Volkswirtschaftliche
Diskussionsbeiträge

VWL

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On the Analysis of Growth Trajectories

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Discussion Paper No. 101-02

ISSN 1433-058x

UNIVERSITÄT SIEGEN
FACHBEREICH WIRTSCHAFTSWISSENSCHAFTEN

Beyond Balanced Growth: On the Analysis of Growth Trajectories

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January, 2002

Paper prepared for the Conference
"Old and New Growth Theories: An Assessment"
in Pisa, October 2001
Second version

*I gratefully acknowledge comments by Christian Groth on an earlier version of this paper. Most of all I have to thank Charles Pugh for his warm hospitality at the Department of Mathematics at the University of California, Berkeley, and for his professional advice and support.

Abstract

The core of the analysis of endogenous growth models typically is the examination optimality and stability of balanced growth trajectories. But the development of a robust and general economic theory of endogenous growth around this concept is limited by the lack of simple tools of analysis. In the paper we broaden the concept of balanced growth and propose two new methods of analysis. The first one invokes the theory of time varying systems and the second one is a compactification approach. Both methods are designed to clarify to what extent the linearization of the system along a path of balanced growth reveals enough information about the dynamics in a neighborhood of the path.

1 Introduction

The mathematical theory of economic growth is complicated enough but at the same time it is very restrictive. In the paper we want to demonstrate that in many well-known models the analysis of a growth path and its neighborhood may be simplified by the use of slightly more sophisticated mathematical tools. Moreover, these tools can be applied to models with a less restrictive, more flexible structure. Technically, the tools are closely related to those usually applied. They are based on linear approximations and involve the computations of eigenvalues.

Growth theory provides an increasing variety of mathematical models which describe forces driving economic growth and focus on some kind of dynamic equilibrium. There have been substantial improvements in the understanding of what an equilibrium is and how it evolves over time. Changes over time considered in models of economic growth are due to savings and investment in capital, accumulation of human capital, innovation in products and processes, creative destruction to name the most important ones. Individual and collective incentives, market structures as well as forms and degree of competition influence scope and direction of these changes. Beyond short run fluctuations we observe regularities in the long run dynamics. They still are assumed to justify the assumption of balanced growth in the long run.

In new growth theories the more elaborate microeconomic foundations of short run market reactions and allocation processes on the one hand and the long run trends on the other hand are interconnected. However, should the model be tractable there have to be limitations to the motion close to the path of balanced growth. If the model can be expressed in ratios of variables the path of balanced growth reduces to a rest point. The analysis of the dynamics in the neighborhood of this point is standard. Obviously, this trick places severe constraints on the degree of freedom in modelling economic dynamics off the path of balanced growth. For this and possibly for other reasons as well Robert Solow[1] judges upon the state of the art: "There is a dangerous lack of robustness in the assumptions that, so far, underlie every version of the theory. ...a model of genuine endogenous growth seems to be achievable only if everything in the model turns out just so." (p. viii). And later on in his book he votes "for less focus on steady

[1]Robert M. Solow (2000) *Growth Theory*, 2. Edition, Oxford University Press.

state paths and more on other kinds of equilibrium trajectories, even if that means dependence on simulations" (p. 182). The goal of this paper is to present mathematical tools which can be applied to generalized models and even to growth paths which show patterns of non-balanced growth.

The recommendation of the first tool is motivated by the fact that apart from the sustainability problem and from overall optimality economists are interested in how the growth path will develop in the near future. This is not only a question of time passing by, moreover it is a question of stocks of resources and knowledge, quantities and diversity of commodities produced, which altogether should not be to different from today. The more we look into the future the less reliable are the quantifications. However, severe changes to be expected in the far future due to current activities - e.g. sustainability problems - should not be neglected in the analysis. A tool which meets these requirements is the compactification of the growth path mapping the far future to a finite interval. Under fairly weak and economically reasonable conditions a compactification should be possible. In mathematical terms it requires normal hyperbolicity of the growth path to carry over to infinity.

