alpha-1} = \frac{(1-\beta)\gamma n + \rho}{\gamma n + \rho}(\gamma n + \delta + \rho). \tag{20a}$$

Equations (19) and (20a) define the steady state values $\hat{a}$ and $\hat{k}$ of a and k, respectively. Equations (18) and (21) can recursively be used to determine $\hat{c}$ and $\hat{\mu}$. Noting that $1+\hat{\mu}\hat{c} > 1$, the solutions for k and a are

$$\begin{aligned} \hat{k} &= \left[\frac{(1+\hat{\mu}\hat{c})\alpha}{(\gamma n + \delta + \rho)(\gamma n)^{\beta/(1-\beta)}}\right]^{\frac{1-\beta}{1-\alpha-\beta}} > \bar{k}, \\ \hat{a} &= \left[\frac{(1+\hat{\mu}\hat{c})\alpha}{(\gamma n + \delta + \rho)(\gamma n)^{(1-\alpha)/\alpha}}\right]^{\frac{\alpha}{1-\alpha-\beta}} > \bar{a}, \end{aligned} \tag{23}$$

with $\bar{k}$ and $\bar{a}$ as provided in equations (13). Thus, the steady state values of k and a in case of a golden rule of thumb solution are below the optimum values. The reason is that, as in the decentralized optimization solution considered below, the saving rate is too low due to the externality of learning by doing. Accumulating more capital implies a higher gross output and therefore higher learning effects.

In order to prove that the steady state solution is indeed optimal if $k_0 = \hat{k}$ and $a_0 = \hat{a}$, note that the Hamiltonian is jointly concave in the state and control variables due to the assumption $1 - \alpha - \beta > 0$ and the positivity of the costates. Next, the following limiting transversality conditions have to be checked:

$$\liminf_{t\to\infty} e^{-\rho t}\hat{\lambda}[k(t) - \hat{k}] \geqq 0,$$
$$\liminf_{t\to\infty} e^{-\rho t}\hat{\mu}[a(t) - \hat{a}] \geqq 0.$$

It is straightforward to show that $k(t)$ and $a(t)$ are nonnegative for all t and all feasible control paths. Since the state and costate variables are all positive in the steady state, the transversality conditions are met. This proves that the stationary steady state solution meets the Mangasarian type sufficient conditions for optimality (cf. e.g. Seierstad and Sydsæter, 1987, p. 234–235). Of course, the centralized optimization solution is Pareto efficient.

b) The Decentralized Economy As in the case of the RKC model, it is possible to analyze a simple aggregate version leading to the same aggregate dynamics as an explicitly decentralized equilibrium formulation with perfect competition and perfect foresight (which for the sake of completeness is briefly analyzed in Appendix E). It should be noted, however, that the usually frankly made assumption of perfect foresight in such models is of an extreme nature. For individual agents must know the complete future time paths of the interest and the wage rate, which in turn depend on the time paths of $k(t)$ and $a(t)$ and the production function. In the aggregate formulation, the decentralized solution differs from the centralized solution in that the households and firms do not take the externality generated by learning by doing into account. Nevertheless, agents must be informed about the time path of $a(t)$, which they take to be exogenously given.

Neglecting the equation for the accumulation of the scale adjusted per capita learning index a, the Hamiltonian to the decentralized optimization problem is

$$H = \ln(cL^{\gamma-1}) + \lambda[a^\beta k^\alpha - (\gamma n + \delta)k - c],$$

from which the optimum conditions

$$\frac{\partial H}{\partial c} = \frac{1}{c} - \lambda = 0,$$
$$\dot{\lambda} = \rho\lambda - \lambda(\alpha a^\beta k^{\alpha-1} - \gamma n - \delta).$$

are derived. Using the first condition to eliminate λ in the second equation leads to

$$\frac{\dot{c}}{c} = \alpha a^\beta k^{\alpha-1} - (\gamma n + \delta + \rho).$$

The steady state is defined by setting $\dot{k} = \dot{a} = \dot{c} = 0$. The (private) optimality of the steady state if the initial values are the equilibrium values can be proven similarly as in the case of the centralized economy.

