Fundamentals of Production Theory in International Trade: A Modern Approach Based on Theory of Duality

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Abstract: In terms of convex analysis the revenue function of a country with a given factor endowment may be seen as the support function of the production possibility set. At the same time this revenue function is the so called convex-conjugate of the indicator function of the production possibility set. The task of this paper is to apply duality results of this kind to sums of functions, where Rockafellar (1972) has shown that the operations of addition and the infimal convolution of convex functions are dual to each other. To be more concrete, we refer to the theory of international trade, where the factor endowment of each country is given and the factors of production are internationally immobile. If the country specific outputs sum up to a world production, what is the meaning of the appropriate dual problem? The answer will deal with the national and world-wide problem of revenue maximization. Moreover, we shall discuss the properties of an optimal commodity price vector in relation to the "dual" world output and to the national commodity supply. Similar problems will be analyzed on a national level, where the factors of production can easily be moved from one firm or sector to another. The second part of the paper draws the attention to the inverse production technology, namely the input correspondence. Here, the results based on convex revenue functions are applied to concave cost functions.

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1 Production Theory in View of Convex Analysis

This paper deals with different problems of production theory, where an appropriate interpretation of the problem at hand depends on the assumptions implied by convex analysis. Given the notation of Section 2.1, Section 2.2 is concerned with the following question: are the countries of the world economy able to produce together a certain commodity bundle $\mathbf{x}$? Here, it is assumed, that the countries possess individual production technologies and that the factor endowments are fixed, i.e. the factors of production are immobile. The dual problem presumes a given commodity price vector $\mathbf{p}$, which is valid for each country. This optimization problem is implied by the theory of conjugate functions and seeks for a revenue maximizing commodity bundle among all producible output vectors. Proposition 1 refers to a pair of dual points $(\mathbf{x}, \mathbf{p})$ and makes a statement on the properties of an optimal price vector given the commodity bundle, et vice versa. Both vectors $\mathbf{x}$ and $\mathbf{p}$ can be expressed in terms of a subgradient, whose geometric meaning will be explained at a later stage. By Proposition 2, the aggregate commodity supply consists of the national commodity bundles, which in turn maximize the national revenues (Proposition 3). Proposition 4 states, that the revenue maximizing price vector is valid for each country.

Section 2.3 draws the attention to the output distance function. By means of convex analysis, the revenue function of Section 2.2 and this distance function are polar gauges. They are connected by Mahler's inequality (2.20) while the indicator function and the revenue function of Section 2.2 satisfy the Young-Fenchel inequality (2.4). The two polar gauges represent the same technology and can be transformed into each other under certain regularity conditions. Hence, it is straightforward to analyze a problem similar to that of Section 2.2: find a price vector, which is feasible for all countries by means of their "polar" production technologies, so that total revenue is maximized. Notice that the duality results in Proposition 5 differ from Proposition 1. The price vector is no longer determined by the sum of production possibility sets but by the intersection of their so called polar sets, which are introduced at the beginning of Section 2.3. This is the reason why Mahler's inequality (2.20), which holds true for each country, is not valid on an aggregate level.

Section 2.4 takes a different point of view. On a national level the factors of production are mobile between firms, while the national factor endowment is given. All commodity prices are fixed by the world markets. Now a fictional benevolent dictator intending to maximize the national revenue has the problem of how to allot the factor endowments to the firms. This problem is not too far away from reality as shown by the dual program of profit maximization. If each firm maximizes its profit, they behave together as the above mentioned benevolent dictator. The duality results are similar to Section 2.2 with the exception that the results of convex analysis are now applied to concave functions.

While Section 2 is based on output correspondences decribing the production technologies, Section 3 refers to the inverse input correspondence. In this regard the analysis switches over from convex revenue functions to concave cost functions. Accordingly, the duality results describe an optimal matching of input vectors and factor price vectors. One important difference is that the results of cost minimization are now discussed on the basis of mobile inputs, so that the factor prices are valid for each firm.

The crucial problem of Section 3.3 will be the observation that no input requirement set contains the input vector $\mathbf{v}_b = \mathbf{0}$ given any commodity bundle $\mathbf{x}_b \neq \mathbf{0}$, that is, no positive output can be produced with zero inputs. The result will be described by a modification of Mahler's inequality. Nevertheless, under analytical aspects the duality results of Propositions 13–16 are in line with the outcomes of Section 2.3 and 2.4. But from an economic point of view the last two sections deal with completely different optimization problems.

2 Dual Operations Based on Output Correspondences

2.1 Production Possibility Sets

In order to represent ν not necessarily distinct production technologies we make use of ν families of production possibility sets $\left(P_b(\mathbf{v}_b)|\ \mathbf{v}_b \in \mathbb{R}^m\right)$ with $b = 1, \ldots, \nu$. Each member $P_b(\mathbf{v}_b)$ denotes the collection of all commodity bundles $\mathbf{x}_b$ capable of being produced by the firm b or alternatively in the economy b by using the inputs $\mathbf{v}_b$. Every activity $(\mathbf{v}_b, \mathbf{x}_b)$, which is compatible with the given respective technology, satisfies $\mathbf{x}_b \in P_b(\mathbf{v}_b)$. The set valued technology P_b is called the output correspondence of firm b.

The *indicator function*[1] is the perhaps easiest representation of the production possibility set $P_b(\mathbf{v}_b) \subseteq \mathbb{R}^n$. For every feasible activity it is $\delta(\mathbf{x}_b|P_b(\mathbf{v}_b)) = 0$. In all other cases the indicator function is set to $\delta(\mathbf{x}_b|P_b(\mathbf{v}_b)) = +\infty$. Given the input vector $\mathbf{v}_b$, the shorter form $\delta_b \equiv \delta(\cdot|P_b(\mathbf{v}_b))$ is used for the sake of brevity. By assumption each production possibility set is nonempty, closed and convex for each feasible input vector, so that its indicator function δ_b is proper, closed and convex.

$$\operatorname{cl} \delta_b = \delta_b \tag{2.1}$$

[1] The unusual properties of proper, n-proper, closed, and polar functions are explained in the appendix. Moreover the basic principles of conjugate and polar function as well as the indicator function (5.60), the gauge function (5.61) and the support function (5.62) are introduced. Besides the infimal convolution (5.58) these terms are indispensable for the understanding of this text. Please take a look at the glossary in the appendix.

By Theorem 3,[2] the proper indicator function δ_b determines a *convex-conjugate function* δ_b^* being not only closed and convex but also proper.

$$\begin{aligned} \delta_b^*(\mathbf{p}) &= \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b - \delta_b(\mathbf{x}_b) \mid \mathbf{x}_b \in \mathbb{R}^n\right\} \\ &= \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b \mid \mathbf{x}_b \in P_b(\mathbf{v}_b)\right\} \end{aligned} \tag{2.2}$$

This *support function* of the production possibility set $P_b(\mathbf{v}_b)$ corresponds to the *revenue function* of firm b, which is reflected by the following notation:

$$\delta_b^* \equiv \delta^*(\cdot \mid P_b(\mathbf{v}_b)) \equiv r_b(\cdot, \mathbf{v}_b) \tag{2.3}$$

With the aid of the *convex-biconjugate function* we return – again by Theorem 3 – to the initial indicator function $\delta_b^{**} = \operatorname{cl}\delta_b = r_b^*(\cdot, \mathbf{v}_b)$, where the closure operation may be omitted by (2.1). As shown by Rockafellar (1972) this one-to-one correspondence, i.e. $\delta_b \to \delta_b^* \to \delta_b^{**} = \delta_b$, holds true in the class of all closed proper convex functions. The definition of conjugate functions immediately yields the *Young-Fenchel inequality* satisfied particularly for the indicator function and the revenue function.

$$\delta_b(\mathbf{x}_b) + r_b(\mathbf{p}, \mathbf{v}_b) \geqq \mathbf{p}^\mathsf{T}\mathbf{x}_b \qquad \forall\, \mathbf{x}_b,\ \forall \mathbf{p} \tag{2.4}$$

As Section 2.2 applies these results to sums of functions we need an idea of how to sum up production possibility sets. Starting with a given factor allocation $\mathfrak{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_\nu)$, the aggregate production technology

$$P_\Sigma(\mathfrak{v}) \coloneqq P_1(\mathbf{v}_1) + \cdots + P_\nu(\mathbf{v}_\nu) = \{\mathbf{x}_1 + \cdots + \mathbf{x}_\nu \mid \mathbf{x}_b \in P_b(\mathbf{v}_b)\ \ \forall\, b\} \tag{2.5}$$

is described by the indicator function $\delta_B \equiv \delta(\cdot \mid P_\Sigma(\mathfrak{v}))$. If each set $P_b(\mathbf{v}_b)$ is convex, then $P_\Sigma(\mathfrak{v})$ is also convex. Moreover, Berge (1963, Corollary 2, p. 161) has proved, that the sum of finitely many compact sets is again compact, i.e. in particular $\delta_B = \operatorname{cl}\delta_B$. Notice, that (2.5) constitutes more than the set $P_\Sigma(\mathfrak{v})$. The relation $\mathbf{x} \in P_\Sigma(\mathfrak{v})$ implies that there is a feasible commodity allocation $\mathfrak{x}$, i.e.

$$\exists\, \mathfrak{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_\nu):\ \mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_\nu \quad \text{with} \quad \mathbf{x}_b \in P_b(\mathbf{v}_b) \quad \forall\, b \tag{2.6}$$

As long as the factors of production are mobile within an economy between firms the consideration of all feasible factor allocations leads to the production possibility set of the entire economy.

$$P(\mathbf{v}) = \{\mathbf{x} \mid \mathbf{x} \in P_\Sigma(\mathfrak{v}),\ \mathfrak{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_\nu),\ \mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_\nu\}$$

[2] All of the cited theorems can be found in the appendix.

2.2 Duality of Feasible Activities and Revenue Maximization

In what follows a given factor allocation $\mathfrak{v}$ is supposed. This assumption corresponds to internationally given factor endowments with immobile factors of production. To stress this point we speak of the economy b in this section. On the contrary in section 2.4 we seek for a factor allocation which maximizes the revenue. This problem arises within an economy with mobile factors. Therefore it is more useful to speak of the firm b in that context. As the firms are not differentiated in this paper by the commodities they produce, the term sector will not be used. Nevertheless, analogue to (2.5) each technology P_b can be thought of as an aggregate technology of different firms.

A commodity bundle $\mathbf{x}$ is called *producible* at the given factor allocation provided a commodity allocation $\mathfrak{x}$ exists such that the optimal value of the following infimal convolution is finite. The convolution of the function $\delta_1, \ldots, \delta_\nu$ is indicated by $\Box \cdots \Box$, where the definition is taken from (5.58) in the appendix. In accordance with (2.5) we have

$$(2.7)\qquad \begin{aligned}(\delta_1 \Box \cdots \Box \delta_\nu)(\mathbf{x}) &= \inf\{\delta_1(\mathbf{x}_1) + \cdots + \delta_\nu(\mathbf{x}_\nu) \,|\, \mathbf{x}_1 + \cdots + \mathbf{x}_\nu = \mathbf{x}\} \\ &= \delta(\mathbf{x}|P_\Sigma(\mathfrak{v})) \equiv \delta_B(\mathbf{x})\end{aligned}$$

This function takes a finite value, $\delta_B(\mathbf{x}) = 0$, if and only if each economy b realizes a feasible activity $(\mathbf{x}_b, \mathbf{v}_b)$, i.e. $\delta_b(\mathbf{x}) = 0$, $b = 1, \ldots, \nu$. Given the factor allocation $\mathfrak{v}$, we now analyze the convex-conjugate function

$$(2.8)\qquad (\delta_1 \Box \cdots \Box \delta_\nu)^*(\mathbf{p}) = \sup\left\{\mathbf{p}^\top \mathbf{x} - (\delta_1 \Box \cdots \Box \delta_\nu)(\mathbf{x}) \,|\, \mathbf{x} \in \mathbb{R}^n\right\}$$

or

$$\delta_B^*(\mathbf{p}) = \sup\left\{\mathbf{p}^\top \mathbf{x} - \delta_B(\mathbf{x}) \,|\, \mathbf{x} \in \mathbb{R}^n\right\},$$

which are equivalent by definition. The next step refers to Theorem 6 (with $f_b = \delta_b$). It emphasizes that the operations of addition (2.9) and the infimal convolution (2.7) of convex functions are dual to each other.

$$(2.9)\qquad \begin{aligned}&\delta_B^*(\mathbf{p}) \equiv (\delta_1 \Box \cdots \Box \delta_\nu)^*(\mathbf{p}) = \delta_1^*(\mathbf{p}) + \cdots + \delta_\nu^*(\mathbf{p}) \\ :\Longleftrightarrow \quad &r(\mathbf{p}, \mathfrak{v}) = r_1(\mathbf{p}, \mathbf{v}_1) + \cdots + r_\nu(\mathbf{p}, \mathbf{v}_\nu)\end{aligned}$$

On an international level with immobile factors of production, i.e. a given factor allocation, the world wide maximum revenue $r(\cdot, \mathfrak{v})$ results from the sum of the national maximum revenues. Although this outcome seems to be trivial, it constitutes the initial point for the further analysis. To stress the aspect of duality of (2.7) and (2.9), it may be helpful to apply again Theorem 6. If the effective domains $\operatorname{Dom} r_b(\cdot, \mathbf{v}_b)$, $b = 1, \ldots, \nu$, have a relative interior point in common, then $r^*(\cdot, \mathfrak{v}) = (\delta_1 \Box \cdots \Box \delta_\nu)$. At first glance this is an astonishing result, because the definition of the convex-conjugate function $r^*(\cdot, \mathfrak{v})$ yields

$$\begin{aligned} r^*(\mathbf{x}, \mathfrak{v}) &= \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} - r(\mathbf{p}, \mathfrak{v}) \mid \mathbf{p} \in \mathbb{R}^n \right\} \\ &= \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} - r_1(\mathbf{p}, \mathbf{v}_1) - \cdots - r_\nu(\mathbf{p}, \mathbf{v}_\nu) \mid \mathbf{p} \in \mathbb{R}^n \right\} \end{aligned}$$

and this formula seems to be far away from (2.7). Nevertheless, an optimal price vector with $r^*(\mathbf{x}, \mathfrak{v}) = 0$ is the same as having $(\delta_1 \square \cdots \square \delta_\nu)(\mathbf{x}) = 0$. This guarantees the existence of an optimal commodity allocation solving (2.7).

