

**Volkswirtschaftliche  
Diskussionsbeiträge**



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and a Golden Rule of Thumb**

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**Discussion Paper No. 94-01**

ISSN 1433-058x

**UNIVERSITÄT GESAMTHOCHSCHULE SIEGEN  
FACHBEREICH WIRTSCHAFTSWISSENSCHAFTEN**



# Economic Growth, the Mathematical Pendulum, and a Golden Rule of Thumb

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March 12, 2001

## Abstract

It is argued that due to their general instability dynamic optimization models cannot be used as positive theories of economic growth. The argument is substantiated by (numerical) examples. A simple rule of thumb is provided as an alternative to the RKC model. This rule is shown to perform well from a normative and to be reasonable from a positive point of view. The model is consistent with empirically estimated rates of convergence if a broad concept of capital is used.

**Keywords:** Economic Growth – Dynamic Optimization vs. Rules of Thumb

**JEL-Classification:** O41; D90

## 1 Introduction

Solow (1956) and other early writers on economic growth used a fixed saving rate as a simple rule of thumb describing economic behavior. In the 1960's, dynamic optimization methods were primarily used in normative applications of growth theory aiming at deriving conditions for optimal economic growth but not alleging that reality could be explained by such hypotheses. E.g., Cass (1965) and Arrow (1968) employed dynamic optimization from the viewpoint of *centralized* economies or governments in a position to *control the economy completely*. While positive interpretations of dynamic optimization models are not new (cf. e.g. Cass and Shell, 1976a), they have become the unquestioned standard in growth theory at least since the seminal papers on endogenous growth by Romer (1986, 1990) and Lucas (1988) have been published (cf. e.g. Blanchard and Fischer, 1989; Barro and Sala-i-Martin, 1995, and nearly any of the innumerable recent publications on economic growth). In the meantime, even drastic applications of this paradigm can be found. E.g., Eicher et al. (2000) have interpreted economic crises as being the result of the decisions of dynamically optimizing agents. By the properties of dynamic optimization methods which in general require knowledge of the terminal values of the (co-)state variables, this approach implies that these optimizing agents must have known the respective *crisis* in advance.

The reason for positive interpretations of dynamic optimization models seems to be the basic persuasion of neoclassical economists that households and firms do always behave fully rational. If rationality is understood to imply the calculation of optimum solutions under all circumstances, this requires the application of Pontryagin's maximum principle or related methods in dynamic settings as e.g. the theory of economic growth. One of the objectives of the present paper is to show that the assumption of rational behavior in dynamic economic theories is much more questionable than it is in static models. The basic reason for this assertion is simple. While the optimum solutions of many static optimization problems are stable for several adjustment processes, the optimum solutions of dynamic optimization problems are generally unstable.

Starting with Simon (1955, 1957), the hypothesis of unbounded rationality has been criticized by various authors for several reasons. Rational decision-making is taken to be subject to various restrictions (limited time span, uncertainty, market imperfections etc.), the result not being overall maximizing behavior, but a sequence of more or less independent and possibly inconsistent decisions. A major problem of research is the formulation of alternatives to the hypothesis of rational behavior, because there is an infinity of possibilities. A good example is Simon's (1955) well-known *satisficing* hypothesis. As a further example note that in applied macroeconomics, due to its empirical and practical orientation, there is no major direct concern with rationality hypotheses (Arrow, 1987). Moreover, as Krusell and Smith (1996) have shown, unless effort costs of using a sophisticated optimum solution are extremely low, in macroeconomic equilibrium a positive fraction of *rational* agents will use a simple rule of thumb (e.g. a fixed saving rate). Since this is not the place for going further into detail here, the reader is referred to Conlisk (1996) as a recent review of bounded rationality in economics.

It is the author's conviction that unbounded rationality could only be a reasonable hypothesis in positive economics if it were possible to show that its implications are robust against deviations from rationality. As Buhr and Christiaans (2001) have emphasized, unbounded rationality is incompatible with modern brain research results. The capabilities of human senses are limited and the development of the telencephalon in the human brain is still in the state of immaturity. The diencephalon still plays a decisive role in shaping the actions of human beings. But the functions of the diencephalon are structured in the form of programs which may be interpreted as a set of rules. This substantiates the importance of rules of thumb in modeling human behavior instead of assuming unbounded rationality. It should be noted that such rules are not only of importance for positive, but also for normative economics. For example, the theory of investment evaluation, particularly cost-benefit analysis, is mainly concerned with the derivation of simple decision rules from rational problem solutions. A good rule of thumb from a normative point of view is a rule which generates a value of the objective function close to the optimum value. If such a rule appears to be simple and plausible, it may also be reasonable to use it as a working hypothesis about actual economic behavior. As has been shown by Buhr and Christiaans (2001) drawing on the example of a monopolistic firm by Baumol and Quandt (1964), it is possible that a simple and plausible rule of thumb leads to the true optimum. Such a result can raise the reliance on the positive implications of basic neoclassical models.

There is an important difference between rules of thumb for static and dynamic economic models. In a static model, one only has to look for some kind of method to find

the roots of a (system of) equation(s) describing the optimum solution. This approach is similar to mathematical procedures such as Newton's method for solving a system of equations. An important example of a mathematical method for solving static optimization problems which can in fact be given an economic interpretation as a stable resource allocation process is the gradient method which has been extensively analyzed in Arrow et al. (1958). While the gradient method is explicitly concerned with the determination of the optimum solution for a known economic structure, the monopolistic rule of thumb mentioned in the preceding paragraph does even work without explicit knowledge of the involved equation. In any case, these examples show that it is possible at least on principle to assume that economic agents behave like being near an optimum in static problems. It is important to note that these static problems are augmented by dynamic adjustment processes, the stability of which implies that the solution may be approached.

The optimum in dynamic models, however, is not just the root of a system of equations but a solution of a system of differential equations constituting a boundary value problem which is even difficult to approximate by a mathematical method if the underlying model is completely known. Part of this difficulty stems from the fact that the optimum paths are usually saddle paths. The slightest error in determining the correct initial values will take the proposed solution further and further away from the optimum. Clearly, these problems are reinforced if the underlying model is not completely known, that is, if the agents do not know the true structure of the economy. Since reality is much more complex than even the most difficult model of economic behavior, it is more than likely that agents are subject to errors with regard to the initial values necessary to follow a saddle path in a dynamic optimization model. Moreover, it is difficult to imagine a *meta-dynamic* adjustment process which could augment the dynamic model in the same way as the gradient process can be used to augment a static optimization problem, e.g. Thus, even if individuals were well trained in Pontryagin's maximum principle and if one disagreed with the arguments of behavioral economists against unbounded rationality, a *positive* theory of economic growth cannot be reasonably based on unstable dynamic processes.

The general saddle path property of solutions to dynamic optimization problems concerns their representation in *open loop*, which is usually obtained from the application of Pontryagin's maximum principle. If closed form pure *feedback* solutions – which provide the optimum values of the control variables as functions of the state variables in the form of policy functions – were available, this problem would cease to exist. Such closed form feedback solutions are difficult if not impossible to obtain even for extremely simplified problems, however. Hence, it will be argued that households and firms cannot be supposed to behave according to the optimum solutions of dynamic optimization problems. For the open loop representation is generally unstable while the feedback representation is usually not available. The best that can be expected is that the agents follow some kind of simplified feedback rules, which may or may not come close to the real optimum. I will propose a simple rule of thumb that can be shown to perform reasonably well from a normative point of view and which is plausible enough to be used as a working hypothesis in positive growth economics.<sup>1</sup>

While this paper is rather critical about the standard dynamical optimization approach

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<sup>1</sup>Simple decision rules which come close to the optimum rules or even coincide with them in the special case in which a parameterized model does admit an analytic solution have already been discovered in the context of stochastic discrete time growth models, cf. e.g. Christiano (1990, p. 101).

to economic growth, it nevertheless concludes with some good news for neoclassical growth economics. The proposed rule of thumb often has implications similar to those of the standard optimization model. E.g., the steady state of the dynamic optimization approach and the rule of thumb growth model may coincide; all the results that are implied by the supply side in the steady state are therefore also the same. Moreover, as is shown in a companion paper (Christiaans, 2001), in the presence of learning by doing the external effects of learning by doing in a rule of thumb economy can be internalized by the steady state tax cum subsidy policy implied by a decentralized dynamic optimization approach. This result raises the reliance on the implications of neoclassical growth economics. It should be noted, however, that much empirical work needs to be done in order to decide whether households actually use rules of thumb similar to the one proposed here. As a matter of fact, the number of possible consumption hypotheses is infinite, and of course the most of them will lead to different conclusions than the hypothesis of unbounded rationality.