3.3 Comparison of Solutions and Optimum Tax Cum Subsidy Policy

The dynamic equations of the three models considered are summarized as follows:

1. Centralized optimization solution:

$$\begin{aligned} \dot{k} &= a^{\beta}k^{\alpha} - (\gamma n + \delta)k - c \\ \dot{a} &= a^{\beta}k^{\alpha} - \gamma n a \\ \dot{c} &= (1 + \mu c)\alpha a^{\beta}k^{\alpha-1}c - (\gamma n + \delta + \rho)c \\ \dot{\mu} &= (\rho + \gamma n)\mu - (\mu + 1/c)\beta a^{\beta-1}k^{\alpha} \end{aligned} \tag{24}$$

2. Decentralized optimization solution:

$$\begin{aligned} \dot{k} &= a^{\beta}k^{\alpha} - (\gamma n + \delta)k - c \\ \dot{a} &= a^{\beta}k^{\alpha} - \gamma n a \\ \dot{c} &= \alpha a^{\beta}k^{\alpha-1}c - (\gamma n + \delta + \rho)c \end{aligned} \tag{25}$$

3. Golden rule of thumb solution:

$$\begin{aligned} \dot{k} &= \alpha a^{\beta}k^{\alpha} - (\gamma n + \delta + \rho)k \\ \dot{a} &= a^{\beta}k^{\alpha} - \gamma n a \end{aligned} \tag{26}$$

Comparing the equations of motion for the decentralized optimization solution and the golden rule of thumb solution, it is straightforward to show that the resulting equilibrium values of k, a, and c are the same. That is, in both of these cases, the steady state values of k and a are $\bar{k}$ and $\bar{a}$ provided in (13), which are too low in comparison with the centralized optimization solution. Since the steady state value of c is implied by those of k and a, it suffices to concentrate on these variables.

As to the optimum policy, consider at first the equations (25) concerning the decentralized optimization solution. These imply the same time path of k, a, and c as the equations (24) if the rental price of capital, $r + \delta = \alpha a^{\beta}k^{\alpha-1}$, is subsidized at the rate μc implied by equations (24).[5] That is, capital owners should receive more than the private rental price of capital. It is important in this respect that the costate μ (the shadow price of learning) can be shown to be positive for all t. This follows from the fact that the optimum steady state value is $\hat{\mu} > 0$, which cannot be approached if $\mu(\bar{t}) = 0$ for some $\bar{t}$ since $\mu(t)$ is continuous and the fourth equation in (24) would imply $\dot{\mu}(\bar{t}) < 0$.

In this model, the external effects of learning by doing can be internalized by subsidizing the rental price of capital. Two comments on this result are in order. First, the source of learning by doing is not the accumulation of capital but production. Thus, it would be natural to expect that subsidizing production was the optimum policy. But subsidizing the rental price of capital and therefore capital accumulation in this model amounts to subsidizing production, because there is only one sector which employs the entire labor

[5] Note that the optimal path for initial values off the steady state has not been analyzed. I refrain from doing this because I am interested in a simple policy solution with respect to the golden rule of thumb, and it will be shown that it suffices to impose the optimum steady state tax cum subsidy policy.

force of exogenous size and the entire capital stock. In a two sector model with learning by doing in one of the sectors it would be necessary to subsidize production of this sector, thereby raising the relative price of the respective commodity (cf. e.g. Bardhan, 1971, who neglects capital accumulation, however). Second, a balanced budget of the government without further distortions requires to finance the subsidy with a lump-sum tax. Since there is no labor/leisure choice in this economy, a wage tax amounts to a lump-sum tax. It is important that an efficient subsidy is not feasible without an accompanying taxation, that is, an optimum tax cum subsidy policy is required.