Taking the definition of subgradients (5.63) into account, Theorem 7 yields immediately

1 Proposition *If the indicator function* $\delta_B \equiv (\delta_1 \square \cdots \square \delta_\nu)$ *is proper and convex, the subsequent three conditions on* $\hat{\mathbf{x}}$ *are equivalent to each other.*

(2.10a) $\hat{\mathbf{p}} \in \partial \delta_B(\hat{\mathbf{x}})$

(2.10b) $\hat{\mathbf{p}}^\mathsf{T}\mathbf{x} - \delta_B(\mathbf{x})$ *achieves its supremum in* $\mathbf{x}$ *at* $\hat{\mathbf{x}}$*; see (2.8)*

(2.10c) $\delta_B(\hat{\mathbf{x}}) + \delta_B^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$

As the indicator function is closed anyway, $\operatorname{cl} \delta_B = \delta_B$, *two more conditions can be added to the list.*

(2.10d) $\hat{\mathbf{x}} \in \partial \delta_B^*(\hat{\mathbf{p}})$

(2.10e) $\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} - \delta_B^*(\mathbf{p})$ *achieves its supremum in* $\mathbf{p}$ *at* $\hat{\mathbf{p}}$*; see (2.9)*

A pair $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ satisfying all five conditions of Proposition 1 is called a pair of dual points. The proposition will now be discussed in detail by some remarks which particularly refer to the commodity supply and the common price vector.

The proposition presupposes a given factor allocation $\mathfrak{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_\nu)$ with $\mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_\nu$. Notice in (2.10b) and (2.10c), that $\delta_B(\hat{\mathbf{x}}) = 0$ or $\hat{\mathbf{x}} \in P_\Sigma(\mathfrak{v})$ is equivalent to the existence of a feasible commodity allocation $\hat{\mathfrak{x}} = (\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_\nu)$ according to (2.6).

The assumed properties of δ_B hold true because each indicator function δ_b satisfies the assumptions of Theorem 2. The aggregate technology especially allows for the possibility of inaction, $\delta_B(\mathbf{0}) = \delta_1(\mathbf{0}) + \cdots + \delta_\nu(\mathbf{0}) = 0$, because each production possibility set $P_b(\mathbf{v}_b)$ includes the possibility of inaction $\mathbf{x}_b = \mathbf{0}$.

Regarding (2.10d) recall particularly $\delta_B^* \equiv r(\cdot, \mathfrak{v})$. Thus, if the revenue function is differentiable at $\hat{\mathbf{p}}$,

$$\hat{\mathbf{x}} = \nabla_\mathbf{p} r(\hat{\mathbf{p}}, \mathfrak{v}) \,.$$

This statement on the pair of dual points $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ is now generalized with respect to the aggregate commodity supply $\hat{\mathbf{x}}$ of (2.10d). In order to break $\hat{\mathbf{x}}$ down into its constituents $\hat{\mathbf{x}}_b$, $b = 1, \ldots, \nu$, we make use of Theorem 9 so that the right hand side of (2.10d) will be expressed by a sum of subgradients of the individual revenue functions $\delta_b^* \equiv r_b$.

2 Proposition *If the convex effective domains* $\mathrm{Dom}\,\delta_b^*$, $b = 1, \ldots, \nu$, *have a relative interior point in common, then*

$$\partial\delta_B^*(\mathbf{p}) = \partial\delta_1^*(\mathbf{p}) + \cdots + \partial\delta_\nu^*(\mathbf{p}) \qquad \forall\, \mathbf{p}$$

In the case of Proposition 2 an allocation $\hat{\mathfrak{x}}$ exists for every commodity vector $\hat{\mathbf{x}} \in \partial\delta_B^*(\hat{\mathbf{p}})$ so that the total revenue maximizing supply is composed of the country specific commodity supply.

$$\exists\, \hat{\mathfrak{x}} = (\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_\nu) : \ \textstyle\sum_b \hat{\mathbf{x}}_b = \hat{\mathbf{x}} \quad \text{and} \quad \hat{\mathbf{x}}_b \in \partial\delta_b^*(\hat{\mathbf{p}}) \quad \forall\, b$$

Under the condition of Proposition 2 the allocation $\hat{\mathfrak{x}}$ solves (2.7) provided the pair $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ satisfies the five conditions of Proposition 1. The proof results from (2.10c). If $\hat{\mathfrak{x}}$ is no optimal solution to (2.7), then

$$\begin{aligned} & \hat{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{x}} - \delta_B^*(\hat{\mathbf{p}}) \stackrel{(2.10c)}{=} \delta_B(\hat{\mathbf{x}}) \stackrel{(2.7)}{<} \delta_1(\hat{\mathbf{x}}_1) + \cdots + \delta_\nu(\hat{\mathbf{x}}_\nu) \\ \stackrel{(2.9)}{\Longleftrightarrow} \quad & \hat{\mathbf{p}}^{\mathsf{T}}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_\nu) + \delta_1^*(\hat{\mathbf{p}}) + \cdots + \delta_\nu^*(\hat{\mathbf{p}}) < \delta_1(\hat{\mathbf{x}}_1) + \cdots + \delta_\nu(\hat{\mathbf{x}}_\nu) \end{aligned}$$

This inequality is contradicted by $\hat{\mathbf{x}}_b \in \partial\delta_b^*(\hat{\mathbf{p}})$ for all b being equivalent to $\hat{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{x}}_b = \delta_b(\hat{\mathbf{x}}_b) + \delta_b^*(\hat{\mathbf{p}})$.

Assuming differentiable revenue functions, the commodity supply of each country results from

$$\hat{\mathbf{x}}_b = \nabla_{\mathbf{p}} r_b(\hat{\mathbf{p}}, \mathbf{v}_b) \quad \forall\, b.$$

The following proposition is based on Corollary 7.3. It hands in the explanation why it is meaningful to call $\hat{\mathbf{x}}_b$ a revenue maximum commodity supply.

3 Proposition ("Commodity Supply") *Let the production possibility set* $P_b(\mathbf{v}_b)$ *be nonempty, closed and convex and* $\mathbf{p}$ *be an arbitrary price vector. Then* $\partial\delta_b^*(\mathbf{p}) = \partial r_b(\mathbf{p}, \mathbf{v}_b)$ *– provided this set is not empty – consists of all commodity bundles* $\mathbf{x}_b$ *where the linear function* $\mathbf{p}^{\mathsf{T}}\mathbf{x}_b$ *attains its maximum over* $P_b(\mathbf{v}_b)$.

As having been noted in Corollary 7.1, the subdifferentials of the indicator function $\partial\delta_B$ in (2.10a) and the revenue function $\partial\delta_B^* \equiv \partial r(\cdot, \mathfrak{v})$ in (2.10d) denote inverse correspondences.

$$\hat{\mathbf{p}} \in \partial\delta_B(\hat{\mathbf{x}}) \iff \hat{\mathbf{x}} \in \partial\delta_B^*(\hat{\mathbf{p}}) = \partial r_1(\hat{\mathbf{p}}, \mathbf{v}_1) + \cdots + \partial r_\nu(\hat{\mathbf{p}}, \mathbf{v}_\nu)$$

The subdifferential on the left hand side corresponds to the normal cone of the set $P_\Sigma(\mathfrak{v})$ at $\hat{\mathbf{x}}$. This cone includes all price vectors $\mathbf{p}$ being normal to $P_\Sigma(\mathfrak{v})$ at $\hat{\mathbf{x}}$. Proposition 4 states that an optimal price vector $\hat{\mathbf{p}}$ given by Proposition 1 is also optimal for each involved economy b. The main difference between the two propositions is that Proposition 1 refers to the indicator function δ_B of the aggregate technology, while the following proposition is based on the individual indicator functions δ_b. Consequently, $\hat{\mathbf{p}}$ is also normal to each production possibility set $P_b(\mathbf{v}_b)$ at the respective point $\hat{\mathbf{x}}_b$.

4 Proposition ("World Market Prices") *A price vector* $\hat{\mathbf{p}}$ *satisfies*

$$\hat{\mathbf{p}} \in \bigcap_{b=1,\ldots,\nu} \partial\delta_b(\hat{\mathbf{x}}_b) \tag{2.11}$$

if and only if the allocation $\hat{\mathfrak{x}}$ *solves the problem (2.7) and (2.10a) holds good.*

Proof: Starting with (2.10c), which is equivalent to (2.10a), yields

$$\begin{aligned} & \delta_B(\hat{\mathbf{x}}) + \delta_B^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} \\ \iff \quad & (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) + (\delta_1\Box \cdots \Box\delta_\nu)^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} \quad \text{cf. (2.9)} \\ \iff \quad & (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) + \delta_1^*(\hat{\mathbf{p}}) + \cdots + \delta_\nu^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} \quad \text{by Theorem 5} \end{aligned}$$

An optimal allocation $\hat{\mathfrak{x}}$, which solves the problem (2.7), satisfies $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_\nu$, so that

$$\delta_1(\hat{\mathbf{x}}_1) + \cdots + \delta_\nu(\hat{\mathbf{x}}_\nu) + \delta_1^*(\hat{\mathbf{p}}) + \cdots + \delta_\nu^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_\nu). \tag{2.12}$$

According to the Young-Fenchel inequality (5.59) it is $\delta_b(\hat{\mathbf{x}}_b) + \delta_b^*(\hat{\mathbf{p}}) \geqq \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}_b$ for all b, so that

$$\delta_b(\hat{\mathbf{x}}_b) + \delta_b^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}_b \qquad \forall\, b \tag{2.13}$$

applies by (2.12). By Proposition 1, (2.13) is equivalent to $\hat{\mathbf{p}} \in \partial\delta_b(\hat{\mathbf{x}}_b)$ for all b, so that (2.11) results. On the opposite, if (2.11) holds good, then (2.13) and, therefore, (2.12) or

$$\delta_1(\hat{\mathbf{x}}_1) + \cdots + \delta_\nu(\hat{\mathbf{x}}_\nu) + (\delta_1\Box \cdots \Box\delta_\nu)^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} \tag{2.14}$$

(again by Theorem 5) are satisfied. Moreover the inequality implied by (2.7)

$$\begin{aligned} & (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) - \delta_1(\hat{\mathbf{x}}_1) - \cdots - \delta_\nu(\hat{\mathbf{x}}_\nu) \leqq 0 \\ \iff \quad & (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) + \delta_1^*(\hat{\mathbf{p}}) + \cdots + \delta_\nu^*(\hat{\mathbf{p}}) \leqq \hat{\mathbf{p}}^\mathsf{T}(\mathbf{x}_1 + \cdots + \mathbf{x}_\nu) \\ \iff \quad & (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) + (\delta_1\Box \cdots \Box\delta_\nu)^*(\hat{\mathbf{p}}) \leqq \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} \end{aligned}$$

faces the Young-Fenchel inequality according to (2.8).

$$(\delta_1 \square \cdots \square \delta_\nu)(\hat{\mathbf{x}}) + (\delta_1 \square \cdots \square \delta_\nu)^*(\hat{\mathbf{p}}) \geqq \hat{\mathbf{p}}^\mathsf{T} \hat{\mathbf{x}}$$

Hence, (2.10c) or equivalently (2.10a) results. Together with (2.14) it follows

$$\delta_1(\hat{\mathbf{x}}_1) + \cdots + \delta_\nu(\hat{\mathbf{x}}_\nu) = (\delta_1 \square \cdots \square \delta_\nu)(\hat{\mathbf{x}}),$$

that is, the allocation $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_\nu)$ is optimal for the problem (2.7). ■

2.3 Polar Production Possibility Sets

2.3.1 Duality of Polar Sets and Functions

This section refers again to a given factor allocation $\mathfrak{v}$ and a given commodity bundle $\mathbf{x}$. The main problem will be the revenue maximization problem (2.24), which may be seen as opposite to (2.9), but now the commodity bundle $\mathbf{x}$ is given, while formerly the price vector was fixed. However, under technical aspects it will be shown that (2.24) corresponds to the convex-conjugate function of (2.7). Before continuing with the analysis, the principle of polar sets and functions has to be introduced. It should be kept in mind, that two sets being polar to each other embody the same information. Similar to conjugate functions there is a one-to-one correspondence in the class of all closed convex set containing the origin, i.e. $C \to C^\circ \to C^{\circ\circ} = C$. Hence, the results of the examination of one set are also implied by the other one. A similar statement holds true regarding two polar functions.

The polar production possibility set

$$P_b^\circ(\mathbf{v}_b) := \left\{\mathbf{p} \in \mathscr{P} | \ \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1 \ \ \forall\, \mathbf{x}_b \in P_b(\mathbf{v}_b)\right\} \tag{2.15}$$

is a closed convex set containing the origin, hence it is *star shaped*. It consists of all price vectors, such that the revenue does not exceed the value 1 in any case. Graphically the boundary of the set $P_b^\circ(\mathbf{v}_b)$ can be represented by determining all price vectors such that the respective hyperplanes $\mathbf{p}^\mathsf{T}\mathbf{x}_b = 1$ are tangent to the production possibility set $P_b(\mathbf{v}_b)$ at $\mathbf{x}_b$. The properties of the polar set imply its indicator function $\delta(\cdot | P_b^\circ(\mathbf{v}_b))$ to be proper, closed (thus (2.1) holds analogously) and convex. Under the assumption of a given factor allocation the simplified notation

$$\delta(\cdot | P_b^\circ(\mathbf{v}_b)) \equiv \delta_b^\circ$$

emphasizes at the same time that δ_b° is the polar function of δ_b. This function is defined by

$$\delta_b^\circ(\mathbf{p}) = \inf \left\{\lambda \geqq 0 | \ \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1 + \lambda \delta_b(\mathbf{x}_b) \ \ \forall\, \mathbf{x}_b\right\}, \tag{2.16}$$

so that polar functions have the following property:

$$\mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1 + \delta_b(\mathbf{x}_b)\,\delta_b^\circ(\mathbf{p}) \qquad \forall\,\mathbf{x}_b \in \operatorname{Dom}\delta_b,\ \forall\,\mathbf{p} \in \operatorname{Dom}\delta_b^\circ$$

In the case at hand it is particularly

$$\mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1 \qquad \forall\,\mathbf{x}_b \in P_b(\mathbf{v}_b),\ \forall\,\mathbf{p} \in P_b^\circ(\mathbf{v}_b) \tag{2.17}$$

This relationship has a useful interpretation, if we assume that the polar production possibility set $P_b^\circ(\mathbf{v}_b)$ is known. In this case the commodity bundle $\mathbf{x}_b$ is producible if and only if $\mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1$ for all price vectors of the set $P_b^\circ(\mathbf{v}_b)$. Technically, this statement can be expressed by the bipolar set $P_b^{\circ\circ}$, which is defined by

$$P_b^{\circ\circ}(\mathbf{v}_b) \coloneqq \left\{\mathbf{x}_b \in \mathbb{R}^n \mid \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq 1\ \ \forall\,\mathbf{p} \in P_b^\circ(\mathbf{v}_b)\right\}.$$

For a closed convex set containing the origin, one can show that the bipolar set equals the initial production possibility set, $P_b^{\circ\circ}(\mathbf{v}_b) = P_b(\mathbf{v}_b)$.