The organization of the paper is as follows. In section 2, I argue that dynamic optimization models are generally unstable and can therefore not be applied as positive theories of economic growth, just as a physicist does not use the saddle path of the mathematical model of the pendulum to predict that a pendulum will usually be staying straight up. The consequences of instability are illustrated by means of numerical examples. A simple rule of thumb which generalizes the classical savings function is introduced in section 3 as an alternative and compared to the Ramsey-Koopmans-Cass approach to optimal economic growth as interpreted by Blanchard and Fischer (1989), e.g. It is shown that this rule of thumb performs reasonably well both from a normative and from a positive point of view. An extended model taking capital depreciation and technological progress into account is briefly analyzed with regard to empirically estimated rates of convergence. It is shown that the model is in a sense more naturally consistent with these empirical estimates than the Ramsey-Koopmans-Cass approach. The final section offers concluding remarks and discusses some generalizations. E.g., the proposed rule of thumb is a reasonable alternative also in more general settings than in the basic neoclassical model.

## 2 The General Instability of Dynamic Optimization Solutions

### 2.1 Optimal Economic Growth and the Mathematical Pendulum

The application of Pontryagin's maximum principle to problems of economic growth implies the solution of either a Hamiltonian system (HS) (in case of no discounting) or a perturbed Hamiltonian system (PHS) (in case of discounting) of differential equations. As Kurz (1968) has shown by generalizing a classical result of Poincaré, if  $\lambda$  is an eigenvalue of a linearized HS (or PHS) around an equilibrium, then so is  $-\lambda$  (or  $-\lambda + \rho$ , where  $\rho > 0$  designates the rate of time preference). If there are no eigenvalues with zero real part, these results imply that equilibria of HS or PHS are saddle-points, which are unstable by definition. If  $\rho > 0$ , stable equilibria are impossible. Further research by Levhari and Liviatan (1972) and Samuelson (1972) led to the result that in concave models of economic growth saddle-points are indeed the rule rather than the exception even if  $\rho = 0$ .

Kurz (1968) provided an interesting interpretation of the *Hahn problem*, according to which the balanced growth path in competitive equilibrium models is unstable. While Hahn (1966) attributed the instability to the existence of heterogeneous capital goods, Kurz (1968) has shown that instability may even arise in the standard one sector neoclassical growth model. The equilibrium in the Solow (1956) growth model is known to be globally asymptotically stable given a fixed saving rate,  $s$ . If, however, one interprets this saving rate  $s$  as being the result of a dynamic optimization problem, the stability property is the result of having a *feedback solution* for consumption per capita,  $c(k) = (1 - s)f(k)$ , rather than an *open-loop solution* of the original problem ( $k$  denotes capital per capita and  $f$  is the per capita production function).<sup>2</sup> The stability of the model is therefore just a reflection of the fact that for different  $k(0)$  one chooses different values of the shadow prices and therefore of consumption per capita  $c(k(0))$  which put the system on the stable arm of the saddle path. The main implication of this result for the present paper is that while dynamic optimization problems have generally unstable open-loop solutions, the equivalent feedback solutions, if available, may be stable.

Somewhat surprisingly, however, it has become usual in economics to refer to the saddle-points in open-loop representation as being *stable* rather than *unstable*. There is a whole branch of literature dealing with the so-called (*global*) *asymptotic stability of optimal control*, cf. e.g. Cass and Shell (1976b) and Brock and Scheinkman (1976). It is important to recognize that these authors call the solutions of dynamic optimization problems *stable* if the optimum solution converges to a steady state along the stable manifold of a saddle point. The equilibrium of the underlying dynamical system, a saddle point, is by no means stable but unstable by definition. This questionable kind of *saddle-path stability*, however, provides the basis for the positive interpretation of dynamic optimization growth models mentioned before. It is the author's conviction that it is not possible to make an unstable model stable by redefining the properties of a saddle point. While feedback solutions could provide a way out of this dilemma, it should be noted they are usually not available, at least not in closed form.

It is interesting to note that the situation is similar to the instability of neoclassical models of money and growth. By now, such models are usually interpreted as being stable by following the procedure of Sargent and Wallace (1973) who replaced the usual (backward) solution of a (system of) differential equation(s) by a so-called *forward solution*. This approach requires a variable such as the price level to be a discontinuous function of time in order to jump to its appropriate value yielding a saddle-path stable solution. Note that the *jump variable* must suddenly move to an *exactly specified value*. Otherwise, the slightest error will move the system further and further away from the saddle path. In dynamic optimization, the jump variables are the controls or the costates which are required to take on exactly the initial value(s) necessary to reach the stable manifold.

The problems involved may be revealed more clearly by considering a similar situation in physics, the mathematical model of a (frictionless) pendulum. A special case of this model can be described by the following pair of differential equations emanating from

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<sup>2</sup>The task of detecting the utility function for which a constant saving rate  $s$  is the optimum solution is known as the *inverse optimal growth problem*.

the Hamiltonian function  $H(x, y) = y^2/2 + \int_0^x \sin(\tau) d\tau$  (cf. e.g. Perko, 1996, p. 172):

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -\sin(x).\end{aligned}$$

The phase portrait of this system is shown in figure 1.

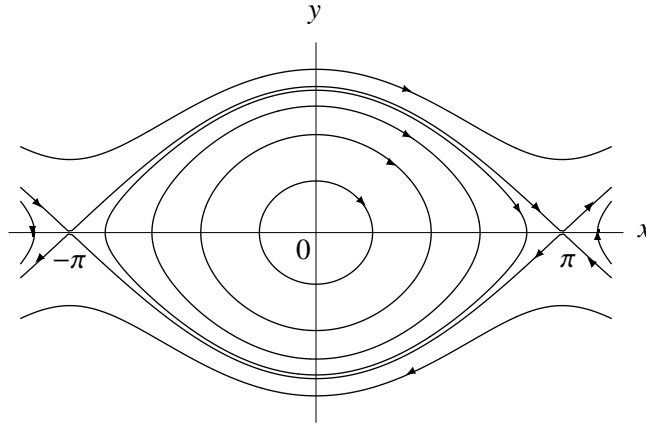


Figure 1: Phase Portrait of the Undamped Pendulum

The variable  $x$  describes the angle by which the pendulum deviates from its stable equilibrium position hanging straight down, and  $y$  is the change of this angle in time. The origin in figure 1 corresponds to this stable equilibrium while the saddle-points  $(\pm\pi, 0)$  are the unstable equilibria where the pendulum is straight up. The closed trajectories encircling the origin describe the periodic motions associated with a pendulum swinging back and forth. The separatrices connecting the saddles are the paths where the pendulum approaches its unstable vertical position as  $t \rightarrow \pm\infty$ . The trajectories outside these separatrix loops correspond to motions where the pendulum goes over the top.<sup>3</sup>

No physicist would conclude from the existence of a saddle-path approaching the points  $(\pm\pi, 0)$  that a pendulum will usually be staying straight up. But so do economists. Consider the standard Ramsey-Koopmans-Cass (RKC) model which is nowadays usually interpreted as a *positive* theory of economic growth (cf. e.g. Blanchard and Fischer, 1989; Barro and Sala-i-Martin, 1995). Under assumptions specified in detail in section 3, this model implies the following pair of differential equations obtained from a PHS of necessary optimum conditions:

$$\begin{aligned}\dot{c} &= \frac{1}{\theta}(f'(k) - n - \rho)c, \\ \dot{k} &= f(k) - nk - c,\end{aligned}\tag{1}$$

where  $c$  and  $k$  denote consumption respectively capital per capita,  $f$  the per capita production function,  $n$  the growth rate of population,  $\theta$  the inverse intertemporal elasticity of substitution, and  $\rho$  the rate of time preference.