How can golden rule of thumb consumers be influenced in their behavior in order to reach the optimum steady state? The optimum long-run rate of subsidy in case of optimizing households to the rental price of capital is given by $\hat{\mu}\hat{c} > 0$ in equation (22). As has been indicated before, since there are only two types of income in this economy, it is most natural to tax labor income if capital income is subsidized in order to obtain a balanced budget. The appropriate balanced budget tax rate τ_w if the rental price of capital is subsidized at the constant rate $\tau_r = \hat{\mu}\hat{c}$ can be inferred from the identity of net national product and national income:

$$Y - \delta K = (1 + \tau_r)(r + \delta)K - \delta K + (1 - \tau_w)wL$$
$$\iff \qquad Y = (1 + \hat{\mu}\hat{c})\alpha a^\beta k^{\alpha-1}K + (1 - \tau_w)(1 - \alpha)a^\beta k^\alpha L^{\gamma-1}L,$$

where $(1 - \alpha)a^\beta k^\alpha L^{\gamma-1}$ is the wage rate. In terms of scale adjusted per capita output, $Y/L^\gamma = y = a^\beta k^\alpha$, this equation reads

$$a^\beta k^\alpha = (1 + \hat{\mu}\hat{c})\alpha a^\beta k^\alpha + (1 - \tau_w)(1 - \alpha)a^\beta k^\alpha$$
$$\iff \qquad \tau_w = \hat{\mu}\hat{c}\frac{\alpha}{1 - \alpha}.$$

The tax rate thus obtained ensures a balanced budget of the government since capital owners get as subsidies what workers loose as taxes. It is also important to note that $\tau_w < 1$, that is the tax rate is not unreasonably high:

$$\hat{\mu}\hat{c}\frac{\alpha}{1-\alpha} \overset{(22)}{=} \frac{\beta\gamma n\alpha}{[(1-\beta)\gamma n + \rho](1-\alpha)} = \frac{\alpha\beta\gamma n}{\alpha\beta\gamma n + (1-\alpha-\beta)\gamma n + (1-\alpha)\rho} < 1,$$

since $1 - \alpha - \beta > 1$.

If the wage rate is taxed at the rate τ_w, consumption according to the golden rule of thumb is

$$C = (1 - \tau_w)wL + \rho K = \left(1 - \hat{\mu}\hat{c}\frac{\alpha}{1 - \alpha}\right)(1 - \alpha)a^\beta k^\alpha L^\gamma + \rho K,$$

or in terms of scale adjusted per capita variables:

$$c = [1 - \alpha(1 + \hat{\mu}\hat{c})]a^\beta k^\alpha + \rho k.$$

Using this consumption function in equation (10) yields

$$\dot{k} = \alpha(1 + \hat{\mu}\hat{c})a^\beta k^\alpha - (\gamma n + \delta + \rho)k.$$

Setting $\dot{k} = \dot{a} = 0$ and comparing with (24) shows that the long-run equilibrium now coincides with the optimum steady state. The stability of the model without a wage tax is not affected by the inclusion of the constant factor $1 + \hat{\mu}\hat{c}$.

In summary, a wage tax at the rate $\hat{\mu}\hat{c}\alpha/(1-\alpha)$ combined with a balanced budget subsidy of the capital rental implies that the golden rule of thumb leads asymptotically to the steady state of the centralized optimum solution. The rate $\hat{\mu}\hat{c}$ coincides with the optimal steady state tax rate in case of a decentralized optimization solution. This result to a certain degree raises the reliability of policy recommendations derived from optimizing models, since in the author's opinion the hypothesis that households follow the golden rule of thumb is more plausible than that they follow unstable saddle paths. It could also be suspected that one expects too much from governments by supposing that they calculate and raise a time varying optimal tax rate. The recommendation to raise constant taxes which are optimal in a steady state may therefore be a good rule of thumb for economists advising governments. Finally, it should be noted that in general the optimal tax and subsidy rates depend on the intertemporal elasticity of substitution, σ, which has been assumed to be equal to one. In general, the steady states of the decentralized optimization and the golden rule of thumb model and therefore the optimal subsidies in these two cases do not coincide. But who knows actual values of σ precisely enough to calculate optimum solutions depending on this value?

4 On the Stylized Facts about Economic Growth

A preliminary test of the golden rule of thumb model with learning by doing is to examine it in light of the so-called stylized facts about economic growth as introduced by Kaldor (1961, facts 1–4 below) and extended by Romer (1989, facts 5–9). The model is in a sense consistent with all of these stylized facts and able to explain most of them. This is easily shown if it is supposed that the economy starts in the vicinity of the steady state, which allows to concentrate on the steady state properties. The facts are grouped into those applying to a single country and those concerned with a comparison between several countries. Although it is pretty clear with respect to the first group of facts that any non-scale equilibrium growth model should be be able to generate them, they are included here for the sake of completeness. Unless otherwise specified, the analysis regards the steady state.