As stressed by Rockafellar (1972, Theorem 15.4), the possibility of inaction $\delta_b(\mathbf{0}) = 0$ implies $\delta_b^\circ(\mathbf{0}) = 0$ as well as $\delta_b^{\circ\circ} = \operatorname{cl}\delta_b$. Therefore, we are permitted to write $\delta_b^{\circ\circ} = \delta_b$ because δ_b is closed.
The convex-conjugate function of δ_b° results from

$$\begin{aligned}\delta_b^{\circ *}(\mathbf{x}_b) &= \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b - \delta_b^\circ(\mathbf{p}) \mid \mathbf{p} \in \mathscr{P}\right\}\\ &= \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b \mid \mathbf{p} \in P_b^\circ(\mathbf{v}_b)\right\}.\end{aligned}$$

This support function corresponds to the *output distance function*[3] $t_{Ob}(\cdot,\mathbf{v}_b) = \delta_b^{\circ *}$ by Theorem 4, where $t_{Ob}(\cdot,\mathbf{v}_b)$ is defined to be the distance function of the production possibility set $P_b(\mathbf{v}_b)$. Therefore, we have $t_{Ob}(\cdot,\mathbf{v}_b) \equiv \gamma(\cdot|P_b(\mathbf{v}_b))$ in the notation of (5.61). A detailed discussion of the economic importance of this output distance function can be found in Färe (1988). An intuitive idea of this function results from the observation that $P_b(\mathbf{v}_b)$ is a nonempty closed convex set containing the origin. In this case $\mathbf{x}_b \in P_b(\mathbf{v}_b)$ holds good if and only if $\delta_b^{\circ *}(\mathbf{x}_b) = t_{Ob}(\mathbf{x}_b,\mathbf{v}_b) \leqq 1$ applies. Hence, *the output distance function may be seen as generalized production function in the context of more than one output.* Interestingly, a similar relationship can be transferred to the revenue function, since the polar set $P_b^\circ(\mathbf{v}_b)$ has the same properties as $P_b(\mathbf{v}_b)$. Thus, the relation $\mathbf{p} \in P_b^\circ(\mathbf{v}_b)$ is satisfied if and only if $\delta_b^*(\mathbf{p}) = r_b(\mathbf{p},\mathbf{v}_b) \leqq 1$.

Technically, the relationship between the revenue function $r_b(\cdot,\mathbf{v}_b) \equiv \delta_b^*$ and the output distance function $t_{Ob}(\cdot,\mathbf{v}_b) = \delta_b^{\circ *}$ is given by an extremely simple equation. As the indicator function δ_b is

[3] In economics the mathematical gauge function is called a distance function.

a nonnegative closed convex function, with $\delta_b(\mathbf{0}) = 0$, we gain

$$\delta_b^{*\circ} = \delta_b^{\circ *} \tag{2.18}$$

by Rockafellar (1972, Corollary 15.5.1). The revenue function and the output distance function are polar gauges, where the definition of polar functions by (2.16) reduces to

$$r_b(\mathbf{p}, \mathbf{v}_b) = \inf\left\{\lambda \geqq 0 | \; \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq \lambda t_{Ob}(\mathbf{x}_b, \mathbf{v}_b) \;\; \forall\, \mathbf{x}_b\right\} \tag{2.19}$$

because of the linear homogeneity of $t_{Ob}(\cdot, \mathbf{v}_b)$. Thus the revenue function and the output distance function satisfy *Mahler's inequality* which is cited in Theorem 5:

$$\begin{aligned} (2.20) \qquad & \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq r_b(\mathbf{p}, \mathbf{v}_b)\, t_{Ob}(\mathbf{x}_b, \mathbf{v}_b) \qquad && \forall\, \mathbf{x}_b \in \operatorname{Dom} r_b(\cdot, \mathbf{v}_b),\; \forall\, \mathbf{p} \in \operatorname{Dom} t_{Ob}(\cdot, \mathbf{v}_b) \\ :\Longleftrightarrow \quad & \mathbf{p}^\mathsf{T}\mathbf{x}_b \leqq \delta_b^*(\mathbf{p})\, \delta_b^{*\circ}(\mathbf{x}_b) && \forall\, \mathbf{x}_b \in \operatorname{Dom} \delta_b^*,\; \forall\, \mathbf{p} \in \operatorname{Dom} \delta_b^{*\circ} \end{aligned}$$

This result has been discussed in detail in Bobzin (1999). There the attention is drawn to the case where Mahler's inequality is satisfied as an equation. It is worthwhile to notice the difference between the Young-Fenchel inequality (2.4) and the preceding Mahler's inequality.

2.3.2 Dual Operations Regarding Polar Production Possibility Sets

Having introduced the revenue function and the output distance function to be polar gauges, we now turn to the polar production possibility sets. Given the factor allocation $\mathfrak{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_\nu)$, each country b is characterized by a unique set $P_b^\circ(\mathbf{v}_b)$ of price vectors. Therefore, seeking for a price vector $\mathbf{p}$ *common to all countries* is the same as calculating the intersection of the polar sets $P_b^\circ(\mathbf{v}_b)$, $b = 1, \ldots, \nu$, which is abbreviated to

$$P_\cap^\circ(\mathfrak{v}) := P_1^\circ(\mathbf{v}_1) \cap \cdots \cap P_\nu^\circ(\mathbf{v}_\nu)$$

for the sake of clarity. Alternatively, the relation $\mathbf{p} \in P_\cap^\circ(\mathfrak{v})$ holds true if and only if $\delta_1^\circ(\mathbf{p}) + \cdots + \delta_\nu^\circ(\mathbf{p}) = \delta(\mathbf{p}|P_\cap^\circ(\mathfrak{v})) = 0$. Similar to (2.7) this is a sum of indicator functions, but now the functions are evaluated at the same point $\mathbf{p}$. Hence, the second formula of Theorem 6 (with $f_b = \delta_b^\circ$) determines the dual operation. If we assume that the polar production possibility sets P_b°, $b = 1, \ldots, \nu$, have a relative interior point in common, then

$$(\delta_1^\circ + \cdots + \delta_\nu^\circ)^*(\mathbf{x}) = (\delta_1^{\circ *} \Box \cdots \Box \delta_\nu^{\circ *})(\mathbf{x}) \tag{2.21}$$

where the infimum of

$$(\delta_1^{\circ *} \Box \cdots \Box \delta_\nu^{\circ *})(\mathbf{x}) = \inf\left\{\delta_1^{\circ *}(\mathbf{x}_1) + \cdots + \delta_\nu^{\circ *}(\mathbf{x}_\nu) |\; \mathbf{x}_1 + \cdots + \mathbf{x}_\nu = \mathbf{x}\right\} \tag{2.22}$$

is attained for each $\mathbf{x}$. To get an idea of the economic meaning we firstly rewrite the preceding equation.

(2.23) $$(\delta_1^{\circ *} \Box \cdots \Box \delta_\nu^{\circ *})(\mathbf{x}) = \inf \{ t_{O1}(\mathbf{x}_1, \mathbf{v}_1) + \cdots + t_{O\nu}(\mathbf{x}_\nu, \mathbf{v}_\nu) | \ \mathbf{x}_1 + \cdots + \mathbf{x}_\nu = \mathbf{x} \}$$

Hence, a commodity allocation is needed which minimizes the sum of the individual output distance functions. As the advantage of an allocation being optimal in this sense is not immediate[4], the attention is now drawn to the right hand side of (2.21).

(2.24) $$\begin{aligned}(\delta_1^\circ + \cdots + \delta_\nu^\circ)^*(\mathbf{x}) &= \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} - \delta_1^\circ(\mathbf{p}) - \cdots - \delta_\nu^\circ(\mathbf{p}) | \ \mathbf{p} \in \mathcal{P} \right\} \\ &= \sup \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} | \ \mathbf{p} \in P_1^\circ(\mathbf{v}_1) \cap \cdots \cap P_\nu^\circ(\mathbf{v}_\nu) \right\} \\ &= \delta^*\big(\mathbf{x} | P_\cap^\circ(\mathfrak{v})\big)\end{aligned}$$

The attraction of this problem is obvious. We seek for a price vector $\mathbf{p}$ *common to all countries*, which maximizes the total revenue $\mathbf{p}^\mathsf{T}\mathbf{x}$ for the given output vector $\mathbf{x}$ holding the factor allocation $\mathfrak{v}$ fixed. The properties of such a price vector will be given in Proposition 5. The differences between (2.24) and the revenue maximization problem (2.9) will be discussed at the end of this section.

From the properties of polar sets it is known that their intersection is closed and convex and that it contains the origin, $\mathbf{0} \in P_\cap^\circ(\mathfrak{v})$. Thus, the indicator function $\delta(\mathbf{p}|P_\cap^\circ(\mathfrak{v}))$ is a proper closed convex function. Moreover, by Theorem 3, its convex-conjugate function $\delta^*(\cdot|P_\cap^\circ(\mathfrak{v})) \equiv (\delta_1^\circ + \cdots + \delta_\nu^\circ)^*$ is not only closed and convex but also proper so that the biconjugate function satisfies

(2.25) $$\delta^{**}(\cdot|P_\cap^\circ(\mathfrak{v})) = \operatorname{cl} \delta(\cdot|P_\cap^\circ(\mathfrak{v})) = \delta(\cdot|P_\cap^\circ(\mathfrak{v})) = \delta_1^\circ + \cdots + \delta_\nu^\circ.$$

Now Theorem 7 can be applied to the total revenue $\delta^*(\cdot|P_\cap^\circ(\mathfrak{v}))$ – as it is given in the initial equation (2.21) – where (2.25) has to be noted.

5 Proposition *As $\delta^*(\cdot|P_\cap^\circ(\mathfrak{v}))$ is a proper closed convex function, the following five conditions on the pair of points $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ are equivalent to each other.*

(2.26a) $\hat{\mathbf{p}} \in \partial\delta^*(\hat{\mathbf{x}}|P_\cap^\circ(\mathfrak{v}))$

(2.26b) $\hat{\mathbf{p}}^\mathsf{T}\mathbf{x} - \delta^*(\mathbf{x}|P_\cap^\circ(\mathfrak{v}))$ *achieves its supremum in $\mathbf{x}$ at $\hat{\mathbf{x}}$*

(2.26c) $\delta^*(\hat{\mathbf{x}}|P_\cap^\circ(\mathfrak{v})) + \delta(\hat{\mathbf{p}}|P_\cap^\circ(\mathfrak{v})) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$

(2.26d) $\hat{\mathbf{x}} \in \partial\delta(\hat{\mathbf{p}}|P_\cap^\circ(\mathfrak{v}))$

(2.26e) $\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} - \delta(\mathbf{p}|P_\cap^\circ(\mathfrak{v}))$ *achieves its supremum in $\mathbf{p}$ at $\hat{\mathbf{p}}$; see (2.24)*

[4] For an interpretation of (2.23) it may be useful to take the case of one country, $\nu = 1$, into account. The output $\mathbf{x}_1 = \mathbf{x}$ is producible, i.e. $\mathbf{x}_1 \in P_1(\mathbf{v}_1)$, if and only if $\delta_1^{\circ *}(\mathbf{x}) = \inf\{t_{O1}(\mathbf{x}_1, \mathbf{v}_1) | \ \mathbf{x}_1 = \mathbf{x}\} = t_{O1}(\mathbf{x}, \mathbf{v}_1) \leqq 1$.

Recall in (2.26e) or equivalently (2.24) that the supremum is finite, $\delta(\hat{\mathbf{p}}|P^\circ_\cap(\mathfrak{v})) = 0$, if and only if there is a price vector $\hat{\mathbf{p}} \in P^\circ_\cap(\mathfrak{v})$ common to all economies. In this case (2.26c) determines the revenue maximum $\delta^*(\hat{\mathbf{x}}|P^\circ_\cap(\mathfrak{v})) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$.

Regarding (2.26d), it is worthwhile to notice that the subdifferential can be divided into a sum of normal cones by Corollary 9.1,

[a] $$\partial\delta(\hat{\mathbf{p}}|P^\circ_\cap(\mathfrak{v})) = \partial\delta^\circ_1(\hat{\mathbf{p}}) + \cdots + \partial\delta^\circ_\nu(\hat{\mathbf{p}}),$$

provided the convex sets $P^\circ_b(\mathbf{v}_b)$ have a relative interior point in common. Thus, if (2.26d) applies, there is a commodity allocation $\hat{\mathfrak{x}}$ solving (2.23)[5] such that

(2.27) $$\hat{\mathfrak{x}} = (\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_\nu), \quad \hat{\mathbf{x}} = \hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_\nu \quad \text{and} \quad \hat{\mathbf{x}}_b \in \partial\delta^\circ_b(\hat{\mathbf{p}}) \quad \forall\, b.$$

Each optimal commodity bundle $\hat{\mathbf{x}}_b$ corresponds to a vector being normal to the respective polar set $P^\circ_b(\mathbf{v}_b)$ of country b at the point $\hat{\mathbf{p}}$. In order to prove that the commodity bundle $\hat{\mathbf{x}}_b$ maximizes country b's revenue at the common price vector $\hat{\mathbf{p}}$ it is useful to apply Proposition 5 to the function $\delta^{\circ*}_b \equiv \delta^*(\cdot|P^\circ_b(\mathbf{v}_b))$. In this case (2.26d) becomes $\hat{\mathbf{x}}_b \in \partial\delta^\circ_b(\hat{\mathbf{p}})$, which is equivalent to $t_{Ob}(\hat{\mathbf{x}}_b, \mathbf{v}_b) \equiv \delta^{\circ*}_b(\hat{\mathbf{x}}_b) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}_b$ resulting from (2.26c) with $\delta^\circ_b(\hat{\mathbf{p}}) = 0$. As has been shown in Bobzin (1999, Proposition 2.1), this result applies if and only if $\hat{\mathbf{x}}_b$ solves the problem of revenue maximization (2.3). The pair of polar points $(\hat{\mathbf{x}}_b, \hat{\mathbf{p}})$ satisfies Mahler's inequality (2.20) as an equation.

Regarding Proposition 5, which is based on the Young-Fenchel inequality, $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ is called a *pair of dual points*. However, Proposition 2.1 in Bobzin (1999) refers to polar gauges and, therefore, is based on Mahler's inequality. Hence, $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ is called a *pair of polar points*.