<sup>3</sup>The more realistic mathematical model of a damped pendulum has a similar but more complicated structure. The major distinction is that the stable equilibrium with the pendulum hanging straight down is not only stable but asymptotically stable. The equilibria with the pendulum straight up remain saddle points, however.



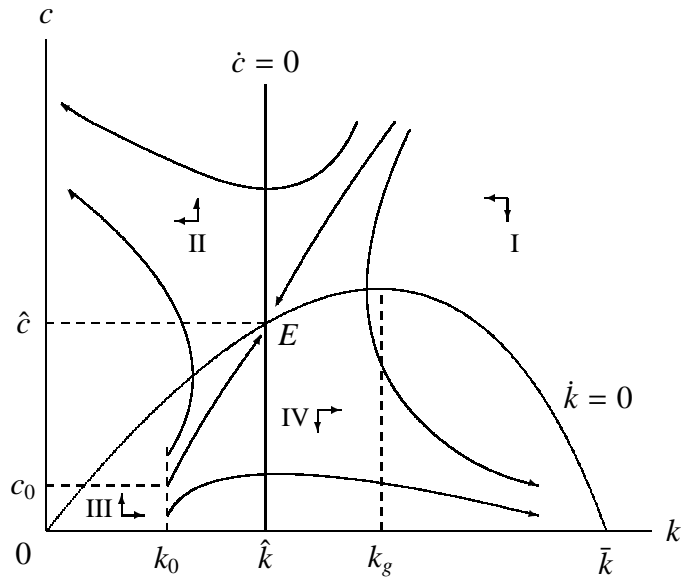


Figure 2: Phase Portrait of the Ramsey-Koopmans-Cass Model

As can be seen in figure 2, the unique equilibrium  $E$  of system (1) in the positive orthant is a saddle-point, the stable arms of which represent the optimum (and alleged competitive) solution. In order to reach the equilibrium  $E$  asymptotically from an initial situation with capital per head  $k_0$ , it is necessary to jump on the stable arm by choosing consumption  $c_0$  and then following the Keynes-Ramsey rule of consumption specified in equation (1). But the slightest error in the initial condition  $c_0$  – the determination of which usually requires numerical procedures – will lead the economy further and further away from the saddle-path. Supposing that a competitive economy – in which agents certainly are mistaken with respect to some decisions – follows the differential equations (1) therefore implies that, with probability measure one, this economy will end up either in quadrant II with the capital stock being eaten up in finite time, or to converge to the asymptotically *stable* equilibrium  $(\bar{k}, 0)$  with no consumption. But this is an interpretation of the RKC model which I never came about.

There is one important difference between the mathematical pendulum and the RKC model. While there seems to be no obvious reason why the pendulum should follow a saddle-path in order to be staying straight up eventually, following the saddle-path is a necessary optimum condition in the RKC model. The analogy between both of the models can be made complete, however, by asking a physicist how he would evaluate the probability that all people having access to a pendulum could set it in a certain initial position and give it the appropriate impulse to follow the saddle-path and reach the vertical unstable position. Even if a high prize for all successful participants was awarded, the physicist would probably not risk to make himself ridiculous by stating a finite positive probability.

Of course, one may argue that people always do re-optimize after some time of adjustment in order to prevent from going too far away from the saddle-path. It should be kept in mind, however, that every model can only be a rough approximation of reality, and that real life optimization problems are much more complicated than the elementary problem considered here. Even the calculation of the appropriate initial value  $c_0$  requires

numerical methods in general, and a recalculation would be necessary very often. In fact, such a nearly continuous re-optimization comes close to the determination of a *feedback* solution (as opposed to the open-loop solution of the Maximum Principle). It is the author's conviction that the numerical problems involved with the instability of saddle-paths in applications are often overlooked. In order to clarify this point, an even simpler problem which admits explicit solutions is analyzed in the next subsection with emphasis on the problem of numerical errors.

## 2.2 Infinite Horizon Utility Maximization

Consider the following model of consumer optimization:<sup>4</sup>

$$\begin{aligned} \max_{c \in \bar{C}[0, \infty)} \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \quad \theta > 0, \\ \text{subject to} \\ \dot{b} = w + rb - c, \quad b(0) = b_0, \quad \liminf_{t \rightarrow \infty} e^{-rt} b(t) \geq 0, \\ c(t) \geq 0, \end{aligned} \tag{2}$$

where  $c$  denotes consumption,  $b$  real asset holdings yielding an interest rate  $r$ ,  $\theta$  the inverse intertemporal elasticity of substitution,  $\rho > 0$  the consumer's rate of time preference, and  $w$  the wage rate (the household supplies one unit of labor every period). For simplicity,  $\theta$ ,  $\rho$ ,  $r$  and  $w$  are all assumed to be constant. For the problem to have a non-trivial solution, it is assumed that  $\liminf_{t \rightarrow \infty} e^{-rt} b(t) > 0$  if  $c(t) \equiv 0$ . The symbol  $\bar{C}[0, \infty)$  denotes the set of piecewise continuous real valued functions defined on  $t \in [0, \infty)$ .

For simplicity,  $\theta = 1$  is assumed, which leads to the instantaneous utility function  $\lim_{\theta \rightarrow 1} (c^{1-\theta} - 1)/(1-\theta) = \ln(c)$ . A standard application of Pontryagin's maximum principle<sup>5</sup> implies the Keynes-Ramsey rule of consumption and therefore the system of differential equations

$$\begin{aligned} \dot{c} &= (r - \rho)c, \\ \dot{b} &= w + rb - c, \end{aligned}$$

with boundary values  $b(0) = b_0$  and  $\liminf_{t \rightarrow \infty} e^{-rt} b(t) \geq 0$ . A straightforward calculation shows that the unique solution satisfying both the necessary and the Mangasarian sufficient conditions is

$$\begin{aligned} c^*(t) &= \rho \left( b_0 + \frac{w}{r} \right) e^{(r-\rho)t}, \\ b^*(t) &= \left( b_0 + \frac{w}{r} \right) e^{(r-\rho)t} - \frac{w}{r}. \end{aligned}$$

It is easily seen that the following feedback representation of the solution is possible:

$$c^*(t) = \rho \left[ b^*(t) + \frac{w}{r} \right]. \tag{3}$$

<sup>4</sup>Note that it is not possible to impose the terminal condition  $\liminf_{t \rightarrow \infty} b(t) \geq 0$ , since for  $\rho > r$  the problem would not have a solution then. For an alternative to using the non-negativity of the present value of assets as a terminal condition, cf. Arrow and Kurz (1970, S. 155–156).

<sup>5</sup>A comprehensive source of optimal control methods for economists is Seierstad and Sydsæter (1987).