1. Average labor productivity grows with no evident tendency for the growth rates to decline over time: $w_{Y/L} = (\gamma - 1)n > 0$, cf. eq. (6).

2. The capital-output ratio is stationary: $w_{K/Y} = w_K - w_Y = 0$, cf. eq. (5).

3. The shares of labor and capital in total net output are stationary:

$$\frac{wL}{Y-\delta K} = \frac{(\partial Y/\partial L)L}{Y-\delta K} = \frac{(1-\alpha)Y}{Y-\delta K} = \frac{1-\alpha}{1-\delta(K/Y)} = \text{const.},$$

   cf. fact 2, and

$$\frac{rK}{Y-\delta K} = \frac{(\partial Y/\partial K-\delta)K}{Y-\delta K} = \frac{\alpha Y-\delta K}{Y-\delta K} = \frac{\alpha-\delta(K/Y)}{1-\delta(K/Y)} = \text{const.}$$

Two other important stylized facts (stationary interest rate and $w_{K/L} = w_{Y/L} > 0$) are implied by those already mentioned and are therefore not discussed explicitly.

Whether the model is compatible with the stylized facts concerning the comparison between several countries is a more complicated question. These facts are the following:

4. Average labor productivity and its growth rate vary considerably between countries.
5. There is no simple relationship between the initial value of output per capita Y/L and the subsequent growth rates $w_{Y/L}$. Poor countries do not necessarily grow faster than rich countries.
6. Growth accounting always finds a residual; the growth rate of factor inputs is not large enough to explain the growth rate of output.
7. The migration of both skilled and unskilled workers tends to be toward high-income countries.
8. The growth in the volume of international trade is positively correlated with the growth in output.
9. There is a negative correlation between population growth rates and the level of per capita income.

It will be shown that the golden rule of thumb model with learning by doing is able to explain all the stylized facts which fall in the realm of a one sector model and and that it is consistent with the other facts in the sense that it is conceivable that these other facts could be explained by a two sector version of the model.

With respect to an application of the model to a world consisting of many countries, it is important to be specific about the externality associated with learning by doing. While it has been noted in section 2 that there is a high degree of knowledge spillovers, recent empirical findings of Branstetter (2001), who is mainly concerned with knowledge spillovers from R&D, however, nevertheless suggest that these spillovers are primarily intranational and not international in scope. Thus, it is possible to consider the learning index A as being external on the firm level but internal on the country level. This assumption enables an endogenous explanation of different productivity levels across countries and of comparative advantages in case of international trade.

Since the growth rate of average labor productivity, $w_{Y/L} = (\gamma - 1)n$, is constant and positive, there is no catching-up of poor to rich countries in this model. Thus, it is able to explain differences in average labor productivities between countries. The first part of fact 4 can be generated by simply assuming that production in one country started sooner than in another. While it would even be possible to explain different growth rates by assuming different parameters, an actual explanation requires more than an argument based on exogenous constants. This issue will be examined below. Fact 5 follows immediately from the argument just provided. In the absence of growth of the learning index, A, output and capital would both grow at the rate n in long-run equilibrium. With a positive growth rate of A, both grow faster than labor. Thus, even fact 6 can be explained with the model.

It would be natural to expect that migration tends to countries with a high wage rate. Since the wage rate $w = (1 - \alpha)a^{\beta}k^{\alpha}L^{\gamma-1}$ grows at the same rate $(\gamma - 1)n$ as output per capita in the steady state, a country with a higher per capita output has also a higher wage rate. Thus, fact 7 can be partially explained (the model does not distinguish between skilled and unskilled workers).