2.3.3 Comparison of some Results

Regarding the common price vector $\hat{\mathbf{p}}$, two outcomes can be compared to each other:

[b] $\hat{\mathbf{p}} \in \partial\delta^*(\hat{\mathbf{x}}|P^\circ_\cap(\mathfrak{v}))$ by (2.26a)

[c] $\hat{\mathbf{p}} \in \partial\delta(\hat{\mathbf{x}}|P_\Sigma(\mathfrak{v}))$ by (2.10a)

This result is noteworthy because the indicator function $\delta(\cdot|P^\circ_\cap(\mathfrak{v}))$ and the support function $\delta^*(\cdot|P_\Sigma(\mathfrak{v}))$ are based on completely different principles. Putting $t_O(\cdot, \mathfrak{v}) \equiv \delta^*(\cdot|P^\circ_\cap(\mathfrak{v}))$ analogous to the revenue

[5] Suppose the allocation $\hat{\mathfrak{x}}$ in (2.27) does not solve (2.22), then (2.21), (2.22) and (2.24) imply

$$\hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} - \delta^\circ_1(\hat{\mathbf{p}}) - \cdots - \delta^\circ_\nu(\hat{\mathbf{p}}) < \delta^{\circ*}_1(\hat{\mathbf{x}}_1) + \cdots + \delta^{\circ*}_\nu(\hat{\mathbf{x}}_\nu)$$

But $\hat{\mathbf{x}}_b \in \partial\delta^\circ_b(\hat{\mathbf{p}})$ is the same as $\hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}_b = \delta^\circ_b(\hat{\mathbf{p}}) + \delta^{\circ*}_b(\hat{\mathbf{x}}_b)$ contradicting the above inequality.

function, it can be summarized that

$$\begin{aligned}
\delta(\hat{\mathbf{x}}|P_\Sigma(\mathfrak{v})) &= (\delta_1\Box \cdots \Box\delta_\nu)(\hat{\mathbf{x}}) &&= \delta_1(\hat{\mathbf{x}}_1)+\cdots+\delta_\nu(\hat{\mathbf{x}}_\nu) \\
\delta^*(\hat{\mathbf{p}}|P_\Sigma(\mathfrak{v})) &= (\delta_1\Box \cdots \Box\delta_\nu)^*(\hat{\mathbf{p}}) &&= \delta_1^*(\hat{\mathbf{p}})+\cdots+\delta_\nu^*(\hat{\mathbf{p}}) \qquad = r(\hat{\mathbf{p}},\mathfrak{v}) \\
\delta(\hat{\mathbf{p}}|P_\cap^\circ(\mathfrak{v})) & &&= \delta_1^\circ(\hat{\mathbf{p}})+\cdots+\delta_\nu^\circ(\hat{\mathbf{p}}) \\
\delta^*(\hat{\mathbf{x}}|P_\cap^\circ(\mathfrak{v})) &= (\delta_1^\circ+\cdots+\delta_\nu^\circ)^*(\hat{\mathbf{x}}) &&= \delta_1^{\circ*}(\hat{\mathbf{x}}_1)+\cdots+\delta_\nu^{\circ*}(\hat{\mathbf{x}}_\nu) = t_O(\hat{\mathbf{x}},\mathfrak{v})
\end{aligned}$$

Regarding a graphical representation it is important to note, that [a], [c] and even (2.11) in Proposition 4 refer to indicator functions. Thus [a] represents the normal cone of the intersection $P_\cap^\circ(\mathfrak{v})$ at $\hat{\mathbf{p}}$. Similarly, [c] corresponds to the normal cone of the set $P_\Sigma(\mathfrak{v})$ at $\hat{\mathbf{x}}$. On the contrary, [b] is based on a support function. Rewriting the subgradients as $\binom{\hat{\mathbf{p}}}{-1}$, these vectors form the so called normal cone in the sense of Clark.[6] This cone is normal to the epigraph of $\delta^*(\cdot|P_\cap^\circ(\mathfrak{v}))$ at $\begin{pmatrix}\hat{\mathbf{x}}\\ \delta^*(\hat{\mathbf{x}}|P_\cap^\circ(\mathfrak{v}))\end{pmatrix}$.

Finally, it has to be stressed that the outcome of (2.18) cannot be passed on to sums of functions. By (2.18), the revenue function and the output distance function of each country b are polar gauges. Symbolically this statement can be made clear by putting $C = P_b(\mathbf{v}_b)$:

$$t_{Ob}(\cdot,\mathbf{v}_b) = \delta^*(\cdot|C^\circ) = \gamma(\cdot|C) \quad \longleftrightarrow \quad r_b(\cdot,\mathbf{v}_b) = \delta^*(\cdot|C) = \gamma^\circ(\cdot|C)$$

If we want to apply this result to the sum of output distance functions, then it is important to know, that $P_\cap^\circ(\mathfrak{v}) = D^\circ$ is satisfied by Rockafellar (1972, Corollary 16.5.2) provided the set D is defined as follows:

$$D := \operatorname{conv}\{P_b(\mathbf{v}_b)|\ b=1,\ldots,\nu\} = \bigcup_{\substack{\lambda_b\geqq 0\\ \lambda_1+\cdots+\lambda_\nu=1}} \{\lambda_1 P_1(\mathbf{v}_1)+\cdots+\lambda_\nu P_\nu(\mathbf{v}_\nu)\}$$

Hence, the sets D and $P_\Sigma(\mathfrak{v})$ differ, so that $P_\cap^\circ(\mathfrak{v})$ and $P_\Sigma(\mathfrak{v})$ are *no* polar sets as well as $t_O(\cdot,\mathfrak{v})$ and $r(\cdot,\mathfrak{v})$ are *no* polar gauge functions.

$$\begin{aligned}
t_O(\cdot,\mathfrak{v}) &= \delta^*(\cdot|P_\cap^\circ(\mathfrak{v})) = \delta^*(\cdot|D^\circ) = \gamma(\cdot|D) \\
r(\cdot,\mathfrak{v}) &= \delta^*(\cdot|P_\Sigma(\mathfrak{v})) \neq \delta^*(\cdot|D) = \gamma(\cdot|D^\circ) = \gamma^\circ(\cdot|D)
\end{aligned}$$

In the above sense the result of (2.20) is not valid on an aggregate level.

2.4 Duality of Revenue Maximization and Profit Maximization

In what follows it is presumed that the factors can be moved from one firm to another. The commodity price vector $\mathbf{p}$ is constant and, therefore, dropped from notation for the sake of clarity. The problem of this

[6] In general, a subgradient $\hat{\mathbf{y}}$ of a convex function f at a point $\hat{\mathbf{x}}$, i.e. $\hat{\mathbf{y}} \in \partial f(\hat{\mathbf{x}})$, satisfies $f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) + \hat{\mathbf{y}}^\mathsf{T}(\mathbf{x}-\hat{\mathbf{x}})$ for all $\hat{\mathbf{x}}$. At the same time the normal cone to the epigraph epi f at $\binom{\hat{\mathbf{x}}}{f(\hat{\mathbf{x}})}$ is given by $\left\{\binom{\hat{\mathbf{y}}}{\xi}\middle|\ \binom{\hat{\mathbf{x}}}{\mu}^\mathsf{T}\binom{\hat{\mathbf{y}}}{\xi} \leqq \binom{\hat{\mathbf{x}}}{f(\hat{\mathbf{x}})}^\mathsf{T}\binom{\hat{\mathbf{y}}}{\xi}\ \forall \binom{\mathbf{x}}{\mu} \in \text{epi} f\right\}$. Rewriting the restriction with $\xi = -1$ shows that $\binom{\hat{\mathbf{y}}}{-1}$ is an element of this normal cone. Hence, $\binom{\hat{\mathbf{y}}}{-1}$ is normal to epi f at $\binom{\hat{\mathbf{x}}}{f(\hat{\mathbf{x}})}$.

section is to allocate the given factor endowment $\mathbf{v}$ to the firms, $b = 1, \dots, \nu$, such that the total revenue is maximized.[7]

$$(2.28) \qquad (r_1 \Box \cdots \Box r_\nu)(\mathbf{v}) = \sup \{ r_1(\mathbf{v}_1) + \cdots + r_\nu(\mathbf{v}_\nu) | \ \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v} \}$$

Dually, for a given vector of factor prices $\mathbf{q}$ the concave-conjugate function

$$(2.29) \qquad \begin{aligned} (r_1 \Box \cdots \Box r_\nu)_*(\mathbf{q}) &= \inf \left\{ \mathbf{q}^\mathsf{T}\mathbf{v} - (r_1 \Box \cdots \Box r_\nu)(\mathbf{v}) | \ \mathbf{v} \in \mathbb{R}^m \right\} \\ &= r_{1*}(\mathbf{q}) + \cdots + r_{\nu *}(\mathbf{q}) \end{aligned}$$

determines those factor endowments $\mathbf{v}$, which realize the minimum of the "negative profit". Here, firm b's profit results from $\pi_b(\cdot, \mathbf{p}) \equiv -r_{b*}$, where

$$r_{b*}(\mathbf{q}) := \inf \left\{ \mathbf{q}^\mathsf{T}\mathbf{v}_b - r_b(\mathbf{v}_b) | \ \mathbf{v}_b \in \mathbb{R}^m \right\}.$$

The concave-biconjugate function leads back to the revenue function r_b, which is closed and concave.

$$r_{b**}(\mathbf{v}_b) = \inf \left\{ \mathbf{q}^\mathsf{T}\mathbf{v}_b - r_{b*}(\mathbf{q}) | \ \mathbf{q} \in \mathbb{R}^m \right\} = \operatorname{cl} r_b(\mathbf{v}_b) = r_b(\mathbf{v}_b)$$

Again the one-to-one correspondence $f \to f_* \to f_{**} = f$ holds true in the class of all n-proper closed concave functions. As long as every r_b is n-proper and concave – excluding increasing economies of scale for each firm –, $(r_1 \Box \cdots \Box r_\nu)$ also shows these properties, particularly

$$(2.30) \qquad (r_1 \Box \cdots \Box r_\nu)(\mathbf{v}) = \inf \left\{ \mathbf{q}^\mathsf{T}\mathbf{v} - (r_1 \Box \cdots \Box r_\nu)_*(\mathbf{q}) | \ \mathbf{q} \in \mathbb{R}^m \right\}$$

After Theorem 7 has been switched over to n-proper concave functions – where the subdifferential ∂f of the convex function f becomes the superdifferential $\Delta(r_1 \Box \cdots \Box r_\nu)$ of the concave function $(r_1 \Box \cdots \Box r_\nu)$ – it follows

6 Proposition *If $(r_1 \Box \cdots \Box r_\nu)$ is an n-proper concave function, the following three conditions on a vector $\hat{\mathbf{v}}$ are equivalent to each other.*

(2.31a) $\quad \hat{\mathbf{q}} \in \Delta(r_1 \Box \cdots \Box r_\nu)(\hat{\mathbf{v}})$

(2.31b) $\quad \hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - (r_1 \Box \cdots \Box r_\nu)(\mathbf{v})$ *achieves its infimum in* $\mathbf{v}$ *at* $\hat{\mathbf{v}}$*; see (2.29)*

(2.31c) $\quad (r_1 \Box \cdots \Box r_\nu)(\hat{\mathbf{v}}) + (r_1 \Box \cdots \Box r_\nu)_*(\hat{\mathbf{q}}) = \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}$

[7] The problem (2.28) is equivalent to

$$(r_1 \Box \cdots \Box r_\nu)(\mathbf{p}, \mathbf{v}) = \sup_{\mathfrak{v}} \{ r(\mathbf{p}, \mathfrak{v}) | \ \mathfrak{v} = (\mathbf{v}_1, \dots, \mathbf{v}_\nu), \ \mathbf{v} = \mathbf{v}_1 + \cdots + \mathbf{v}_\nu \}$$

where the maximum revenue $r(\mathbf{p}, \mathfrak{v})$ regarding the allocation $\mathfrak{v}$ has been defined by (2.9). A simplified version of this problem concerning the Ricardo-Viner model can be found in Dixit, Norman (1980).

If $(\mathrm{cl}(r_1\square \cdots \square r_\nu))(\hat{\mathbf{v}}) = (r_1\square \cdots \square r_\nu)(\hat{\mathbf{v}})$, *then three more conditions can be added to the list.*

(2.31d) $\hat{\mathbf{v}} \in \Delta(r_1\square \cdots \square r_\nu)_*(\hat{\mathbf{q}})$

(2.31e) $\mathbf{q}^\mathsf{T}\hat{\mathbf{v}} - (r_1\square \cdots \square r_\nu)_*(\mathbf{q})$ *achieves its infimum in* $\mathbf{q}$ *at* $\hat{\mathbf{q}}$*; see (2.30)*

(2.31f) $\hat{\mathbf{q}} \in \Delta(\mathrm{cl}(r_1\square \cdots \square r_\nu))(\hat{\mathbf{v}})$

Given a pair of dual points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ satisfying all six conditions, (2.31c) simply states that the total profit equals revenue less factor costs.

$$\pi(\hat{\mathbf{q}}, \mathbf{p}) = -(r_1\square \cdots \square r_\nu)_*(\hat{\mathbf{q}}) = (r_1\square \cdots \square r_\nu)(\hat{\mathbf{v}}) - \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}$$

In accordance with Theorem 9 the condition (2.31d) can be put into a more concrete form.

7 Proposition *If the convex effective domains* n-Dom r_{b*}, $b = 1, \ldots, \nu$, *have a relative interior point in common, then*

$$\Delta(r_1\square \cdots \square r_\nu)_*(\mathbf{q}) = \Delta r_{1*}(\mathbf{q}) + \cdots + \Delta r_{\nu*}(\mathbf{q}) \qquad \forall\, \mathbf{q}$$

For an economic interpretation it is easier to apply Proposition 6 to the revenue function of merely one firm so that $(r_1\square \cdots \square r_\nu)$ becomes r_b. The analogue of (2.31d), $\hat{\mathbf{v}} \in \Delta r_{b*}(\hat{\mathbf{q}})$ is fulfilled if and only if the negative profit in (2.31b), i.e. $\hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - r_b(\mathbf{v})$, attains its minimum at $\hat{\mathbf{v}}_b$. Finally, the dual view can be stressed by examining a pair of dual points $(\hat{\mathbf{q}}, \hat{\mathbf{v}}_b)$ satisfying all six conditions of the translated version of Proposition 6. Now firm b faces two inverse correspondences, namely

$$\hat{\mathbf{q}} \in \Delta r_b(\hat{\mathbf{v}}_b) \iff \hat{\mathbf{v}}_b \in \Delta r_{b*}(\hat{\mathbf{q}}).$$

In the case of differentiable functions the factors $\hat{\mathbf{v}}$ demanded are chosen such that the factor prices equal the marginal revenue.