The second tool is based on the theory of time varying systems developed for problems in electrical engineering. The idea is to disentangle the motion along a growth path from forces transversal to it. In general, these forces will change along the path, or in other words they will change as time passes by. It is then possible to derive conditions under which the time varying linearization provides sufficient information about the qualitative behavior of the system. As a conclusion the information is sufficient if the system does not change too fast.

Both tools work within a framework that is more general than that of balanced economic growth! Sufficient conditions for the applicability are much more generic. No ratios of variables in the neighborhood of a growth path are involved. Nor is the condition of constant growth rates necessary for the analysis. The approach may therefore serve as a step towards a more robust and generic theory of endogenous economic growth.

2 Beyond Balanced Growth

The goal of this section is to redefine the general model of a balanced growth path. At the end we will not give a concise, general definition. We will rather mark out substantial ingredients. Nevertheless, a prototype of a definition will be given.

To begin with, consider a growth path of a single economy. Every single point of such a *growth path* is associated with a particular point of time. It is made up of a description of the state of the economy together with all relevant (and available) decisions made at that time. A sequence of such points forms a growth path. The sequence may be finite running from one particular point in time to another, it may run from a finite starting time to infinite future or it may be double infinite.

Obviously, the restriction to a single economy has no strucural implication at this point. Whatever the scope of consideration is, we may as well think of the world economy as a whole. This will not matter until we connect the description of the economy to a model which deals with the interdependence of particular elements of the description.

Formally, the joint description of state and control may be an element of a fairly general mathematical space. However, due to availability of data and possible problems of computability it seems to be reasonable to confine the model to a subset of a finite dimensional vector space which has the structure of R^n. In other words, we have a list of real numbers measuring quantities of resources allocated to certain uses and the like.

An infinite time horizon raises the question of unboundedness of the growth path. Economists tend to think of unbounded paths at least as far as ideas, knowledge or other abstract quantities are concerned. In contrast to natural scientists economists often disregard bounds on labor force, capital stock goods produced. No doubt, this is questionable, but we do not want to discuss this issue here. However, we want to seize the opportunity to point out that unboundedness creates its own complications in the mathematical theory of dynamical systems.

Whether time is considered to be a discrete or continuous quantity is a less important issue. Although data are available only in discrete time series we may try to fit a model with continuous time. Technically, some issues are easier to deal with in discrete time models, others are simplified by using a continuous time framework.

The evolution of an economy over time is affected by different kinds of periodical or less regular fluctuations. The most obvious to find are seasonal fluctuations. More complicated patterns are generated by business cycles. A growth path of an economy is the more even development which remains after eliminating these fluctuations. This establishes a strucural property of a growth path expressed in terms of empirical data together with a broad understanding of basic economics. We take this as a first constituent of a definition:

Requirement A) A *growth path* is a time series of economic data free of seasonal fluctuations and business cycles.

Growth economists seem to agree that growth paths show some empirical regularities independent of the point of time they are taken at or the country the data stem from. Although there is no complete agreement about a list of such regularities there seems to be consensus to treat some of these regularities almost as facts. (Cf. Temple (1999) for a recent discussion of stilized facts and empirical evidence in economic growth.) Still, a particular model may deliberately disregard certain aspects (e.g. a *real* growth model of a *closed* economy) and corresponding stilized facts.

Nevertheless, compatibility with stilized facts is like a backbone of a growth model. It should therefore be part of the definition of balanced growth.

Requirement B): A model of a *growth path* should be in accordance with stylized facts. The list of stilized facts reproduced by a particular model need not and perhaps cannot not be complete.

The fact that a growth path is supposed to be smoother than raw data is not spectacular. Neither is the assumption of accordance with stilized facts. However, they indicate what a proper definition of *balanced* growth has to add to the basic description of a growth path and what the links to economic theory contribute to the understanding of economic growth. Whereas all the qualifications made so far are standard and rather obvious, the definition of a *balanced growth path* needs substantial conceptual, economic and technical input. The following formulation may constitute a first approach to pin down the concept. It rather demonstrates the need for a precise, complete definition than being one:

Requirement C): A *balanced growth path* is a growth path with a very smooth and regular pattern. Small deviations from the path should not change the dynamic behavior too much.