Fact 8 cannot be reasonably analyzed in a one sector model with no room for international trade. It is conceivable, however, that a two sector extension of the present model is capable of explaining this fact as well as fact 9,[6] which is at odds with the one sector model considered here, and the second part of 4. Suppose that there exist two sectors. Sector 1 produces high-tech goods using capital relatively intensively while sector 2 produces agricultural goods using labor relatively intensively. If α_1 and α_2 denote the shares of capital in sector 1 and sector 2, respectively, it follows that $\alpha_1 > \alpha_2$. The learning elasticity β_1 in sector 1 is greater than β_2 in sector 2 (which corresponds to result (5) of the empirical evidence on learning by doing cited in section 2). In a two country world, the growth rate of labor in the home country, n, shall be smaller than n^* in the foreign country, that is $n < n^*$. Otherwise, both countries are identical. Now suppose for simplicity that under free trade the foreign country, which has a larger labor force, completely specializes in the production of the labor intensive good 2 while the home country completely specializes in the production of the capital intensive good 1. Although equation (14) implies a higher growth rate abroad, equations (15) and (16) show that on balance the growth rate of per capita income may be higher in the home country. This explains fact 9, and since labor productivities now also grow at different rates across countries, the second part of fact 4. A problem with this approach is that nothing has been said about the development of the relative price of high-tech goods in terms of agricultural goods. I refrain at this stage from speculating about this issue which requires an explicit mathematical formulation.

As to the stylized fact 8, the argument provided in the preceding paragraph can only explain why the home country grows faster with international trade. Under the conditions just stated, the foreign country would indeed grow slower. Fact 8 is formulated very unspecific, however. It does not assert that countries grow faster with *free* trade. The two sector model just outlined comes close to the theory of infant industry protection in its modern form which recommends subsidies instead of tariffs (Bardhan, 1971; Christiaans, 1997). Subsidies to export industries may lead to an increasing volume of trade in some countries and to an increasing growth rate of output if learning can be reinforced. It would therefore be possible that in a multi country world even a country such as the foreign country considered in the preceding paragraph grows faster with an increasing volume of international trade. In summary, although the simple one sector model cannot explain all the stylized facts, it is not inconsistent with these facts in the sense that a similar two sector model will probably be capable of an explanation.

There are some further empirical regularities of economic growth which a reasonable growth model should be capable of generating. E.g., the rates of convergence to the steady state have often been estimated to be around 2%–3% (cf. e.g. Mankiw et al., 1992; Barro and Sala-i-Martin, 1995, p. 431). These results have been criticized by Klenow and Rodriguez-Clare (1997) for good reasons, however, who cite other estimates of the rates of convergence up to 11%. Therefore, the analysis of convergence rates is not further pursued here.[7]

[6]The correlation mentioned in fact 9 may be caused mutually. Since n is constant in the present model, there is no room to explain why a higher per capita income should imply a lower population growth rate. The model is only concerned with the inverse causation from population growth rates to output per capita.

[7]Anyway, it is shown in Christiaans (2001) that the basic neoclassical growth model with exogenous technological change and the golden rule of thumb is capable of generating empirically estimated rates of

An important issue is the behavior of the saving rate during the transition to the steady state. Barro and Sala-i-Martin (1995, ch. 12) interpret empirical evidence across countries as indicating that the saving rate tends to rise with per capita income during the transition. Although this result does not deserve to be called a stylized fact (e.g. the saving rate in the U.S. declined since the mid-1950s, cf. Browning and Lusardi, 1996, pp. 1816–1817), a growth model should admit the possibility of rising as well as declining saving rates. While the Solow (1956) model with a fixed saving rate excludes this possibility by definition, the (gross) saving rate in the golden rule of thumb model considered here is

$$s = \frac{Y - C}{Y} \overset{(11)}{=} 1 - \frac{(1-\alpha)Y + \rho K}{Y} = \alpha - \rho \frac{K}{Y} \overset{(7)}{=} \alpha - \rho \frac{k}{y} \overset{(8)}{=} \alpha - \rho \frac{k^{1-\alpha}}{a^{\beta}}.$$

It follows that the saving rate rises if

$$(1-\alpha)w_k < \beta w_a$$

and vice versa. Whether this condition is satisfied depends on the initial values of a and k. E.g., if a_0 is relatively small and far off the locus $\dot{a} = 0$ in figure 1 while k_0 is near the locus $\dot{k} = 0$, the condition should be met. The saving rate certainly rises if k falls and a rises, as is the case for the first part of the trajectory O in figure 1. In any case, it is important to note that the model is able to generate both rising and declining saving rates. This would not be the case if capital, K, instead of cumulated production, A, had been used as the learning index, as is usual in growth models of the learning by doing type (an exception is Wong and Yip, 1999). For if A in equation (2) was substituted by K, the (gross) saving rate would read

$$s = \alpha - \rho \frac{k}{y} = \alpha - \rho k^{1-\alpha-\beta}.$$

Since $1 - \alpha - \beta > 0$, this expression decreases in k, and during the transition to the steady state s declines if k is initially below its long-run equilibrium value (k is monotonously increasing in this case). Thus, using cumulated production instead of capital as the learning index has more advantages than just being more plausible.