If the aggregate factor demand holds $\hat{\mathbf{v}} \in \Delta(r_1\square \cdots \square r_\nu)_*(\hat{\mathbf{q}})$ so that $\hat{\mathbf{q}}$ solves (2.29) by (2.31e), then an allocation $\hat{\mathfrak{v}}$ exists with

$$\hat{\mathfrak{v}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_\nu), \quad \hat{\mathbf{v}} = \hat{\mathbf{v}}_1 + \cdots + \mathbf{v}_\nu \quad \text{and} \quad \hat{\mathbf{v}}_b \in \Delta r_{b*}(\hat{\mathbf{q}}) \quad \forall\, b,$$

which solves the dual problem (2.28). In particular differentiable profit functions imply Hotelling's lemma,

$$\hat{\mathbf{v}}_b = \nabla r_{b*}(\hat{\mathbf{q}}) = -\nabla_{\mathbf{q}}\pi_b(\hat{\mathbf{q}}, \mathbf{p}) \quad \forall\, b.$$

Similar to (2.31d), we can put (2.31a) into a more concrete form where the proof corresponds to that of Proposition 4.

8 Proposition ("Common Factor Prices") *For a price vector* $\hat{\mathbf{q}}$ *,*

(2.32) $$\hat{\mathbf{q}} \in \bigcap_{b=1,\ldots,\nu} \Delta r_b(\hat{\mathbf{v}}_b)$$

holds true if and only if the allocation $\hat{\mathfrak{v}}$ *solves the problem (2.28) and (2.31a) is satisfied.*

Assuming that each revenue function is differentiable at the respective point $\hat{\mathbf{v}}_b$, the factor prices of all firms correspond to the marginal revenue, $\hat{\mathbf{q}} = \nabla r_b(\hat{\mathbf{v}}_b)$ for all b.

3 Dual Operations Based on Input Correspondences

3.1 Input Requirement Sets

The firm b's output correspondence P_b is opposite to the input correspondence L_b. Now the input requirement set $L_b(\mathbf{x}_b)$ includes all input vectors $\mathbf{v}_b$ permitting the production of the commodity bundle $\mathbf{x}_b$. The inverse character of both correspondences is reflected by the following equivalence relation, which is fulfilled for every admissible activity $(\mathbf{x}_b, \mathbf{v}_b)$:

$$\mathbf{x}_b \in P_b(\mathbf{v}_b) \iff \mathbf{v}_b \in L_b(\mathbf{x}_b)$$

To deal with the nonempty closed convex input requirement sets $L_b(\mathbf{x}_b)$ in an appropriate way, the *reciprocal*[8] indicator function $\varrho(\cdot|L_b(\mathbf{x}_b))$ is introduced. Similar to the original indicator function, $\varrho(\mathbf{v}_b|L_b(\mathbf{x}_b)) = 0$ if and only if $\mathbf{v}_b \in L_b(\mathbf{x}_b)$ holds good. On the contrary we now set $\varrho(\mathbf{v}_b|L_b(\mathbf{x}_b)) = -\infty$ instead of $+\infty$ for each inadmissible activity $(\mathbf{x}_b, \mathbf{v}_b)$. Every nonempty input requirement set has an *n-proper* closed concave indicator function.

(3.33) $$\operatorname{cl} \varrho(\cdot|L_b(\mathbf{x}_b)) = \varrho(\cdot|L_b(\mathbf{x}_b))$$

The abbreviated notation $\varrho_b \equiv \varrho(\cdot|L_b(\mathbf{x}_b))$ will be used as long as the commodity vectors $\mathbf{x}_b$ are given. Again $\varrho(\cdot|L_\Sigma(\mathfrak{x})) \equiv \varrho_B$ is the indicator function of the aggregate technology

(3.34) $$L_\Sigma(\mathfrak{x}) := L_1(\mathbf{x}_1) + \cdots + L_\nu(\mathbf{x}_\nu) = \{\mathbf{v}_1 + \cdots + \mathbf{v}_\nu |\, \mathbf{v}_b \in L_b(\mathbf{x}_b) \ \forall\, b\},$$

where the allocation $\mathfrak{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_\nu)$ is held fixed. As the sum of closed sets needs not be closed, $\operatorname{cl} \varrho_B = \varrho_B$ cannot be assumed a priori. In order to avoid an extensive proof of this equation, this fact is explicitly stressed in the subsequent Proposition 9.

[8] The term "reciprocal" will be dropped in all cases of unambiguity.

According to (2.2) the concave-conjugate function ϱ_{b*} is n-proper closed and concave and corresponds to the *reciprocal* support function of the input requirement set:

$$\begin{aligned} \varrho_{b*}(\mathbf{q}) &= \inf\left\{\mathbf{q}^\top \mathbf{v}_b - \varrho_b(\mathbf{v}_b))|\ \mathbf{x}_b \in \mathbb{R}^n\right\} \\ &= \inf\left\{\mathbf{q}^\top \mathbf{v}_b|\ \mathbf{v}_b \in L_b(\mathbf{x}_b)\right\} \end{aligned}$$

From an economic point of view, ϱ_{b*} is the *cost function* denoted by $\varrho_{b*} \equiv c_b(\cdot, \mathbf{x}_b)$. Again the one-to-one correspondence between the indicator function and the cost function ($\varrho_b \leftrightarrow \varrho_{b*}$) is determined by the concave-biconjugate function,

$$c_{b*}(\cdot, \mathbf{x}_b) = \varrho_{**b} = \mathrm{cl}\, \varrho_b = \varrho_b$$

in accordance with (3.33).

3.2 Duality of Feasible Activities and Cost Minimization

Suppose now, that the factor endowment $\mathbf{v}$ of the economy concerned is fixed and that each firm has to produce a certain commodity bundle $\mathbf{x}_b$. According to this commodity allocation $\mathfrak{x}$, the supremal convolution (3.35) gives an answer to the question whether it is possible to determine a factor allocation $\mathfrak{v}$ so that all firms $b = 1, \dots, \nu$ choose an admissible activity $(\mathbf{x}_b, \mathbf{v}_b)$ with the restriction that the factor demand $\mathbf{v}_1 + \cdots + \mathbf{v}_\nu$ equal factor supply $\mathbf{v}$.

$$(3.35) \qquad (\varrho_1 \Box \cdots \Box \varrho_\nu)(\mathbf{v}) = \sup\{\varrho_1(\mathbf{v}_1) + \cdots + \varrho_\nu(\mathbf{v}_\nu)|\ \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v}\}$$

If a factor allocation of the above type exists, then $\varrho_b(\mathbf{v}_b) = 0$ for each firm b and the optimal value of (3.35) is finite. In all other cases at least one ϱ_b takes the value $-\infty$ so that the infimal convolute $(\varrho_1 \Box \cdots \Box \varrho_\nu)$ is also infinite. Therefore, with regard to (3.34) the optimal value may be rewritten as

$$(\varrho_1 \Box \cdots \Box \varrho_\nu) = \varrho(\cdot|L_\Sigma(\mathfrak{x})) \equiv \varrho_B.$$

In accordance with the "concave version" of Theorem 6 (with $f_b = \varrho_b$) the minimal total cost at a factor price vector $\mathbf{q}$ results from

$$(3.36) \qquad \begin{aligned} (\varrho_1 \Box \cdots \Box \varrho_\nu)_*(\mathbf{q}) \equiv \varrho_{B*}(\mathbf{q}) &:= \inf\left\{\mathbf{q}^\top \mathbf{v} - \varrho_B(\mathbf{v})|\ \mathbf{v} \in \mathbb{R}^m\right\} \\ &= \varrho_{1*}(\mathbf{q}) + \cdots + \varrho_{\nu*}(\mathbf{q}) \\ :\Longleftrightarrow \qquad c(\mathbf{q}, \mathfrak{x}) &= c_1(\mathbf{q}, \mathbf{x}_1) + \cdots + c_\nu(\mathbf{q}, \mathbf{x}_\nu), \end{aligned}$$

where again the commodity allocation $\mathfrak{x}$ is given. The two problems (3.35) and (3.36) reflect the analogue duality of (2.7) and (2.8). Assuming the effective domains n-Dom $c_b(\cdot, \mathbf{x}_b)$, $b = 1, \ldots, \nu$ to have a relative interior point in common, then Theorem 6 implies

$$
\begin{aligned}
(3.37)\qquad c_*(\mathbf{v}, \mathfrak{x}) &= \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v} - c(\mathbf{q}, \mathfrak{x}) \mid \mathbf{q} \in \mathbb{R}^m\right\} && \text{conjugate of (3.36)}\\
&= \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v} - c_1(\mathbf{q}, \mathbf{x}_1) - \cdots - c_\nu(\mathbf{q}, \mathbf{x}_\nu) \mid \mathbf{q} \in \mathbb{R}^m\right\}\\
&= \sup\left\{c_{1*}(\mathbf{v}_1, \mathbf{x}_1) + \cdots + c_{\nu*}(\mathbf{v}_\nu, \mathbf{x}_\nu) \mid \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v}\right\} && \text{by Theorem 6}\\
&= \sup\left\{\varrho_1(\mathbf{v}_1) + \cdots + \varrho_\nu(\mathbf{v}_\nu) \mid \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v}\right\}\\
&= (\varrho_1 \square \cdots \square \varrho_\nu)(\mathbf{v}) && \text{cf. problem (3.35)}
\end{aligned}
$$

Having this result in mind, Proposition 9 firstly describes the properties of optimal solutions to (3.36) and (3.37). Afterwards, we turn to an optimal solution to (3.35). Theorem 7 yields immediately

9 Proposition *If the indicator function* $\varrho_B \equiv (\varrho_1 \square \cdots \square \varrho_\nu)$ *is n-proper and concave, then the following three conditions on a vector* $\hat{\mathbf{v}}$ *are equivalent to each other.*

(3.38a) $\hat{\mathbf{q}} \in \Delta\varrho_B(\hat{\mathbf{v}})$

(3.38b) $\hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - \varrho_B(\mathbf{v})$ *achieves its infimum in* $\mathbf{v}$ *at* $\hat{\mathbf{v}}$*; see (3.36)*

(3.38c) $\varrho_B(\hat{\mathbf{v}}) + \varrho_{B*}(\hat{\mathbf{q}}) = \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}$

If $(\mathrm{cl}\,\varrho_B)(\hat{\mathbf{v}}) = \varrho_B(\hat{\mathbf{v}})$, *then three more conditions can be added to the list.*

(3.38d) $\hat{\mathbf{v}} \in \Delta\varrho_{B*}(\hat{\mathbf{q}})$

(3.38e) $\mathbf{q}^\mathsf{T}\hat{\mathbf{v}} - \varrho_{B*}(\mathbf{q})$ *achieves its infimum in* $\mathbf{q}$ *at* $\hat{\mathbf{q}}$*; see (3.37)*

(3.38f) $\hat{\mathbf{q}} \in \Delta(\mathrm{cl}\,\varrho_B)(\hat{\mathbf{v}})$

Similar to Proposition 1 some remarks on the economic content are handed in referring particularly to the factors demanded and the common vector of factor prices.

The preceding proposition presupposes a commodity allocation $\mathfrak{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_\nu)$ and investigates the question whether there are feasible activities for all firms such that they produce altogether the commodity bundle $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_\nu$. Regarding (3.38b) and (3.38c), it is worthwhile to recall that $\varrho_B(\hat{\mathbf{v}}) = 0$ or $\hat{\mathbf{v}} \in L_\Sigma(\mathfrak{x})$ is equivalent to the existence of a feasible factor allocation $\hat{\mathfrak{v}}$, i.e.

$$\exists\, \hat{\mathfrak{v}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_\nu):\ \hat{\mathbf{v}} = \hat{\mathbf{v}}_1 + \cdots + \hat{\mathbf{v}}_\nu \quad \text{and} \quad \hat{\mathbf{v}}_b \in L_b(\mathbf{x}_b)\ \ \forall\, b.$$

While this relationship emphasizes the admissibility of activities, (3.40) stresses their optimality. However, before proceeding with (3.40) we need some further results. Concerning (3.38d), $\varrho_{B*} \equiv c(\cdot, \mathfrak{x})$ has to be taken into account. Hence, the assumption that the cost function is differentiable at $\hat{\mathbf{q}}$ implies a formula similar to Shephard's lemma,

$$\hat{\mathbf{v}} = \nabla_{\mathbf{q}} c(\hat{\mathbf{q}}, \mathfrak{x}). \tag{3.39}$$

With the aid of Theorem 9 this statement becomes more concrete by the following

10 Proposition *Assuming the convex effective domains* n-Dom ϱ_{b*}, $b = 1, \ldots, \nu$, *to have a relative interior point in common yields*

$$\Delta\varrho_{B*}(\mathbf{q}) = \Delta\varrho_{1*}(\mathbf{q}) + \cdots + \Delta\varrho_{\nu*}(\mathbf{q}) \qquad \forall\, \mathbf{q}.$$

Recall that ϱ_{b*} denotes firm b's cost function $c_b(\cdot, \mathbf{x}_b)$. If $\hat{\mathbf{v}} \in \Delta\varrho_{B*}(\hat{\mathbf{q}})$ holds good, there is a factor allocation $\hat{\mathfrak{v}}$ with

$$\hat{\mathfrak{v}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_\nu), \quad \hat{\mathbf{v}} = \hat{\mathbf{v}}_1 + \cdots + \hat{\mathbf{v}}_\nu \quad \text{and} \quad \hat{\mathbf{v}}_b \in \Delta\varrho_{b*}(\hat{\mathbf{q}}) \;\; \forall\, b. \tag{3.40}$$

While $\hat{\mathbf{q}}$ solves (3.36) by (3.38e), the allocation $\hat{\mathfrak{v}}$ solves the dual problem (3.35). The proof follows the same arguments as footnote 5. The statement of (3.36) becomes more obvious by noting the next

11 Proposition ("Factor Demand") *Let the input requirement set $L_b(\mathbf{x}_b)$ be nonempty closed and convex and let $\mathbf{q}$ be an arbitrary vector of factor prices. Then the superdifferential $\Delta\varrho_{b*}(\mathbf{q}) = \Delta c_b(\mathbf{q}, \mathbf{x}_b)$ consists of all points $\mathbf{v}_b$ – provided this set is not empty – where the linear function $\mathbf{q}^\mathsf{T}\mathbf{v}_b$ attains its minimum over $L_b(\mathbf{x}_b)$.*

The statement of this proposition can be compared to (3.39). Assuming differentiable cost functions, the firms' specific factor demand results from

$$\hat{\mathbf{v}}_b = \nabla_{\mathbf{q}} c_b(\hat{\mathbf{q}}, \mathbf{x}_b) \quad \forall\, b,$$

which is frequently called Shephard's lemma. Recall that this outcome results from (3.38d), where the dual condition (3.38a) refers to the inverse correspondence, i.e. explicitly

$$\hat{\mathbf{q}} \in \Delta\varrho_B(\hat{\mathbf{v}}) \iff \hat{\mathbf{v}} \in \Delta\varrho_{B*}(\hat{\mathbf{q}})$$

Geometrically, the left hand side superdifferential corresponds to the normal cone of the set $L_\Sigma(\mathfrak{x})$ at $\hat{\mathbf{v}}$. This cone includes all price vectors $\mathbf{q}$ being normal to $L_\Sigma(\mathfrak{x})$ at the point $\hat{\mathbf{v}}$. Economically, the same factor prices are valid with respect to all firms:

12 Proposition ("Common Prices") *For a price vector* $\hat{\mathbf{q}}$ *,*

$$\hat{\mathbf{q}} \in \bigcap_{b=1,\dots,\nu} \Delta\varrho_b(\hat{\mathbf{v}}_b)$$

is satisfied if and only if the allocation $\hat{\mathfrak{v}}$ *solves (3.35) and (3.38a) holds true.*

The proof parallels that of Proposition 4 and is omitted. With Proposition 12 the factor price vector $\hat{\mathbf{q}}$ is not only normal to the set $L_\Sigma(\mathfrak{x})$ at $\hat{\mathbf{v}}$ but also to each input requirement set $L_b(\mathbf{x}_b)$ at the respective point $\hat{\mathbf{v}}_b$.