In order to simplify the problem further, it is assumed that  $w = 0$ , that is, the household has no wage income. The optimum solutions in open-loop respectively feedback-form are now

$$\begin{aligned}c^*(t) &= \rho b_0 e^{(r-\rho)t}, \\b^*(t) &= b_0 e^{(r-\rho)t}, \\c^*(t) &= \rho b^*(t).\end{aligned}\tag{4}$$

Suppose a household has had a course in Pontryagin's maximum principle and now has calculated his optimum consumption path according to (4). Because of inaccurate information, however, his calculations involve slight errors. E.g., let the discount rate be  $\rho = 0.1$  and let the interest rate identified by a bank be  $r = 0.1$  (= 10, 00%), accurate up to two decimal places as is usual in reality. An exact calculation would lead to  $r = 0.09995$  (= 9, 995%). Initial asset holdings are  $b_0 = 100$ . Consider the open-loop solution at first. Since  $r = \rho$  from the viewpoint of the household, (4) implies  $c(t) = 0.1 \cdot 100 \cdot e^0 = 10$ . The time path of  $b(t)$  then follows the differential equation  $\dot{b} = 0.09995b - 10$  with initial condition  $b_0 = 100$ . The solution is

$$b(t) = -0.05003e^{0.09995t} + 100.05003.$$

Setting  $b(t) = 0$  and solving for  $t$  yields

$$t = \ln(100.05003/0.05003)/0.09995 = 76.04605,$$

that is, the assets – actually planned to last forever – are used up after a period of about 76 years. Compared to eternity, 76 years are nil. The reason is solely the inaccurate third decimal place of the identified interest rate.

Now turn to the feedback-solution. The consumption function is  $c(t) = 0.1b(t)$ , and asset holding follows the differential equation  $\dot{b} = 0.09995b - 0.1b = -0.00005b$  with solution

$$b(t) = 100e^{-0.00005t},$$

which is actually the optimum solution. While the open-loop solution is disastrous due to the inaccurate interest rate used, the feedback solution is not affected at all by this error (apart from the fact that the household will be astonished at his assets getting smaller and smaller while he has planned to hold them constant). The reason is, of course, that the feedback solution in the present example is independent of the interest rate. In general, the solution will not be as good as in this case.

As a further example, suppose that  $r = 0.1$  and  $\rho = 0.085$ . The household does know both values exactly. Initial asset holdings are  $b_0 = 100$ . The optimum solution is  $c^*(t) = 0.085 \cdot 100e^{0.015t}$ . The household, however, decides to choose rounded values of the optimum solution at any point in time. The function *Round* associates with every real number its nearest integer. Thus, consumption at time  $t$  is given by

$$c(t) = \text{Round}(c^*(t)) = \text{Round}\left(0.085 \cdot 100e^{0.015t}\right),$$

and the time path of  $b(t)$  follows the discontinuous differential equation

$$\dot{b} = 0.1b - \text{Round}(c^*(t)).$$

The solution of this equation has been calculated numerically with *Mathematica*.<sup>6</sup> figure 3 shows the results of the rounded solution ( $c(t)$  step curve and  $b(t)$  dashed curve) up to time  $t = 100$  in comparison to the optimum solution (smooth curves for  $c(t)$  and  $b(t)$ ). The numerical errors lead to a disastrous result. At  $t = 59.3625$ , asset holding  $b(t)$  becomes zero, which implies also that  $c(t) = 0$  for  $t \geq 59.3625$ . In contrast, asset holding grows without bound for the optimum solution.

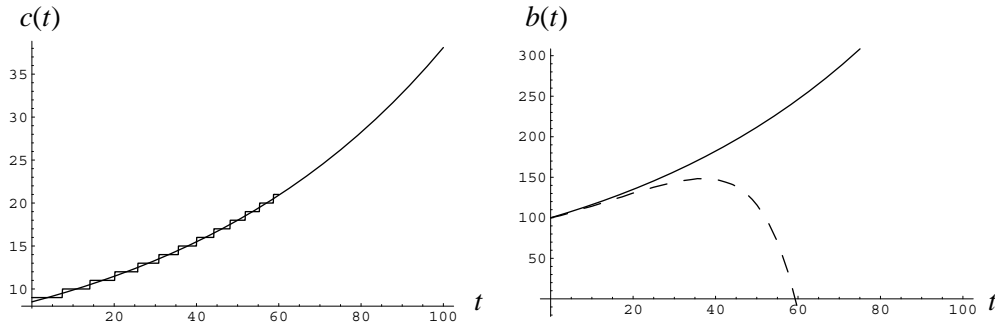


Figure 3: Open-Loop solution and Rounded Open-Loop Solution

The rounded feedback solution for consumption is

$$c(t) = \text{Round}(0.085b(t)),$$

which implies

$$\dot{b} = 0.1b - \text{Round}(0.085b).$$

The numerical solution obtained with *Mathematica* is shown in figure 4,<sup>7</sup> again compared to the optimum solution (solid lines). The figures provide ample evidence on the superiority of the feedback solution compared to the open-loop solution. The rounded feedback solution does not deviate much from the optimum for a period of 100 years. Even for 200 years, it performs reasonable well. Denoting the rounded feedback solution by the index  $r$ , one obtains the following figures:  $b^*(200) = 2008.56$ ,  $b^r(200) = 1956.93$ .

The superiority of feedback solutions follows from the fact that rounding errors do not continue as in the case of open-loop solutions. While the provided examples are only special cases, experiments with other figures reinforce that feedback solutions are generally better.

Finally, a warning is in order. First, it is possible to produce numerical examples in which even the feedback solution is very sensitive to numerical errors (e.g., choose

<sup>6</sup>*Mathematica* is a registered trademark of Wolfram Research, Inc. The following *Mathematica* input has been used to produce figure 3:

```
sol=NDSolve[a'[t]==0.1 a[t]-Round[0.085*100*Exp[0.015 t]],
  a[0]==100,a,t,0,200,MaxSteps->2000]
at=a/.sol; FindRoot[at[t]==0,t,40,41,MaxIterations->100]
Plot[a[t]/.sol,t,0,60,PlotStyle->Dashing[0.04]]
Plot[Round[0.085*100*Exp[0.015 t]],t,0,60]
```

<sup>7</sup>Figure 4 requires the following commands:

```
sol=NDSolve[a'[t]==0.1 a[t]-Round[0.085 a[t]],a[0]==100,
  a,t,0,200,MaxSteps->4000]
Plot[a[t]/.sol,t,0,100,PlotStyle->Dashing[0.04]]
Plot[Round[0.085 a[t]/.sol],t,0,100]
```

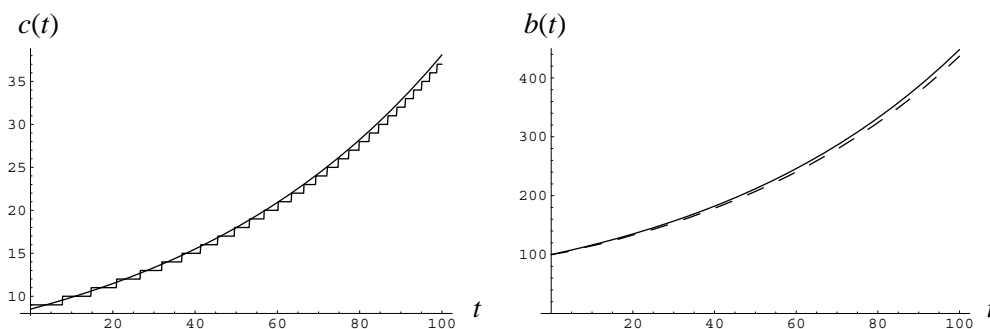


Figure 4: Feedback Solution and Rounded Feedback Solution

$\rho = 0.095$  in the last example). Second, feedback solutions are generally difficult to obtain; note that the model considered here is extremely simple. In fact, it is a usual practice to approximate the feedback solution (which is sometimes called the *policy function*) by using numerical methods for solving the PHS obtained from Pontryagin's maximum principle rather than trying to solve the Hamilton-Jacobi-Bellman equation of dynamic programming,<sup>8</sup> which would lead more directly to the feedback representation of the solution. Moreover, the information structure must be sufficient to support a feedback solution in that the agent must know the values of the state variables at every point in time.