5 Concluding Remarks

The golden rule of thumb has been introduced in Christiaans (2001), where it has been shown in the context of the basic neoclassical growth model that it performs well from a normative point of view and is also a plausible working hypothesis for positive growth economics. The present paper shows that this rule of thumb can also be used in a non-scale growth model of the learning by doing type. The good normative performance of the golden rule of thumb is substantiated because even in the presence of endogenous technological change it may lead to the same steady state as the decentralized dynamic optimization approach. The optimum steady state tax cum subsidy policy with respect to dynamically optimizing consumers can therefore be used to lead rule of thumb consumers

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convergence at least as good as a model with optimizing consumers.

to the long-run optimum. This example shows that some implications of neoclassical economics are remarkably robust with respect to assumptions regarding the demand side. The most important implications of growth models are those with respect to the growth rates of per capita income, and these are mainly determined by the supply side. Nevertheless, this property of neoclassical growth models should not be regarded as a justification for using a consumption hypothesis that leads to an unstable solution.

The *optimum* policy to internalize the external effects of learning by doing in case of golden rule of thumb consumers is not the constant tax-subsidy mix that has been derived but of course a non-constant system of taxes and subsidies which would imply that the dynamic behavior of an economy with rule of thumb consumers resembles that of the centrally planned dynamic optimization economy. It has been noted, however, that recommending the constant policy may be a good rule of thumb for economists advising governments. For determining a constant policy which is optimal only for a steady state is difficult enough in reality, and one should therefore be content with a policy which helps to approach the optimum asymptotically. The additional advantage is that, as has been shown, there is at least one other consumption hypothesis besides that of dynamic optimization for which this policy rule of thumb works well.

From a positive point of view, the non-scale growth learning by doing approach eliminates the basic shortcomings of the model analyzed in Christiaans (2001), which does not endogenously explain technological progress and does not allow for a rising saving rate during the transition to the steady state with an increasing scale adjusted capital-labor ratio. Moreover, the learning by doing model with rule of thumb consumers is consistent with those stylized facts about economic growth which can reasonably be analyzed in the framework of a one sector model. Note in this respect that the model has been formulated in line with empirical evidence on learning by doing. E.g., cumulated production has been used as the learning index with a positive learning elasticity much smaller than one.

A one sector model without international trade neglects some aspects of reality which are extremely important for actual economic growth and development. As has been indicated before, an explanation of all stylized facts about economic growth certainly requires a model with at least two sectors which is capable of explaining comparative advantages between nations, which in turn influence long-run growth. Such a two sector extension of the learning by doing model involves conceptual problems which make the formulation of an adequate model difficult, however. E.g., if the usual assumption of homothetic preferences with respect to the two goods is made, a diversified steady state with two sectors growing at different rates will not exist unless the relative price of the commodity with a higher growth rate of production decreases steadily. Whether in case of specialization the country producing the high-tech good really enjoys a higher growth rate of per capita product depends on other parameters of the model. Because of these difficulties the literature on learning by doing simplifies matters by neglecting either the accumulation of capital (e.g. Bardhan, 1971; Lucas, 1988; Boldrin and Scheinkman, 1988; Christiaans, 1997), restricting attention to a one sector model (e.g. the first model in Sheshinski, 1967a), or considering a small open economy with linear production functions respectively production possibility curves (e.g. the second model in Sheshinski, 1967a, or Wong and Yip, 1999). The approach of Wong and Yip (1999) is similar in many respects to a possible two sector extension of the present model. Apart from employing the usual dynamic optimization approach, however, these authors consider the effects of an otherwise constant population only by means of comparative statics and do not analyze a full general equilibrium of two countries. Further research in this direction seems to be promising.