3.3 Polar Input Requirement Sets

3.3.1 Duality of Reciprocally Polar Sets and Gauges

Following the arguments of Section 2.3.1, the analysis is now converted to *reciprocally* polar sets,

$$L_{b\circ}(\mathbf{x}_b) := \left\{\mathbf{q} \in \mathcal{Q} \mid \mathbf{q}^\top \mathbf{v}_b \geqq 1 \;\; \forall\, \mathbf{v}_b \in L_b(\mathbf{x}_b)\right\},$$

which are again closed and convex. Each set $L_{b\circ}(\mathbf{x}_b)$ consists of all factor price vectors so that the minimum costs of producing $\mathbf{x}_b$ do not fall below 1. In contrast to the polar production possibility sets of (2.15) they do *not* contain the origin $\mathbf{q} = \mathbf{0}$ and they are not star shaped but *aureoled* as the input requirement sets[9] themselves, i.e. $\lambda L_{b\circ}(\mathbf{x}_b) \subseteq L_{b\circ}(\mathbf{x}_b)$ for all $\lambda \geqq 1$. As the inequality in the definition cannot be fulfilled for $\mathbf{v}_b = \mathbf{0}$ we have to assume $\mathbf{x}_b \neq \mathbf{0}$ for the given commodity allocation $\mathfrak{x}$ such that $\mathbf{0} \notin L_b(\mathbf{x}_b)$. Because of the properties of $L_{b\circ}(\mathbf{x}_b)$ its indicator function $\varrho(\cdot|L_{b\circ}(\mathbf{x}_b))$ is n-proper, closed – hence (3.33) holds analogously – and convex and, particularly, $\varrho(\mathbf{0}|L_{b\circ}(\mathbf{x}_b)) = -\infty$. One important result is based on the assumption, that $L_b(\mathbf{x}_b)$ is a closed convex set not containing the origin. If $L_b(\mathbf{x}_b)$ is aureoled as well then the one-to-one correspondence $L_b(\mathbf{x}_b) \leftrightarrow L_{b\circ}(\mathbf{x}_b)$ results from (see Bobzin (1998, p. 159))

$$L_{b\circ\circ}(\mathbf{x}_b) = L_b(\mathbf{x}_b). \tag{3.41}$$

As long as the commodity allocation $\mathfrak{x}$ is held fixed, we make use of the simplified notation

$$\varrho(\cdot|L_{b\circ}(\mathbf{x}_b)) \equiv \varrho_{b\circ}.$$

This suggests the assumption that $\varrho_{b\circ}$ is the *reciprocally* polar function of ϱ_b. As it is difficult to prove a relationship similar to (2.16) regarding aureoled sets not containing the origin, this aspect is omitted. But

[9] The assumption of aureoled input requirement sets $L_b(\mathbf{x}_b)$ corresponds to the assumption of free disposability of inputs. In this sense idle inputs may be thrown away without disturbing the productions process.

it is no problem to comprehend that the inequality analogous to (2.17) remains valid.

$$\mathbf{q}^{\mathsf{T}}\mathbf{v}_b \geqq 1 \qquad \forall\, \mathbf{v}_b \in L_b(\mathbf{x}_b),\ \forall\, \mathbf{q} \in L_{b\circ}(\mathbf{x}_b) \tag{3.42}$$

To grasp the economic meaning of the concave-conjugate function of $\varrho_{b\circ}$

$$\begin{aligned} \varrho_{b\circ *}(\mathbf{v}_b) &\coloneqq \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}_b - \varrho_{b\circ}(\mathbf{q}) \mid \mathbf{q} \in \mathcal{Q}\right\} \\ &= \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v}_b \mid \mathbf{q} \in L_{b\circ}(\mathbf{x}_b)\right\} \end{aligned} \tag{3.43}$$

we firstly define the *reciprocal* gauge of a set C by

$$\psi(\mathbf{z}|C) \coloneqq \sup\left\{\lambda \geqq 0 \mid \mathbf{z} \in \lambda C\right\}.$$

Now $t_{Ib}(\cdot, \mathbf{x}_b) \equiv \psi(\cdot|L_b(\mathbf{x}_b))$ is called the *input distance function*, whose argument is an input vector $\mathbf{v}_b$. In comparison to the output distance function the newly defined input distance function has a similar meaning. It may be seen as generalization of a production function in the case of multi-outputs. While $t_{Ib}(\cdot, \mathbf{x}_b)$ takes the output vector as given, $t_{Ob}(\cdot, \mathbf{v}_b)$ depends on the parameter $\mathbf{v}_b$. Looking back on (3.43), the relationship between the input distance function and the cost function is determined by the support functions of the sets $L_b(\mathbf{x}_b)$ and $L_{b\circ}(\mathbf{x}_b)$ (cf. Bobzin (1998, Corollary III.16.1)):

$$\varrho_{b\circ *}(\mathbf{v}_b) = \psi(\mathbf{v}_b|L_b(\mathbf{x}_b)) = t_{Ib}(\mathbf{v}_b, \mathbf{x}_b) \qquad \forall\, \mathbf{v}_b \in K(L_b(\mathbf{x}_b)) \tag{3.44}$$

$$\varrho_{b*}(\mathbf{q}) = \psi(\mathbf{q}|L_{b\circ}(\mathbf{x}_b)) = c_b(\mathbf{q}, \mathbf{x}_b) \qquad \forall\, \mathbf{q} \in K(L_{b\circ}(\mathbf{x}_b)) \tag{3.45}$$

The enclosed cones K are in general defined by $K(C) \coloneqq \{\lambda\mathbf{z} \mid \mathbf{z} \in C,\ \lambda > 0\}$. Regarding (3.46) and (3.47) these cones are of importance because they ensure for instance

$$\mathbf{v}_b \in K(L_b(\mathbf{x}_b)) \iff t_{Ib}(\mathbf{v}_b, \mathbf{x}_b) > 0.$$

Finally, McFadden (1978) has proved that the cost function in (3.45) and the input distance function in (3.44) are *reciprocally* polar gauges, i.e.

$$c_b(\mathbf{q}, \mathbf{x}_b) = \sup\left\{\lambda \geqq 0 \mid \mathbf{q}^{\mathsf{T}}\mathbf{v}_b \geqq \lambda t_{Ib}(\mathbf{v}_b, \mathbf{x}_b)\ \ \forall\, \mathbf{v}_b \in K(L_b(\mathbf{x}_b))\right\} \quad \forall\, \mathbf{q} \in K(L_{b\circ}(\mathbf{x}_b)) \tag{3.46}$$

$$t_{Ib}(\mathbf{v}_b, \mathbf{x}_b) = \sup\left\{\lambda \geqq 0 \mid \mathbf{q}^{\mathsf{T}}\mathbf{v}_b \geqq \lambda c_b(\mathbf{q}, \mathbf{x}_b)\ \ \forall\, \mathbf{q} \in K(L_{b\circ}(\mathbf{x}_b))\right\} \quad \forall\, \mathbf{v}_b \in K(L_b(\mathbf{x}_b)) \tag{3.47}$$

Hence, both functions satisfy the modified Mahler's inequality

$$\mathbf{q}^{\mathsf{T}}\mathbf{v}_b \geqq c_b(\mathbf{q}, \mathbf{x}_b)\, t_{Ib}(\mathbf{v}_b, \mathbf{x}_b) \qquad \forall\, \mathbf{v}_b \in K(L_b(\mathbf{x}_b)),\ \forall\, \mathbf{q} \in K(L_{b\circ}(\mathbf{x}_b))$$

Analogous to (2.18) it may be useful to know the following result on the cost function and the input distance function as polar gauges.

$$t_{Ib}(\mathbf{v}_b, \mathbf{x}_b) = \varrho_{b\circ *}(\mathbf{v}_b) = \varrho_{b*\circ}(\mathbf{v}_b) = c_{b\circ}(\mathbf{v}_b, \mathbf{x}_b) \qquad \forall\, \mathbf{v}_b \in K(L_b(\mathbf{x}_b))$$

3.3.2 Dual Operations Regarding Polar Input Requirement Sets

Starting with a commodity allocation $\hat{\mathfrak{x}}$, each firm b shows a unique polar input requirement set $L_{b\circ}(\mathbf{x}_b)$ consisting of factor price vectors. If the same price vector is valid for all firms, then it should be an element of the following intersection

$$L_{\cap\circ}(\mathfrak{x}) := L_{1\circ}(\mathbf{x}_1) \cap \cdots \cap L_{\nu\circ}(\mathbf{x}_\nu).$$

The properties of polar sets ensue that their intersection $L_{\cap\circ}(\mathfrak{x})$ is closed and convex. Thus, the indicator function $\varrho(\cdot|L_{\cap\circ}(\mathfrak{x}))$ is closed and concave, however, is it n-proper? Regarding the given commodity allocation $\mathfrak{x}$, $\mathbf{x}_b \neq \mathbf{0}$ has been assumed for all firms. Therefore, each feasible factor allocation has to fulfill $\mathbf{v}_b \neq \mathbf{0}$, $b = 1, \ldots, \nu$. Now, there is no difficulty to determine a sufficiently large price vector $\tilde{\mathbf{q}}$ such that (3.42) is satisfied for all firms, i.e. $\tilde{\mathbf{q}} \in L_{\cap\circ}(\mathfrak{x})$. Thus, $\varrho(\cdot|L_{\cap\circ}(\mathfrak{x}))$ is n-proper.

However, $\mathbf{q} \in L_{\cap\circ}(\mathfrak{x})$ is the same as $\varrho_{1\circ}(\mathbf{q}) + \cdots + \varrho_{\nu\circ}(\mathbf{q}) = \varrho(\mathbf{q}|L_{\cap\circ}(\mathfrak{x})) = 0$. The problem of finding a cost minimum factor price vector – see (3.50) – has a dual problem, which is now determined by the supremal convolution instead of the infimal convolution of Section 2. An application of Theorem 6 (with $f_b = \varrho_{b\circ}$) yields

$$(\varrho_{1\circ} + \cdots + \varrho_{\nu\circ})_*(\mathbf{v}) = \mathrm{cl}(\varrho_{1\circ*}\square \cdots \square\varrho_{\nu\circ*})(\mathbf{v}). \tag{3.48}$$

Neglecting the closure operation on the right hand side of (3.48) implies

$$\begin{aligned}(\varrho_{1\circ*}\square \cdots \square\varrho_{\nu\circ*})(\mathbf{v}) &= \sup\{\varrho_{1\circ*}(\mathbf{v}_1) + \cdots + \varrho_{\nu\circ*}(\mathbf{v}_\nu)|\ \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v}\} \\ &= \sup\{t_{I1}(\mathbf{v}_1, \mathbf{x}_1) + \cdots + t_{I\nu}(\mathbf{v}_\nu, \mathbf{x}_\nu)|\ \mathbf{v}_1 + \cdots + \mathbf{v}_\nu = \mathbf{v}\}\end{aligned} \tag{3.49}$$

by definition. As the sum of input distance functions has no immediate meaning even for an *optimal factor allocation* $\hat{\mathfrak{v}}$, the attention is now drawn to the left hand side of (3.48).

$$\begin{aligned}(\varrho_{1\circ} + \cdots + \varrho_{\nu\circ})_*(\mathbf{v}) &= \inf\left\{\mathbf{q}^\top\mathbf{v} - \varrho_{1\circ}(\mathbf{q}) - \cdots - \varrho_{\nu\circ}(\mathbf{q})|\ \mathbf{q} \in \mathcal{Q}\right\} \\ &= \inf\left\{\mathbf{q}^\top\mathbf{v}|\ \mathbf{q} \in L_{1\circ}(\mathbf{x}_1) \cap \cdots \cap L_{\nu\circ}(\mathbf{x}_\nu)\right\} \\ &= \varrho_*(\mathbf{v}|L_{\cap\circ}(\mathfrak{x}))\end{aligned} \tag{3.50}$$

Hence, we seek for a price vector $\mathbf{q}$ *common to all firms*, which minimizes their total cost.

According to Theorem 3, the concave-conjugate function $\varrho_*(\cdot|L_{\cap\circ}(\mathfrak{x}))$ is not only closed and concave but also n-proper and

$$\varrho_{**}(\cdot|L_{\cap\circ}(\mathfrak{x})) = \mathrm{cl}\,\varrho(\cdot|L_{\cap\circ}(\mathfrak{x})) = \varrho(\cdot|L_{\cap\circ}(\mathfrak{x})).$$

With this information the "concave version" of Theorem 7 can be applied to $\varrho_*(\cdot|L_{\cap\circ}(\mathfrak{x}))$:

13 Proposition *As the total cost* $\varrho_*(\cdot|L_{\cap\circ}(\mathfrak{x}))$ *is an n-proper closed concave function, the subsequent five conditions on the pair of points* $(\hat{\mathbf{v}}, \hat{\mathbf{q}})$ *are equivalent to each other.*

(3.51a) $\hat{\mathbf{q}} \in \Delta\varrho_*(\hat{\mathbf{v}}|L_{\cap\circ}(\mathfrak{x}))$

(3.51b) $\hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - \varrho_*(\mathbf{v}|L_{\cap\circ}(\mathfrak{x}))$ *achieves its infimum in* $\mathbf{v}$ *at* $\hat{\mathbf{v}}$

(3.51c) $\varrho_*(\hat{\mathbf{v}}|L_{\cap\circ}(\mathfrak{x})) + \varrho(\hat{\mathbf{q}}|L_{\cap\circ}(\mathfrak{x})) = \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}$

(3.51d) $\hat{\mathbf{v}} \in \Delta\varrho(\hat{\mathbf{q}}|L_{\cap\circ}(\mathfrak{x}))$

(3.51e) $\mathbf{q}^\mathsf{T}\hat{\mathbf{v}} - \varrho(\mathbf{q}|L_{\cap\circ}(\mathfrak{x}))$ *achieves its infimum in* $\mathbf{q}$ *at* $\hat{\mathbf{q}}$*; see (3.50)*

Regarding (3.51e) it is to be noted that the infimum is finite if and only if $\varrho(\hat{\mathbf{q}}|L_{\cap\circ}(\mathfrak{x})) = 0$ or $\hat{\mathbf{q}} \in L_{\cap\circ}(\mathfrak{x})$ holds good. Hence, (3.51c) determines the cost minimum $\hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}$.