The principal conclusions for the rest of the paper may be summarized as follows:

- Open-loop solutions of dynamic optimization problems are unstable. Therefore, they provide no reasonable basis for a positive theory of economic growth.
- Feedback solutions would be much more reliable if it were reasonable to suppose that people do know such solutions. Unfortunately, the calculation of feedback solutions is far too much involved to allege that people (even economic specialists) do know them.
- The analysis of reasonable and simple feedback rules of thumb is therefore a promising alternative to dynamic optimization solutions. From a normative point of view, it is interesting to analyze such rules with regard to the implied values of the criterion function. If good rules of thumb are available, it is also much more appealing in positive applications to suppose that people follow such rules (perhaps by imitation) rather than that they do calculate the optimum solutions themselves.

### 3 The Ramsey-Koopmans-Cass Model and a Golden Rule of Thumb

#### 3.1 Review of the Ramsey-Koopmans-Cass Model

The Ramsey-Koopmans-Cass (RKC) model of optimal economic growth (along with its variations) has become the workhorse model of modern macroeconomics and especially

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<sup>8</sup>Cf. e.g. Barro and Sala-i-Martin (1995, p. 488–491) on the time-elimination method. The Hamilton-Jacobi-Bellman equation is a partial differential equation, the solution of which is only known for some very special cases.

the theory of economic growth. A decentralized equilibrium version of the model is described in detail in Blanchard and Fischer (1989). Given the assumptions of perfect competition and perfect foresight of the individual agents, this equilibrium version implies the same aggregate dynamics as the original centralized version. It should be emphasized, however, that the usually frankly made assumption of perfect foresight in such models is of an extreme nature. E.g., individual agents must know the complete future time paths of the interest and the wage rate, which in turn depend on the aggregate capital-labor ratio and the production function. Any critical evaluation of the instability of the aggregate model should therefore be reinforced if the decentralized interpretation is considered. For the present purposes, consider the following version of the aggregate RKC model:

$$\begin{aligned} & \max_{c \in \bar{C}[0, \infty)} \int_0^{\infty} \frac{c^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \quad \theta > 0, \\ & \text{subject to} \\ & \dot{k} = f(k) - nk - c, \quad k(0) = k_0, \quad \liminf_{t \rightarrow \infty} k(t) \geq 0, \\ & c(t) \geq 0, \end{aligned} \tag{5}$$

where the objective is the same as in (2) with  $c$  now denoting consumption *per capita*. The variable  $k$  denotes capital per capita,  $f$  is the per capita production function, and  $n$  designates the growth rate of labor, which is exogenously given and constant.

A standard application of Pontryagin's maximum principle implies the differential equations (1), which are repeated here for convenience:

$$\begin{aligned} \dot{c} &= \frac{1}{\theta} (f'(k) - n - \rho)c, \\ \dot{k} &= f(k) - nk - c, \end{aligned} \tag{1}$$

and the solution of these equations is described by the phase diagram in figure 2 provided before. The unique equilibrium  $E$  in the positive orthant is defined by

$$\begin{aligned} f'(\hat{k}) &= n + \rho, \\ \hat{c} &= f(\hat{k}) - n\hat{k}. \end{aligned} \tag{6}$$

The optimum solution is represented by the stable arms of the saddle point  $E$ , the instability of which has been discussed before.

An aspect of the RKC model important for the subsequent discussion is the value  $k = k_g$  at which the isocline  $\dot{k} = 0$  attains its global maximum in figure 2. This value of capital per head corresponds to the well known *golden rule of accumulation* (Phelps, 1967),

$$\left. \frac{dc}{dk} \right|_{\dot{k}=0} = f'(k) - n = 0 \quad \implies \quad k = k_g. \tag{7}$$

It follows from  $\rho > 0$  and  $f''(k) < 0$  that  $k_g > \hat{k}$ . Obviously, per capita consumption  $c$  attains its maximum at  $k_g$  if the restriction  $\dot{k} = 0$  is imposed. Thus,  $k = k_g$  is the aggregate capital-labor ratio corresponding to the maximum steady state per capita consumption. It is straightforward and well known that the golden rule can equivalently be expressed by the condition that savings equal capital income in the steady state. In case of  $\rho = 0$ ,

$\hat{k}$  would equal  $k_g$ .<sup>9</sup> Hence, it comes at no surprise that the steady state path starting and remaining in  $E$  is sometimes called the *modified golden rule path* in case of  $\rho > 0$ . It is important to note that both the golden rule and the modified golden rule path are independent of the special form of the integrand in (5). The parameter  $\theta$  does not appear in the equations (6) determining the equilibrium. This result, however, is special to the model considered here and does not continue to hold if technological progress is introduced,<sup>10</sup> see equations (17) below.

### 3.2 A Golden Rule of Thumb

The literature on the golden rule of accumulation is usually mainly concerned with the implications of this rule for the efficiency of steady state equilibria. It is the purpose of this section to redirect attention to the properties of the golden rule – formulated in terms of a classical savings function – off the steady state, and to generalize it by taking a positive rate of time preference,  $\rho > 0$ , into account. As this generalization is shown to be a good rule of thumb even off the equilibrium growth path and does require only a minimum amount of information, I choose to call it a *golden rule of thumb*.

The well known equation of motion in the neoclassical model of economic growth in the case of a constant saving rate  $s$  is

$$\dot{k} = sf(k) - nk. \quad (8)$$

The constant saving rate associated with the golden rule of accumulation (7) can be calculated to be

$$s_g := \frac{k_g f'(k_g)}{f(k_g)}.$$

It follows from the stability properties of the Solow model that if the representative (or all identical) agent(s) in an economy choose(s) this constant saving rate, the economy will converge to the balanced growth golden rule path with aggregate capital-labor ratio  $k_g$ . Choosing the constant saving rate  $s = s_g$  might therefore be considered as a good rule of thumb in order to asymptotically reach the golden rule path, which is an optimum solution of problem (5) if  $\rho = 0$  and  $k_0 = k_g$ . However, this saving rate  $s_g$  has to be calculated by the households, and this implies that they must know the underlying economic structure including the production function completely.

A less demanding rule to follow is therefore the classical savings function. Let  $S$  denote savings,  $r$  the interest rate,  $K$  the aggregate capital stock, and  $L$  the population, which for simplicity is assumed to equal the labor force. The linearly homogeneous production function is  $Y = F(K, L)$ , or in per capita terms  $y = Y/L = F(K/L, 1) \equiv f(k)$ . In a setting of perfect competition,  $r$  equals the marginal productivity of capital,  $f'(k) = F_K(K, L)$ . The classical savings function is  $S = rK = f'(k)K$ . Since the production function is assumed to be linearly homogeneous, saving the entire capital income is equivalent to consuming

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<sup>9</sup>If  $\rho = 0$ , the integral in (5) does not converge. Therefore, a more general optimality criterion such as catching up-optimality is required.

<sup>10</sup>Cf. e.g. Barro and Sala-i-Martin (1995, p. 72). Note that the steady state in their model is different from the one considered here, because they effectively discount utility with  $e^{-(\rho-n)t}$ . In my opinion, a higher population *per se* should not increase the utility from consumption per capita, which is implied by their formulation, however.

the entire wage income. Denoting aggregate consumption by  $C$  and the wage rate by  $w$ , the classical consumption function is therefore

$$C = wL, \quad \text{respectively} \quad c = w \quad \text{in per capita terms.}$$

While it is known that the neoclassical growth model is stable even if the saving rate is not constant but follows the classical savings function, the aspect that this rule leads asymptotically to the golden rule path has not received much attention in the literature.<sup>11</sup>

Consider the following generalization of the classical consumption function:

$$C(t) = w(t)L(t) + \rho K(t), \quad \text{respectively} \quad c(t) = w(t) + \rho k(t) \quad \text{in per capita terms.} \quad (9)$$

According to this rule, households consume their wage income and a share of their assets (which equal capital in this model) equaling their rate of time preference. That is, households consume more than their wage income if they own a positive stock of assets and discount the future. This kind of behavior appears to be intuitively plausible. Moreover, if  $\rho = r$ , equation (9) resembles the optimum feedback solution (3) of problem (2), which may be considered as a simplified version of the current optimization problem because  $w$  and  $r$  have been treated as constants in (2). It will be shown that (9) performs reasonably well even if applied to the more general problem (5). Thus, the procedure of deriving simple feedback rules under simplifying assumptions and applying them to more general cases seems to be promising.