Appendix

A Derivation of Equation (5)

Logarithmic differentiation of equation (2) implies

$$w_Y = \beta w_A + \alpha w_K + (1-\alpha) w_L.$$

$w_A = \dot{A}/A =$ const. implies that $\dot{A}/A = \ddot{A}/\dot{A}$. Hence, it follows from equation (3) that $w_A = w_Y$. Using $w_K = w_Y$ by the definition of a steady state and $w_L = n$ leads to

$$w_Y = w_K = w_A = \gamma n, \tag{5}$$

where $\gamma := (1-\alpha)/(1-\alpha-\beta)$.

Now consider the case in which the assumption $1-\alpha-\beta > 0$ is violated. If $1-\alpha-\beta = 0$, logarithmic differentiation of equation (2) and $w_Y = w_K = w_A$ in a steady state imply

$$\underbrace{(1-\alpha-\beta)}_{=0} w_Y = 0 = (1-\alpha)n \quad \Longrightarrow \quad n = 0.$$

Similarly, if $n = 0$, the preceding equation can only be met if $1-\alpha-\beta = 0$ or $w_Y = 0$. Thus, if a steady state with positive growth rates of Y, K, and A exists, $1-\alpha-\beta = 0$ if and only if $n = 0$. This is a case of endogenous growth in which the equilibrium growth rate w_Y is not determined by technology alone, as it is usual in scale models (cf. e.g. Romer, 1990). The demand side is required to pin down the equilibrium growth rate in this knife-edge case.

Finally, in case of $1-\alpha-\beta < 0$, the derivation of equation (5) is valid as before. Since $\gamma < 0$, it follows immediately that a steady state with positive growth rates does not exist.

B Derivation of Equation (8)

Dividing equation (2) by L^γ yields

$$y = \frac{Y}{L^\gamma} = A^\beta K^\alpha L^{1-\alpha-\gamma} = \left(\frac{A}{L^\gamma}\right)^\beta \left(\frac{K}{L^\gamma}\right)^\alpha L^{1-\alpha-\gamma} L^{\beta\gamma} L^{\alpha\gamma} = a^\beta k^\alpha L^{1-\alpha-(1-\alpha-\beta)\gamma} = a^\beta k^\alpha,$$

because $(1-\alpha-\beta)\gamma = 1-\alpha$.

C Derivation of Equation (9)

Dividing equation (3) by L^γ gives

$$\frac{\dot{A}}{L^\gamma} = a^\beta k^\alpha. \tag{A1}$$

From the definition of a,

$$\frac{\dot{a}}{a} = \frac{\dot{A}}{A} - \gamma n \quad \text{and} \quad \dot{a} = \frac{\dot{A}}{L^\gamma} - \gamma n a.$$

Substituting (A1) yields (9).

D The Isoclines in Figure 1 and Local Stability Analysis

From equation (9), the isocline $\dot{a} = 0$ is given by the k-axis and

$$a = \left(\frac{k^{\alpha}}{\gamma n}\right)^{1/(1-\beta)}.$$

Thus, if $\alpha + \beta < 1$,

$$\left.\frac{\partial a}{\partial k}\right|_{\dot{a}=0} = \frac{\alpha}{1-\beta}\left(\frac{1}{\gamma n}\right)^{1/(1-\beta)} k^{(\alpha+\beta-1)/(1-\beta)} > 0,$$

and

$$\left.\frac{\partial^2 a}{\partial k^2}\right|_{\dot{a}=0} = \frac{(\alpha+\beta-1)\alpha}{(1-\beta)^2}\left(\frac{1}{\gamma n}\right)^{1/(1-\beta)} k^{(\alpha+2\beta-2)/(1-\beta)} < 0.$$

Moreover,

$$\lim_{k\to 0}\left.\frac{\partial a}{\partial k}\right|_{\dot{a}=0} = \infty, \quad \text{and} \quad \lim_{k\to\infty}\left.\frac{\partial a}{\partial k}\right|_{\dot{a}=0} = 0.$$

Thus, the locus $\dot{a} = 0$ has the concave shape as shown in figure 1.