By Corollary 9.1, the superdifferential $\Delta\varrho(\hat{\mathbf{q}}|L_{\cap\circ}(\mathfrak{x}))$ in (3.51d) can be divided into a sum of superdifferentials

$$\Delta\varrho(\hat{\mathbf{q}}|L_{\cap\circ}(\mathfrak{x})) = \Delta\varrho_{1\circ}(\hat{\mathbf{q}}) + \cdots + \Delta\varrho_{\nu\circ}(\hat{\mathbf{q}})$$

provided the convex sets $L_{b\circ}(\mathbf{x}_b)$ have a relative interior point in common. Moreover, it is known that the relation (3.51d) holds good if and only if there is a factor allocation $\hat{\mathfrak{v}}$ such that

$$\hat{\mathfrak{v}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_\nu), \quad \hat{\mathbf{v}} = \hat{\mathbf{v}}_1 + \cdots + \hat{\mathbf{v}}_\nu \quad \text{and} \quad \hat{\mathbf{v}}_b \in \Delta\varrho_{b\circ}(\hat{\mathbf{q}}) \;\; \forall\, b.$$

While $\hat{\mathbf{q}}$ solves the problem of cost minimization (3.48), we have learned from the arguments in footnote 5 that the factor allocation $\hat{\mathfrak{v}}$ is an optimal solution to the dual problem (3.49).

It follows from the supergradient relation that an optimal input vector $\hat{\mathbf{v}}_b$ is normal to the polar input requirement set $L_{b\circ}(\mathbf{x}_b)$ at $\hat{\mathbf{q}}$. At the same time this input vector is cost minimal for the production of $\mathbf{x}_b$. To see this apply Proposition 13 to $\varrho_{b\circ*}$ instead of $\varrho_*(\cdot|L_{\cap\circ}(\mathfrak{x}))$. The condition (3.51c) becomes then

$$\varrho_{b\circ*}(\hat{\mathbf{v}}_b) + \varrho_{b\circ}(\hat{\mathbf{q}}) = \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}_b \quad \text{with} \quad \varrho_{b\circ}(\hat{\mathbf{q}}) = 0$$

As has been proved in Bobzin (1999, Proposition 3.1), the input distance function $\varrho_{b\circ*} = t_{Ob}(\cdot, \mathbf{x}_b)$ achieves the value $\hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}}_b$ if and only if $\hat{\mathbf{v}}_b$ solves the corresponding problem of cost minimization.

Regarding a graphical representation of the price vector $\hat{\mathbf{q}}$, we have to note the completely different principles of construction regarding the functions $\varrho(\cdot|L_\Sigma(\mathfrak{x}))$ and $\varrho_*(\cdot|L_{\cap\circ}(\mathfrak{x}))$.

$$\hat{\mathbf{q}} \in \Delta\varrho_*(\hat{\mathbf{v}}|L_{\cap\circ}(\mathfrak{x})) \qquad \text{by (3.51a)}$$

$$\hat{\mathbf{q}} \in \Delta\varrho(\hat{\mathbf{v}}|L_\Sigma(\mathfrak{x})) \qquad \text{by (3.38a)}$$

If a geometrical representation is needed, one can orientate oneself to the hints on page 14. The same holds true with respect to the result at the end of Section 2.3.3, i.e. $\hat{\mathbf{q}} \in \bigcap_{b=1,\ldots,\nu} \Delta\varrho_b(\hat{\mathbf{v}}_b)$.

3.4 Duality of Cost Minimization and Profit Maximization

This last section returns to the case of immobile factors of production. To stress this assumption we go back to the notion of countries instead of firms. At the same time the vector of factor prices $\mathbf{q} \in \mathcal{Q}$ is held fixed so that it is dropped from notation for the sake of clarity.

Given the world output $\mathbf{x}$, the first problem is to find a cost minimum commodity allocation $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_\nu)$ so that the country specific outputs $\mathbf{x}_b$ sum up to $\mathbf{x}$. Technically, this problem corresponds to the infimal convolution[10]

(3.52) $$(c_1 \Box \cdots \Box c_\nu)(\mathbf{x}) = \inf\{c_1(\mathbf{x}_1) + \cdots + c_\nu(\mathbf{x}_\nu) \mid \mathbf{x}_1 + \cdots + \mathbf{x}_\nu = \mathbf{x}\}.$$

Regarding a given commodity price vector $\mathbf{p}$, the convex-conjugate function

(3.53) $$\begin{aligned}(c_1 \Box \cdots \Box c_\nu)^*(\mathbf{p}) &= \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x} - (c_1 \Box \cdots \Box c_\nu)(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\right\} \\ &= c_1^*(\mathbf{p}) + \cdots + c_\nu^*(\mathbf{p})\end{aligned}$$

determines those commodity bundles $\mathbf{x}$, which maximize total profit. In doing so the profit of each firm $\pi_b(\cdot, \mathbf{p}) \equiv c_b^*$ is given by

(3.54) $$c_b^*(\mathbf{p}) = \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b - c_b(\mathbf{x}_b) \mid \mathbf{x}_b \in \mathbb{R}^n\right\},$$

where the biconjugate function leads back to the initial cost function, provided the cost function is closed and convex. This one-to-one correspondence of the cost function and the profit function ($c_b \leftrightarrow c_{b*} = \pi_b(\cdot, \mathbf{p})$) is emphasized by

$$c_b^{**}(\mathbf{x}_b) = \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}_b - c_b^*(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\right\} = \operatorname{cl} c_b(\mathbf{x}_b) = c_b(\mathbf{x}_b).$$

If every c_b is proper and convex – excluding increasing economies of scale –, then the infimal convolute $(c_1 \Box \cdots \Box c_\nu)$ shows these properties, too, and

(3.55) $$(c_1 \Box \cdots \Box c_\nu)^{**}(\mathbf{x}) = \operatorname{cl}(c_1 \Box \cdots \Box c_\nu)(\mathbf{x}) = \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x} - (c_1 \Box \cdots \Box c_\nu)^*(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^n\right\}$$

[10] The problem (3.52) can equivalently be written as

$$(c_1 \Box \cdots \Box c_\nu)(\mathbf{x}) = \inf_{\mathbf{x}}\{c(\mathbf{q}, \mathbf{x}) \mid \mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_\nu),\ \mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_\nu\}$$

If each country maximizes its profit in accordance with (3.54), then, by (3.53), they behave together as if they minimize the total cost in the production $\mathbf{x}$ by (3.52). The duality of (3.53) and (3.55) is reflected by the properties of their respective solutions. Theorem 7 yields immediately

14 Proposition *Letting* $(c_1\Box\cdots\Box c_\nu)$ *be a proper convex function, then the following three conditions on the vector* $\hat{\mathbf{x}}$ *are equivalent to each other.*

(3.56a) $\hat{\mathbf{p}} \in \partial(c_1\Box\cdots\Box c_\nu)(\hat{\mathbf{x}})$

(3.56b) $\hat{\mathbf{p}}^\mathsf{T}\mathbf{x} - (c_1\Box\cdots\Box c_\nu)(\mathbf{x})$ *achieves its supremum in* $\mathbf{x}$ *at* $\hat{\mathbf{x}}$*; see (3.53)*

(3.56c) $(c_1\Box\cdots\Box c_\nu)(\hat{\mathbf{x}}) + (c_1\Box\cdots\Box c_\nu)^*(\hat{\mathbf{p}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$

If $(\mathrm{cl}(c_1\Box\cdots\Box c_\nu))(\hat{\mathbf{x}}) = (c_1\Box\cdots\Box c_\nu)(\hat{\mathbf{x}})$, *then three more conditions can be added to this list.*

(3.56d) $\hat{\mathbf{x}} \in \partial(c_1\Box\cdots\Box c_\nu)^*(\hat{\mathbf{p}})$

(3.56e) $\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} - (c_1\Box\cdots\Box c_\nu)^*(\mathbf{p})$ *achieves its supremum in* $\mathbf{p}$ *at* $\hat{\mathbf{p}}$*; see (3.55)*

(3.56f) $\hat{\mathbf{p}} \in \partial(\mathrm{cl}(c_1\Box\cdots\Box c_\nu))(\hat{\mathbf{x}})$

Given a pair of dual points $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$ satisfying all six conditions, (3.56c) merely states that the total profit $(c_1\Box\cdots\Box c_\nu)^*(\hat{\mathbf{p}})$ equals revenue $\hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$ less cost $(c_1\Box\cdots\Box c_\nu)(\hat{\mathbf{x}})$.

An additional information to (3.56d) results from Theorem 9:

15 Proposition *If the convex effective domains* $\mathrm{Dom}\, c_b^*$, $b = 1,\ldots,\nu$, *have a relative interior point in common, then*

$$\partial(c_1\Box\cdots\Box c_\nu)^*(\mathbf{p}) = \partial c_1^*(\mathbf{p}) + \cdots + \partial c_\nu^*(\mathbf{p}) \qquad \forall\, \mathbf{p}$$

An economic interpretation is more convenient, if Proposition 14 is applied to the cost function c_b instead of $(c_1\Box\cdots\Box c_\nu)$. Analogous to (3.56d) the relation $\hat{\mathbf{x}}_b \in \partial c_b^*(\hat{\mathbf{p}})$ is satisfied for a closed cost function if and only if the profit $\hat{\mathbf{p}}^\mathsf{T}\mathbf{x}_b - c_b(\mathbf{x}_b)$ (which is analogous to (3.56b)) achieves its maximum at $\hat{\mathbf{x}}_b$. Thus, at the optimum the subdifferential of the function c_b^* consists of all supplied commodity bundles. Dually, $\hat{\mathbf{p}} \in \partial c_b(\hat{\mathbf{x}}_b)$ holds good.

If (3.56d) is satisfied for the aggregate supply $\hat{\mathbf{x}}$, then a commodity allocation $\hat{\mathfrak{x}}$ exists such that

$$\hat{\mathfrak{x}} = (\hat{\mathbf{x}}_1,\ldots,\hat{\mathbf{x}}_\nu), \quad \hat{\mathbf{x}} = \hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_\nu \quad \text{and} \quad \hat{\mathbf{x}}_b \in \partial c_b^*(\hat{\mathbf{p}}) \;\; \forall\, b.$$

This allocation solves (3.52) again by the same arguments as known from footnote 5. While this result is an immediate consequence of Proposition 15 and, therefore, of (3.56d), the last result refers to the inverse correspondence of (3.56a). It states that the vector of commodity prices $\hat{\mathbf{p}}$ is valid for each country.

16 Proposition *For a price vector* $\hat{\mathbf{p}}$ *we have*

$$\hat{\mathbf{p}} \in \bigcap_{b=1,\dots,\nu} \partial c_b(\hat{\mathbf{x}}_b) \tag{3.57}$$

if and only if the allocation $\hat{\mathbf{x}}$ *solves the problem (3.52) and (3.56a) holds good.*

Again the proof is omitted as it parallels that of Proposition 4. Assuming differentiable cost functions, (3.57) holds true if and only if all firms $b = 1, \dots, \nu$ adjust their marginal costs to the commodity prices, $\hat{\mathbf{p}} = \nabla c_b(\hat{\mathbf{x}}_b)$.

4 Résumé

Once the production technology of a firm or an economy has been described by families of convex sets, the theory of duality is a more powerful tool than commonly known. Given a closed convex set, we equivalently express this set set by the indicator function, the support function (cost or revenue function) or the gauge function (distance functions). All of these functions are closely related to each other, even though they represent different points of view.

In this paper we go one step further ahead obeserving that the sum of convex sets is again convex. Now the above given instruments can be applied on an aggregated level to the production theory of international trade. Here the case of internationally immobile factors of production has to be distinguished from inputs being nationally mobile between firms. The theoretical findings refer to the results of the individual behavior of the firms as well as the results of their common behavior. One of the most important observations is a statement of the following kind: if many firms seek to maximize their common revenue, then it is required that each firm maximizes its own revenue and that all firms face the same commodity price vector. Given a common price vector, it is presumably more important, that profit maximizing firms behave as if they maximize their common revenue.

5 Mathematical Appendix

5.1 Glossary

aureoled The aureoled hull of a set C is defined by $\operatorname{aur} C = \{\lambda \mathbf{x} | \ \mathbf{x} \in C, \ \lambda \geqq 1\}$. Accordingly, the set C is called aureoled, if $C = \operatorname{aur} C$.

closed A proper function f is said to be closed, if its epigraph $\operatorname{epi} f := \left\{ \binom{\mathbf{x}}{\mu} \in \mathbb{R}^{n+1} | \ f(\mathbf{x}) \leqq \mu \right\}$ is closed. The closure of a proper convex function $f \to \operatorname{cl} f$ corresponds to the closure of its epigraph $\operatorname{epi} f \to \operatorname{cl}(\operatorname{epi} f)$ and can be determined analytically by

$$(\operatorname{cl} f)(\mathbf{z}) = \lim_{\lambda \uparrow 1} f((1-\lambda)\mathbf{x} + \lambda \mathbf{z}) \quad \text{with} \quad \mathbf{z} \in \operatorname{rint}(\operatorname{Dom} f).$$

The resulting function $\operatorname{cl} f$ is closed and convex and it differs from f at most at those points lying on the relative boundary of $\operatorname{Dom} f$. Similarly an n-proper function f is said to be closed, if its hypograph $\operatorname{hypo} f := \left\{ \binom{\mathbf{x}}{\mu} \in \mathbb{R}^{n+1} | \ f(\mathbf{x}) \geqq \mu \right\}$ is closed.