If  $\rho > 0$ , a correction of the classical savings function by increasing consumption is necessary from a normative point of view. For any growth path maintaining a value of  $k > k_g$  in the long run is dynamically inefficient (Phelps, 1967, S. 56–61), and any saving rate which is permanently higher than  $s_g$  implies that  $k$  is higher than  $k_g$  eventually. Thus, every reasonable rule of thumb should not converge to a saving rate exceeding  $s_g$ . Since a positive rate of time preference  $\rho > 0$  means that current consumption is preferred to future consumption, even the saving rate  $s_g$  is too high in the long run.

The foregoing discussion substantiates the use of the consumption function (9) from a normative point of view. But even as a positive theory of economic behavior the assumption that households use equation (9) to determine their consumption appears to be more plausible than to assume that they solve a dynamical optimization problem with perfect precision. Spending one's wage income plus a share of assets depending on the rate of time preference is a sensible rule of thumb which does not require the solution of any optimization problem nor any knowledge of the true structure of the economy at all. While households must know this structure if they try to determine their saving rate according to the modified golden rule of accumulation (which leads to the modified golden rule path), it will be shown that they approach the appropriate value automatically by consuming according to (9). In order to do so, only knowledge about the wage rate and the own assets and rate of time preference are required. This is why I choose to call (9) a *golden rule of thumb*.

It should be noted, however, that this rule is only reasonable for a single household if there is some intergenerational altruism. If the household has no *bequest motive* for saving, any reasonable savings rule should not leave a positive amount of assets at the end

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<sup>11</sup>E.g., there is nothing to be found on the importance of the classical savings function off the steady state in the monograph on the golden rule by Phelps (1967). An exception is Burmeister (1980, p. 60).



of the planning horizon. Of course, this problem does not only concern the golden rule of thumb but also the optimum solution of the RKC model or any rule with a constant saving rate. It follows that all such rules are reasonable only if they are applied to an aggregate model with altruistic linkages across generations. Since we are living in a *generation of heirs*, this is certainly an innocuous assumption.

The dynamics of the neoclassical growth model with the consumption function (9) are derived under the usual assumptions on perfect competition. The wage rate is

$$w = f(k) - kf'(k).$$

Substituting into (9) yields

$$c = f(k) + (\rho - f'(k))k.$$

Using this expression in the dynamic equation

$$\dot{k} = f(k) - nk - c$$

one gets

$$\dot{k} = (f'(k) - n - \rho)k. \quad (10)$$

Given the usual neoclassical assumptions on the production function (homogeneous of degree one, positive decreasing marginal productivities, and the Inada-conditions), the right hand side of equation (10) is continuous for  $k > 0$  and there is a unique equilibrium  $\hat{k}$  defined by

$$f'(\hat{k}) = n + \rho. \quad (11)$$

Per capita consumption in the steady state is

$$\hat{c} = f(\hat{k}) + (\rho - f'(\hat{k}))\hat{k} \stackrel{(11)}{=} f(\hat{k}) - n\hat{k}.$$

Thus, the conditions (6) are met in the steady state. If the initial value of  $k$  was  $k_0 = \hat{k}$ , the golden rule of thumb would therefore imply the optimum solution which coincides with the modified golden rule of accumulation. The main advantage of using (9) is, however, that the optimum steady state values will be approached asymptotically even if  $k_0 \neq \hat{k}$  for all  $k_0 > 0$ . To prove this, simply note that the equilibrium  $\hat{k}$  is unique and the right hand side of (10) is continuous in the positive region. The derivative with respect to  $k$ , evaluated at the equilibrium, is

$$\left. \frac{\partial \dot{k}}{\partial k} \right|_{k=\hat{k}} = f''(\hat{k})\hat{k} + f'(\hat{k}) - n - \rho \stackrel{(11)}{=} f''(\hat{k})\hat{k} < 0,$$

which together with the uniqueness implies the global stability of  $\hat{k}$  for  $k > 0$ . (Readers interested in a formal proof should consider the Lyapunov function  $V = 0.5(k - \hat{k})^2$ .)

### 3.3 Numerical Comparison of Solution Concepts

The golden rule of thumb (9) implies convergence to the steady state of the RKC model. In order to evaluate the out of equilibrium performance of this rule, the values of the objective functional in (5) are calculated numerically using four different hypotheses about

the consumption decision. The following parameterization and numerical values have been used:

$$\begin{aligned} f(k) &= ak^\alpha, \quad a = 10, \alpha = 0.3, \\ u(c) &= \ln(c) \quad (\theta = 1), \quad \rho = 0.02, n = 0.01. \end{aligned}$$

The calculations are summarized as follows:

1. Optimum solution (RKC): The equations (1) now read

$$\begin{aligned} \dot{c} &= (3k^{-0.7} - 0.03)c, \\ \dot{k} &= 10k^{0.3} - 0.01k - c, \end{aligned}$$

which implies the steady state values

$$\hat{k} = 719.686, \quad \hat{c} = 64.77174.$$

In order to get sufficiently different numerical results, the far off equilibrium initial value  $k_0 = 100$  has been chosen. By numerical procedures, the initial value of consumption per capita which puts the system on the stable arm of the saddle point is  $c_0 = 21.4888$ .<sup>12</sup> The value of the objective functional has been calculated for a period of 140 years, in which the steady state values are nearly approached. If  $c(t)$  now denotes the optimum consumption path, the solution is

$$\int_0^{140} \ln(c(t))e^{-0.02t} dt \approx 182.403.$$

2. Golden rule of thumb: All subsequent cases lead to a Bernoulli equation in  $k$  of the form

$$\dot{k} = bk^\alpha - dk,$$

where the parameters  $b$  and  $d$  differ from case to case. The solution of this differential equation is

$$k(t, k_0) = \left[ \left( k_0^{1-\alpha} - \frac{b}{d} \right) e^{-d(1-\alpha)t} + \frac{b}{d} \right]^{1/(1-\alpha)}, \quad (12)$$

and all the subsequent results can be derived from this solution. Equation (10) now reads

$$\dot{k} = \alpha ak^\alpha - (n + \rho)k = 3k^{0.3} - 0.03k,$$

that is,  $b = \alpha a = 3$  and  $d = n + \rho = 0.03$ . Using  $\alpha = 0.3$ , (12) yields the solution

$$k(t, k_0) = \left[ (k_0^{0.7} - 100)e^{-0.021t} + 100 \right]^{1/0.7}$$

which implies the path of consumption per capita

$$c(t, k_0) = 7k(t, k_0)^{0.3} + 0.02k(t, k_0).$$

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<sup>12</sup>The calculation has been carried out with Juan M. Aguirregabirias free software *Dynamics Solver*. The numerical values have been further processed with *Mathematica*. The input files for *Dynamics Solver* and *Mathematica* are available from the author upon request.