A similar procedure shows that, if $\alpha + \beta < 1$, the isocline $\dot{k} = 0$ goes through the origin, is positively sloped, and convex with $\lim_{k\to 0} \left.\frac{\partial a}{\partial k}\right|_{\dot{k}=0} = 0$ and $\lim_{k\to\infty} \left.\frac{\partial a}{\partial k}\right|_{\dot{k}=0} = \infty$. These results imply that, in the positive orthant, a unique equilibrium $(\bar{k}, \bar{a})$ exists. As can be seen from figure 1, this equilibrium is globally asymptotically stable for all strictly positive initial values $(k_0, a_0) > (0, 0)$.

The importance of the condition $1 - \alpha - \beta > 0$ for stability is easily seen by considering the linear approximation of system (9), (12) around the equilibrium. The Jacobian is

$$J = \left.\begin{pmatrix} \partial\dot{a}/\partial a & \partial\dot{a}/\partial k \\ \partial\dot{k}/\partial a & \partial\dot{k}/\partial k \end{pmatrix}\right|_{\dot{a}=\dot{k}=0} = \begin{pmatrix} (\beta-1)\gamma n & \gamma n + \delta + \rho \\ \alpha\beta\gamma n & (\alpha-1)(\gamma n + \delta + \rho) \end{pmatrix},$$

where the conditions $\bar{a}^{\beta-1}\bar{k}^{\alpha} = \gamma n$ and $\alpha\bar{a}^{\beta}\bar{k}^{\alpha-1} = \gamma n + \delta + \rho$ which are met for the equilibrium values $\bar{a}$ and $\bar{k}$ have been used. It is straightforward to show that $1 - \alpha - \beta > 0$ implies $\mathrm{Tr}(J) < 0$ and $|J| > 0$. Hence, the Routh-Hurwitz-conditions are satisfied and the equilibrium is (locally) asymptotically stable.

E Decentralized Formulation of the Model in Section 3.2 b)

The symbols are the same as in the main text but are interpreted as variables concerning a specific representative household. The objective of the representative household whose size grows at rate n is to maximize his utility

$$\int_0^{\infty} \ln(cL^{\gamma-1})e^{-\rho t}\, dt. \tag{A2}$$

If $B(t)$ denotes assets of the household at time t, his flow budget constraint is (dropping the time index):

$$C + \dot{B} = wL + rB,$$

or – dividing by L^{γ} – in terms of scale adjusted per person variables:

$$c + \frac{\dot{B}}{L^{\gamma}} = wL^{1-\gamma} + rb. \tag{A3}$$

Using $\dot{b}/b = \dot{B}/B - \gamma n$, the term $\dot{B}/L^{\gamma}$ can be expressed as $\dot{B}/L^{\gamma} = \dot{b} + \gamma nb$. Assuming that all households are identical, scale adjusted assets per person b of the representative household coincide with the scale adjusted assets per capita of the economy. Since physical capital is the only asset in this closed economy, b must also coincide with scale adjusted capital per capita: $b = k$. Substituting these results into (A3) leads to

$$\dot{k} = wL^{1-\gamma} + (r - \gamma n)k - c. \tag{A4}$$

The maximization of (A2) subject to (A4) and appropriate boundary conditions implies the Keynes-Ramsey rule

$$\frac{\dot{c}}{c} = r - \gamma n - \rho, \tag{A5}$$

where the household considers L, r and w as exogenous functions of time about which he has perfect foresight, however.

Firms are assumed to maximize profits; since learning by doing is external to the firms, the interest rate and the wage rate in equilibrium are given by

$$r = \frac{\partial Y}{\partial K} - \delta = \alpha a^{\beta} k^{\alpha-1} - \delta,$$
$$w = \frac{\partial Y}{\partial L} = (1-\alpha) a^{\beta} k^{\alpha} L^{\gamma-1}.$$

Substituting into (A4) and (A5), respectively, leads to

$$\dot{k} = a^{\beta} k^{\alpha} - (\gamma n + \delta)k - c,$$
$$\frac{\dot{c}}{c} = \alpha a^{\beta} k^{\alpha-1} - (\gamma n + \delta + \rho),$$

which are exactly the equations of motion implied by the approach in section 3.2 b).

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