Dom/n-Dom The effective domain $\operatorname{Dom} f$ consists of all points $\mathbf{x}$ of the domain of f where $f(\mathbf{x}) < +\infty$. On the contrary the effective domain $\text{n-Dom} f$ denotes the set of all points $\mathbf{x}$ of the domain of f where $f(\mathbf{x}) > -\infty$.

normal A vector $\mathbf{y}$ is normal to a convex set C at a point $\mathbf{a} \in C$, if $(\mathbf{x} - \mathbf{a})^{\mathsf{T}} \mathbf{y} \leqq 0$ is satisfied for all $\mathbf{x} \in C$. The set of all vectors $\mathbf{y}$ normal to C at $\mathbf{a}$ is called the normal cone to C at $\mathbf{a}$.

proper/n-proper A function f is said to be proper, if it attains nowhere the value $-\infty$ and if it is finite for at least one point. A function f is called n-proper, if $-f$ is proper.

relative interior The relative interior of a convex set $C \subset \mathbb{R}^n$ is denoted by $\operatorname{rint} C$ and corresponds to the interior of the affine hull of C.

star shaped The star shaped hull of a set C is defined by $\star C = \{\lambda \mathbf{x} | \ \mathbf{x} \in C, \ 0 \leqq \lambda \leqq 1\}$. Accordingly, the set C is called star shaped, if $C = \star C$.

5.2 Dual Operations

Regarding notation it is helpful to bear some symbols in mind. All functions marked by a superior star $*$ denote so called convex-conjugate functions. On the contrary concave-conjugate functions are marked by a lowered star. Polar sets and functions are awarded a superior circle $\circ$. Lowered circles characterize reciprocally polar sets and functions. The infimal convolution, which is in the centre of interest, is denoted by $\Box$. This operation is defined by

$$(f_1 \Box \cdots \Box f_m)(\mathbf{x}) := \inf\{ f_1(\mathbf{x}_1) + \cdots + f_m(\mathbf{x}_m) | \ \mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_m \} \tag{5.58}$$

The following theorems are taken from Rockafellar (1972) without their proof. They serve as basis of the analysis and remain here without any comment.

1 Theorem (Rockafellar, Theorem 5.4) *Let f_i ($i = 1, \ldots, m$) be proper convex functions on $\mathbb{R}^n$. Then the infimal convolute $(f_1 \square \cdots \square f_m)$ is a convex function on $\mathbb{R}^n$.*

2 Theorem (Rockafellar, Theorem 9.3) *Let f_i ($i = 1, \ldots, m$) be proper convex functions on $\mathbb{R}^n$. If every f_i is closed and $f_1 + \cdots + f_m$ is not identically $+\infty$, then $f_1 + \cdots + f_m$ is a closed proper convex function. If the f_i are not all closed, but all effective domains* Dom f_i *have a relative interior point in common, then*

$$\mathrm{cl}(f_1 + \cdots + f_m) = \mathrm{cl}\, f_1 + \cdots + \mathrm{cl}\, f_m\,.$$

To emphasize the aspect of duality it may be useful to denote the subsequently used spaces by $X = \mathbb{R}^n = X^*$. The convex-conjugate function[11] $f^*\colon X^* \to [-\infty, +\infty]$ of a function $f\colon X \to [-\infty, +\infty]$ is defined by

$$f^*(\mathbf{x}^*) \coloneqq \sup\left\{\mathbf{x}^{*\mathsf{T}}\mathbf{x} - f(\mathbf{x}) \,|\, \mathbf{x} \in X\right\}.$$

Hence, the pair of functions (f, f^*) satisfies the Young-Fenchel inequality

$$f^*(\mathbf{x}^*) + f(\mathbf{x}) \geqq \mathbf{x}^{*\mathsf{T}}\mathbf{x} \qquad \forall\, \mathbf{x} \in X,\ \ \forall\, \mathbf{x}^* \in X^*. \tag{5.59}$$

3 Theorem (Rockafellar, Theorem 12.2) *Let f be a convex function. The convex-conjugate function f^* is then closed and convex, proper if and only if f is proper. Moreover, $f^* = (\mathrm{cl}\, f)^*$ and $f^{**} = \mathrm{cl}\, f$.*

In order to represent a set $C \subset X$ by functions we make use of the indicator function $\delta(\cdot|C)\colon X \to [0, +\infty]$ with

$$\delta(\mathbf{x}|C) \coloneqq \begin{cases} 0 & \text{if } \mathbf{x} \in C \\ +\infty & \text{otherwise} \end{cases} \tag{5.60}$$

Besides that the gauge function $\gamma(\cdot|C)\colon X \to [-\infty, +\infty]$ and the support function $\delta^*(\cdot|C)\colon X^* \to [-\infty, +\infty]$ will be of major importance.

$$\gamma(\mathbf{x}|C) \coloneqq \inf\left\{\lambda \geqq 0 \,|\, \mathbf{x} \in \lambda C\right\} \tag{5.61}$$

$$\delta^*(\mathbf{x}^*|C) \coloneqq \sup\left\{\mathbf{x}^{*\mathsf{T}}\mathbf{x} \,|\, \mathbf{x} \in C\right\} = \sup\left\{\mathbf{x}^{*\mathsf{T}}\mathbf{x} - \delta(\mathbf{x}|C) \,|\, \mathbf{x} \in X\right\} \tag{5.62}$$

4 Theorem (Rockafellar, Theorem 14.5) *Let C be a closed convex set containing the origin. The polar set $C^\circ \coloneqq \left\{\mathbf{x}^* \in X^* \,|\, \mathbf{x}^\mathsf{T}\mathbf{x}^* \leqq 1\ \ \forall\, \mathbf{x} \in C\right\}$ is then another closed convex set containing the origin and $C^{\circ\circ} = C$. The gauge function $\gamma(\cdot|C)$ equals the support function $\delta^*(\cdot|C^\circ)$. Dually, the gauge function $\gamma(\cdot|C^\circ)$ corresponds to the support function $\delta^*(\cdot|C)$.*

[11] The biconjugate function f^{**}, which will be used at a later stage, corresponds to $(f^*)^*$.

5 Theorem (Rockafellar, Corollary 15.1.2) *Let C be a closed convex set containing the origin. Then $\gamma(\cdot|C)$ and $\delta^*(\cdot|C) = \gamma(\cdot|C^\circ) = \gamma^\circ(\cdot|C)$ are polar gauges satisfying* Mahler's inequality*:*

$$\mathbf{x}^{\mathsf{T}}\mathbf{x}^* \leqq \gamma(\mathbf{x}|C)\,\gamma(\mathbf{x}^*|C^\circ) \qquad \forall\,\mathbf{x} \in C,\ \forall\,\mathbf{x}^* \in C^\circ$$

6 Theorem (Rockafellar, Theorem 16.4)[12] *Let $f_i\colon X \to \mathbb{R}$ $(i = 1,\dots,m)$ be proper convex functions. Then*

$$(f_1 \Box \cdots \Box f_m)^*(\mathbf{x}^*) \equiv f_1^*(\mathbf{x}^*) + \cdots + f_m^*(\mathbf{x}^*)$$
$$(\mathrm{cl}\ f_1 + \cdots + \mathrm{cl}\ f_m)^*(\mathbf{x}^*) \equiv \mathrm{cl}(f_1^* \Box \cdots \Box f_m^*)(\mathbf{x}^*)$$

The closure operation can be omitted from the second formula, if the effective domains Dom f_i *have a relative interior point in common, and*

$$(f_1 + \cdots + f_m)^*(\mathbf{x}^*) = \inf\left\{ f_1^*(\mathbf{x}_1^*) + \cdots + f_m^*(\mathbf{x}_m^*) |\ \mathbf{x}_1^* + \cdots + \mathbf{x}_m^* = \mathbf{x}^* \right\}$$

where the infimum is attained for each $\mathbf{x}^*$.

The subsequent results refer to subgradients, where a vector $\mathbf{y}$ is said to be a subgradient of the function f at a point $\hat{\mathbf{x}} \in X$ if

(5.63) $$f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \qquad \forall\,\mathbf{x} \in X$$

The set of all subgradients of f at $\hat{\mathbf{x}}$ is called the subdifferential of f at $\hat{\mathbf{x}}$ and is denoted by $\partial f(\hat{\mathbf{x}})$.[13] If (5.63) is satisfied for the opposite case, where $\geqq$ is substituted by $\leqq$, then $\mathbf{y}$ is called a supergradient of f at $\hat{\mathbf{x}}$, which is denoted by $\mathbf{y} \in \Delta f(\hat{\mathbf{x}})$.

7 Theorem (Rockafellar, Theorem 23.5) *For any proper convex function f and any vector $\hat{\mathbf{x}}$, the following three conditions on a vector $\hat{\mathbf{x}}^*$ are equivalent to each other.*

(a) $\hat{\mathbf{x}}^* \in \partial f(\hat{\mathbf{x}})$

(b) $\mathbf{x}^{\mathsf{T}}\hat{\mathbf{x}}^* - f(\mathbf{x})$ *achieves its supremum in $\mathbf{x}$ at $\hat{\mathbf{x}}$*

(c) $f(\hat{\mathbf{x}}) + f^*(\hat{\mathbf{x}}^*) = \hat{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{x}}^*$

[12] Some economic applications of this result can be found in Aubin (1979).

[13] In the special case where f is the indicator function of a nonempty set C, i.e. $f = \delta(\cdot|C)$, the subdifferential $\partial\delta(\mathbf{x}|C)$ is the normal cone to C at $\mathbf{x}$. The subdifferential is empty if $\mathbf{x} \notin C$.

If $(\mathrm{cl}\, f)(\hat{\mathbf{x}}) = f(\hat{\mathbf{x}})$, *three more conditions can be added to the list.*

(a*) $\hat{\mathbf{x}} \in \partial f^*(\hat{\mathbf{x}}^*)$

(b*) $\hat{\mathbf{x}}^\mathsf{T}\mathbf{x}^* - f^*(\mathbf{x}^*)$ *achieves its supremum in* $\mathbf{x}^*$ *at* $\hat{\mathbf{x}}^*$

(a**) $\hat{\mathbf{x}}^* \in \partial(\mathrm{cl}\, f)(\hat{\mathbf{x}})$

7.1 Corollary (Rockafellar, Corollary 23.5.1) *Let* f *be a proper closed convex function. Then the subdifferentials of the functions* f *and* f^* *are inverse in the sense of multivalued mappings.*

$$\mathbf{x} \in \partial f^*(\mathbf{x}^*) \iff \mathbf{x}^* \in \partial f(\mathbf{x})\,.$$

7.2 Corollary (Rockafellar, Corollary 23.5.2) *Letting* f *be a proper convex function and* $\mathbf{x}$ *be a point where* f *is subdifferentiable, then* $(\mathrm{cl}\, f)(\mathbf{x}) = f(\mathbf{x})$ *and* $\partial(\mathrm{cl}\, f)(\mathbf{x}) = \partial f(\mathbf{x})$.

7.3 Corollary (Rockafellar, Corollary 23.5.3) *Let C be a nonempty, convex set. Then, for each vector* $\mathbf{x}^*$*, the subdifferential of the support function* $\partial\delta^*(\mathbf{x}^*|C)$ *consists of points* $\hat{\mathbf{x}}$ *where the linear function* $\mathbf{x}^\mathsf{T}\mathbf{x}^*$ *achieves its maximum over C. Notice, that* $\partial\delta^*(\mathbf{x}^*|C)$ *may be empty.*

8 Theorem (Rockafellar, Theorem 23.7) *Let* f *be a proper convex function and* $\hat{\mathbf{x}}$ *be a point where* f *is subdifferentiable, but* f *does not achieve its minimum at* $\hat{\mathbf{x}}$*. Then the normal cone of the set* $C = \{\mathbf{x} |\ f(\mathbf{x}) \leqq f(\hat{\mathbf{x}})\}$ *at* $\hat{\mathbf{x}}$ *is the closure of the convex cone generated by* $\partial f(\hat{\mathbf{x}})$.
Moreover, if $\hat{\mathbf{x}} \in \mathrm{int}(\mathrm{Dom}\, f)$*, then* $\mathbf{x}^*$ *is normal to C at* $\hat{\mathbf{x}}$ *if and only if there is a* $\lambda \geqq 0$ *such that* $\mathbf{x}^* \in \lambda\partial f(\hat{\mathbf{x}})$.

9 Theorem (Rockafellar, Theorem 23.8)[14] *Let* $f_1, \ldots, f_m$ *be proper convex functions on* $\mathbb{R}^n$*, and let* $f = f_1 + \cdots + f_m$*. Then*

$$\partial f(\mathbf{x}) \supset \partial f_1(\mathbf{x}) + \cdots + \partial f_m(\mathbf{x}) \qquad \forall\, \mathbf{x}\,.$$

Moreover, if the convex effective domains $\mathrm{Dom}\ f_i$ *have a relative interior point in common, then*

$$\partial f(\mathbf{x}) = \partial f_1(\mathbf{x}) + \cdots + \partial f_m(\mathbf{x}) \qquad \forall\, \mathbf{x}\,.$$

9.1 Corollary (Rockafellar, Corollary 23.8.1) *Let* $C_1, \ldots, C_m$ *be the convex sets in* $\mathbb{R}^n$ *whose relative interiors have a point in common. Then the normal cone to* $C_1 \cap \cdots \cap C_m$ *at any given point* $\mathbf{x}$ *is* $K_1 + \cdots + K_m$*, where* K_i *is the normal cone to* C_i *at* $\mathbf{x}$.

$$\partial\delta(\mathbf{x}|\ C_1 \cap \cdots \cap C_m) = \partial\delta(\mathbf{x}|C_1) + \cdots + \partial\delta(\mathbf{x}|C_m)$$

[14] Ioffe, Tihomirov (1979, p. 47–50), call this result the Moreau-Rockafellar theorem.

References

AUBIN, J. P., *Mathematical Methods of Game and Economic Theory*. Amsterdam: North-Holland, 1979 (Studies in Mathematics and its Applications, 7).

BERGE, C., *Topological Spaces*. 3 ed. New York: Macmillan, 1963.

BOBZIN, H., *Indivisibilities: Microeconomic Theory with Respect to Indivisible Goods and Factors*. Heidelberg: Physica, 1998 (Contributions to Economics).

BOBZIN, H., Dualities in the Functional Representations of a Production Technology. *Discussion Paper No. 79–99,* University of Siegen, Germany, (1999).

DIXIT, A. and NORMAN, V., *Theory of International Trade*. Cambridge: Cambridge University Press, 1980.

FÄRE, R., *Fundamentals of Production Theory*. Berlin: Springer, 1988 (Lecture Notes in Economics and Mathematical Systems, 311).

IOFFE, A. D. and TIHOMIROV, V. M., *Theory of Extremal Problems*. Amsterdam: North-Holland, 1979 (Studies in Mathematics and its Applications, 6).

MCFADDEN, D., *Cost, Revenue, and Profit Functions*. Amsterdam: North-Holland, pp. 3–109, 1978.

ROCKAFELLAR, R. T., *Convex Analysis*. Princeton: Princeton University Press, 1972.

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