Substituting into the objective functional and integrating numerically yields<sup>13</sup>

$$\int_0^{140} \ln(c(t, 100))e^{-0.02t} dt \approx 180.417.$$

3. Constant saving rate according to the modified golden rule of accumulation: The optimum steady state saving rate is

$$\hat{s} = 1 - \frac{\hat{c}}{a\hat{k}^\alpha} = 1 - \frac{64.77174}{71.9686} = 0.1.$$

Using  $b = \hat{s}a = 1$  and  $d = n = 0.01$ , the equation of motion (8) becomes

$$\dot{k} = k^{0.3} - 0.01k,$$

with solution

$$k(t, k_0) = \left[ (k_0^{0.7} - 100)e^{-0.007t} + 100 \right]^{1/0.7}.$$

Using the consumption function  $c = 0.9y = 9k(t, k_0)^{0.3}$ , an analogous procedure as before leads to

$$\int_0^{140} \ln(c(t, 100))e^{-0.02t} dt \approx 177.711.$$

4. Other constant saving rates: In both of the following cases is  $b = sa$  and  $d = n$ . Choosing a saving rate  $s = 0.08$  below the value  $\hat{s} = 0.1$  of the modified golden rule of accumulation implies

$$\int_0^{140} \ln(c(t, 100))e^{-0.02t} dt \approx 176.706.$$

Choosing  $s = 0.12 > \hat{s}$  leads to

$$\int_0^{140} \ln(c(t, 100))e^{-0.02t} dt \approx 178.463.$$

These results substantiate the good performance of the golden rule of thumb (9) compared to other consumption rules. It has to be noted that the saddle-path is only optimal if the planning horizon is infinite, and 140 years are very far from infinity. As has been remarked before, however, the saddle path solution is not far from the steady state after 140 years. Although the distance from the steady state of the other paths is larger, computations for longer horizons do not yield considerable deviations. For all solutions leading to the optimum steady state differ only slightly near the equilibrium, and due to the positive rate of time preference, differences between the value of the integrand in the far future

<sup>13</sup>For cases 2, 3, and 4, it is not necessary to compute the path of consumption per capita numerically as in case 1 (the RKC model). Of course, such a numerical procedure is possible and in case 2 yields the value

$$\int_0^{140} \ln(c(t, 100))e^{-0.02t} dt \approx 181.992.$$

have virtually no consequences with respect to the value of the objective functional. The path belonging to the constant saving rate  $s < \hat{s}$  approaches an equilibrium where per capita consumption is too low in the long run. Therefore, it cannot perform better than the golden rule of thumb even if the planning horizon gets longer. The path belonging to the constant saving rate  $s > \hat{s}$  approaches an equilibrium where per capita consumption is too high in the long run (note that  $s = 0.12 < 0.3$ , which is the golden rule value of the saving rate for  $\rho = 0$ ). While this implies with respect to the present calculations that this path performs somewhat better for larger horizons, further calculations reveal that the value of the objective functional remains below the one for the golden rule of thumb. It is noteworthy, however, that both the golden rule of thumb and the constant saving rate  $s > \hat{s}$  perform better than the modified golden rule of accumulation,  $s = \hat{s}$ , even if larger planning horizons are considered. The modified golden rule of accumulation, whose determination requires the solution of an optimization problem, is not as golden as the proposed golden rule of thumb. The latter does not require any optimization at all.

In summary, the numerical calculations reinforce that the proposed golden rule of thumb, which has analytically been shown to converge to the steady state optimum solution, works very well from a normative point of view. It is therefore also a reasonable working hypothesis for positive growth economics. For even if households had methods to calculate the optimum solution at their disposal, they would have to compare the additional utility of using the dynamic optimum solution with the losses associated with its costly calculation. Moreover, the proposed rule seems to be plausible even for households who do not care about calculating optimum solutions at all. Of course, only empirical evidence can help to further clarify whether actual economic behavior can indeed be described by such a rule. As a first step in this direction, it is examined in the next subsection whether a growth model with the golden rule of thumb is consistent with empirical estimates of convergence rates.

### 3.4 Convergence in a Generalized Model

The rate of convergence  $\mu$  is defined as the negative change of the gap between the capital-labor ratio  $k$  in period  $t$  and its long-run equilibrium value  $\hat{k}$  in relation to the remaining gap:

$$\mu := -\frac{d(\hat{k} - k(t))}{\hat{k} - k(t)} = \frac{\dot{k}(t)dt}{\hat{k} - k(t)} \approx \frac{\dot{k}(t)}{\hat{k} - k(t)}. \quad (13)$$

This rate  $\mu$  need not be defined with respect to the capital-labor ratio but may also concern per capita output,  $y$ , or other variables. It can be shown that in case of a Cobb-Douglas production function the rate of convergence for  $y$  is the same as that for  $k$  (cf. e.g. Barro and Sala-i-Martin, 1995, p. 37).<sup>14</sup> Note that (13) is concerned with convergence *conditional* on the steady state of a country since convergence to a specific long-run equilibrium  $\hat{k}$  which need not be the same for all countries is analyzed. In contrast, absolute convergence of several countries would occur if the steady state value  $\hat{k}$  was the same for these countries.

The rate of convergence has often been estimated empirically, the best known study

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<sup>14</sup>The rate of convergence is often called the *speed of convergence*. Note that speed has the dimension [distance/time] while  $\mu$  as defined in (13) is dimensionless.

being Mankiw et al. (1992). As a kind of empirical regularity, the rate of convergence takes on values between 2% (Mankiw et al., 1992) and 3% per year (Barro and Sala-i-Martin, 1995, p. 431). Although some other studies have estimated much higher rates of convergence up to 11% (cf. Klenow and Rodriguez-Clare, 1997, p. 605–606), many researchers believe that a reasonable model of economic growth should be capable of generating rates of convergence about 2%–3%.

In order to compare the rates of convergence implied by the previously discussed growth models with the empirical estimates, it is necessary to generalize the models by taking technological progress and the depreciation of capital into account. The most simple way to do this is to assume a constant rate of depreciation and exogenous Harrod-neutral technological progress at a constant rate. In the sequel, the golden rule of thumb model, the Solow model with a constant (gross) saving rate, and the RKC model are considered. In general, a setting of perfect competition is assumed. Since the derivations are standard, the presentation will be brief; for details cf. e.g. Barro and Sala-i-Martin (1995).

Let  $\delta$  be the constant rate of capital depreciation and  $\gamma$  the exogenous rate of Harrod-neutral technological progress. The aggregate production function with constant returns to scale reads now

$$Y = F(K, AL),$$

where

$$A(t) = e^{\gamma t}.$$

The product  $AL$  is called the *effective amount of labor*. Instead of the per capita production function,  $y = f(k)$ , the production function per effective labor will be used:

$$y = F(K/(AL), 1) =: f(k),$$

where  $y = Y/(AL)$  and  $k = K/(AL)$ . The equation of motion for  $k$  with technological progress and capital depreciation becomes

$$\dot{k} = f(k) - (n + \delta + \gamma)k - c, \quad (14)$$

where  $c := C/(AL) = ce^{-\gamma t}$ .

In case of the golden rule of thumb (9), it has to be noted that the wage rate is now  $w = [f(k) - kf'(k)]e^{\gamma t}$ . Substituting (9) into (14) and noting that  $k = ke^{-\gamma t}$  yields

$$\dot{k} = (f'(k) - n - \rho - \delta - \gamma)k. \quad (15)$$

For the Solow model with a constant saving rate  $s$ , equation (14) takes the form

$$\dot{k} = sf(k) - (n + \delta + \gamma)k. \quad (16)$$

Finally, the Keynes-Ramsey rule with technological progress and depreciation reads

$$\frac{\dot{c}}{c} = \frac{\dot{c}}{c} - \gamma = \frac{1}{\theta}(f'(k) - n - \rho - \delta) - \gamma.$$

Together with equation (14) the following differential equations describe the RKC model:

$$\begin{aligned} \dot{c} &= \frac{1}{\theta}(f'(k) - n - \rho - \delta - \gamma\theta)c, \\ \dot{k} &= f(k) - (n + \delta + \gamma)k - c. \end{aligned} \quad (17)$$

The comparison of equations (15) and (17) reveals that, in the presence of technological change, the golden rule of thumb does not lead to the steady state of the RKC model anymore. For the optimum long-run equilibrium now depends on the utility-parameter  $\theta$  and is not the same as the equilibrium of the golden rule of thumb model unless  $\theta = 1$ . Of course, no rule of thumb which is independent of the form of the utility function can in general lead to an equilibrium depending on this function. It is remarkable, instead, that in case of  $\theta = 1$  both equilibria do coincide.

As is usual in the literature on the rate of convergence, the production function is assumed to be of the Cobb-Douglas type,

$$Y = aK^\alpha(AL)^{1-\alpha}.$$

Instead of using definition (13) directly,  $\mu$  is calculated with respect to capital per unit of effective labor,  $k$ . Moreover, it is usual to calculate the rate of convergence by using differential equations linearized around the long-run equilibrium. The Taylor-approximation of equation (15) around the long-run equilibrium value  $\hat{k}$  is

$$\dot{k} \approx f''(\hat{k})\hat{k}(k - \hat{k}),$$

which according to (13) yields

$$\mu_{\text{thumb}} = -f''(\hat{k})\hat{k}.$$

The Cobb-Douglas function implies

$$f''(k) = (\alpha - 1)\alpha ak^{\alpha-2}, \quad \hat{k} = \left( \frac{\alpha a}{n + \rho + \delta + \gamma} \right)^{1/(1-\alpha)}.$$

By substitution into  $\mu_{\text{thumb}}$  one gets

$$\mu_{\text{thumb}} = (1 - \alpha)(n + \rho + \delta + \gamma).$$

For the derivation of the rates of convergence in the other two cases the reader is referred to Barro and Sala-i-Martin (1995). For the Solow model,  $\mu$  becomes

$$\mu_{\text{Solow}} = (1 - \alpha)(n + \delta + \gamma).$$

In case of the RKC model,  $\mu$  is given by

$$2\mu_{\text{RKC}} = \sqrt{\zeta^2 + 4\frac{1-\alpha}{\theta}(n + \rho + \delta + \gamma\theta) \left[ \frac{n + \rho + \delta + \gamma\theta}{\alpha} - n - \delta - \gamma \right]} - \zeta,$$

where  $\zeta := \rho - (1 - \theta)\gamma$ . It is assumed that  $\zeta > 0$  in order to guarantee that the objective functional converges.<sup>15</sup>

In order to compare the rates of convergence of the three models, the following reasonable parameter values are assumed:

$$\alpha = 0.3, \quad \rho = 0.02, \quad n = 0.01, \quad \gamma = 0.02, \quad \delta = 0.05.$$

<sup>15</sup>Note that Barro and Sala-i-Martin (1995) effectively discount utility with  $e^{-(\rho-n)t}$ . Substituting  $\rho$  by  $\rho - n$  in the formula for  $\mu_{\text{RKC}}$  therefore yields the formula derived by Barro and Sala-i-Martin (1995, p. 80, where the symbol  $x$  instead of  $\gamma$  is used).

Moreover, the calculation of  $\mu_{\text{RKC}}$  requires a value of  $\theta$ , the reciprocal of the intertemporal elasticity of substitution. E.g.,  $\theta = 1$  yields:

$$\mu_{\text{thumb}} = 0.07, \quad \mu_{\text{Solow}} = 0.056, \quad \text{and} \quad \mu_{\text{RKC}} = 0.124.$$

Compared to the empirical values around 2%–3%, all these figures, especially  $\mu_{\text{RKC}}$ , are much too high. While  $\mu_{\text{RKC}}$  is smaller for other values of  $\theta$ , it is not possible to reach a value about 2%–3%.

Inspection of the formulas for  $\mu$  reveals that the rates of convergence decrease in the capital share  $\alpha$ . This result has led growth theorists to assume that  $\alpha$  is much higher than 0.3. Such an assumption is only reasonable if a broad concept of capital which includes *human capital* is employed. E.g., Barro and Sala-i-Martin (1995, ch. 2) assume  $\alpha = 0.75$  combined with  $\theta = 3$  (or more generally: somewhat greater than two), which implies the value  $\mu_{\text{RKC}} = 0.018$  if  $e^{nt}$  appears in the objective functional. In the present setting, one gets

$$\mu_{\text{thumb}} = 0.025, \quad \mu_{\text{Solow}} = 0.02, \quad \text{and} \quad \mu_{\text{RKC}} = 0.016.$$

Thus, if the the assumption of a broad concept of capital is made, the golden rule of thumb model fits the empirical convergence rates of 2%–3% extremely well. Of course, the other parameter values influencing the rates of convergence, e.g.  $\delta$  and  $\rho$ , could also take on different values. The simple calculation here is therefore not suited to conclude that the golden rule of thumb model fits the data better than the other models. What I want to emphasize is just that the golden rule of thumb model (as well as the Solow model) implies these rates of convergence somewhat more naturally than the RKC model. For all rates of convergence just stated have reasonable values, but there are less parameters that can freely be chosen in  $\mu_{\text{Solow}}$  and  $\mu_{\text{thumb}}$  as in  $\mu_{\text{RKC}}$ , where  $\theta = 3$  has been assumed (although the value of  $\theta$  has also consequences for the behavior of the saving rate and should therefore not lie outside a certain range). The procedure of Barro and Sala-i-Martin (1995) can be described as follows: All parameter values apart from  $\alpha$  and  $\theta$  are fixed in correspondence with empirical values. This results in a system of two equations in  $\alpha$  and  $\theta$  (the equation for  $\mu_{\text{RKC}}$  and an equation for the long-run saving rate not included here) which is solved in order to get the desired values of the rate of convergence and the long-run saving rate. Since the two equations are linearly independent, the odds are good to get positive solutions for  $\alpha$  and  $\theta$ . After all, this is not an explanation of reality but the calibration of a model.

## 4 Concluding Remarks

It has been argued on the basis of stability considerations that dynamic optimization models cannot be used as positive theories of economic growth. As an alternative approach, a simple feedback rule of thumb has been introduced and has been shown to perform reasonably well from a normative point of view. Although this rule of thumb is just one of an infinite number of possible hypotheses on the consumption behavior, it appears to be plausible enough to be used as a working hypothesis in positive growth economics. This view has been reinforced by showing that a growth model formulated with this rule of thumb is capable of generating plausible results as regards conditional convergence rates, e.g.

With respect to all the models considered here it must be admitted that none of them is really able to explain economic growth. At first, the rates of convergence are reasonable only provided that the capital share,  $\alpha$ , is about 0.75, which does not correspond to the original intention of the model where  $K$  is physical capital. Second, exogenous technological change has been assumed; thus, there is no possibility to really explain long-run growth in per capita incomes. Third, the saving rate should be allowed to rise during the transition to the steady state. This is not possible in the Solow model by definition while in the golden rule of thumb model the saving rate can be shown to fall with a rising ratio of capital to effective labor. The theory of endogenous (non-scale) growth has succeeded in overcoming some of these shortcomings.

The next step in investigating the golden rule of thumb is therefore to examine whether it can successfully be used in a more general model with endogenous technological change. This task is pursued in a companion paper (Christiaans, 2001), where a non-scale growth model with learning by doing and rule of thumb consumers is analyzed. It turns out that the model is reasonably consistent with the so-called *stylized facts* about economic growth introduced by Kaldor (1961) and extended by Romer (1989) and that the saving rate can fall as well as rise during the transition process.

With respect to the rationality hypothesis of neoclassical economics, it has been argued that, while static decision problems can on principle be augmented by a dynamic adjustment process which in case of stability raises the probability that people do behave similar to being in an optimum, it is hardly possible to imagine a metadynamic adjustment process for dynamic optimization problems. But if there are doubts about the capability of people to solve sophisticated decision problems exactly and the true optimum is unstable, alternatives are needed. The golden (feedback) rule of thumb analyzed here has some implications similar to those of the dynamic optimization approach. The steady state of the rule of thumb growth model is the same as that of the RKC model if there is no technological change or if the intertemporal elasticity of substitution is equal to one. Therefore, all the results that are implied by the supply side in the steady state do also coincide. Moreover, as is shown in Christiaans (2001), in the presence of learning by doing the steady state tax cum subsidy policy implied by a decentralized dynamic optimization approach can be used to internalize the external effects in a rule of thumb economy. Thus, it is at least possible that some implications of dynamic neoclassical models continue to hold if the rationality assumption is violated, although the situation remains more difficult in case of dynamic problems than in case of static ones. These are the good news for neoclassical economics implied by a critical approach.

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