In general a formalization of this requirement is considered the core of a definition of *balanced growth.* A precise but still verbal form of the requirement is the phrase: Along a path of balanced growth all growth rates are constant. A formal version of this requirement may be given by[2]

Convention 1 (Balanced Growth) *Let $x(t)$, $t \epsilon R_+$, be a path in R^n. $x(t)$ is called balanced with growth rates $g_1, \ldots, g_n$, $g_i \geq 0 \, \forall i$, if*

$$\dot{x}_i = g_i x_i \,, \quad g_i \geq 0 \, \forall i$$

and $g_i > 0$ for at least one i.

Growth along a balanced growth path is called even if there is a common positive growth rate g and if all g_i are either equal to g or equal to zero. It is called uneven if there are at least two different positive growth rates.

There are two important issues we want to draw the readers' attention to. Often traditional growth models are presented in per capita form. This 'reduced form' then usually has a rest point - with zero growth - which corresponds to the path of balanced growth. However, the analysis of the reduced dynamical system is not fully equivalent to the analysis of the full growth model. In general exponential stability of the rest point does not imply stability of the full balanced growth path![3] From the economic point of view this is a minor problem in traditional (old) growth models. It may well be that we are only interested in the long run behavior of the per capita model from the beginning. In New Growth models this is a more complicated issue.

Reductions by taking ratios may hide interesting details and may even be misleading with respect to the analysis of stability and transition. The following little formal example demonstrates that the ratio of two variables may converge over time, whereas the amplitude of fluctuations around the trend increases! Consider

$$x(t) = t \,, \quad y(t) = t^{1/2} \sin(t)$$

The ratio $y(t)/x(t)$ converges to zero. However, the amplitude of the fluctuation around $y = 0$ goes to infinity of order t^α. This phenomenon occurs in economic models and one should deal with it carefully.[4]

[2]We hesitate to call this a definition, because a definition should be part of a theory which is not available in this case.

[3]Deardorf (1970) was the first to analyze this issue in the model by Solow and Swan. It is not always stability we are looking for. It may well be that economic policy instruments put the economy right away to balanced growth like in the unstable *Ak* model we will discuss later.

[4]The problem is often disregarded e.g. Benhabib and Perli (1994) in their analysis of the Lucas model, Barro and Sala-I-Martin (1995) do not mention it in their book, ect.

A further aspect is the dependence of such a concept on the choice of coordinates. Consider an even balanced growth path with positive growth rate g for $k \geq 2$ coordinates, say for $x_1, \ldots, x_n$. By a rotation in R^n we can transform the growth path into the first coordinate axis. It may then turn out, that all other growth rates except for the first one are undefined, i.e. not finite. The other way around, the system with

$$\dot{x}_1 = x_1 \quad \text{and} \quad \dot{x}_j = x_j^{1/2} \, \forall j > 1$$

has a solution $x_1(t) = e^t$ and $x_j = 0 \, \forall j > 1$ which is a linear path with growth rate g in direction of the path. Yet, the path does not satisfy the conventional definition of balanced growth given above. [5] However, almost any rotation of this path yields proper even balanced growth! The reason is that even balanced growth is a one-dimensional property along a line. Only the formulation looks n-dimensional! It does not contain any assumption about the growth rates perpendicular to the path under consideration, unless coordinates are perpendicular to the path.

There is another minor problem related to the choice of coordinates one can easily cope with. Adding a constant to a variable with positive growth rate shifts a path by the same constant but turns a balanced growth path into a (shifted) unbalanced one! Hence we may at least want to make the concept independent of such shifts.

Convention 2 (Affine Balanced Growth) *Let $x(t)$, $t \epsilon R_+$, be a path in R^n. $x(t)$ is called affine balanced with growth rates $g_1, \ldots, g_n$, $g_i \geq 0 \forall i$, and shift $\tilde{x}^0 = (x_1^0, \ldots, x_n^0)$, if*

$$\dot{x}_i = g_i(x_i - x_i^0)\,, \quad g_i \geq 0 \, \forall i$$

and $g_i > 0$ for at least one i for some constant vector x.

Although this seems to be a necessary formal generalization, we will not make use of the term *affine balanced growth* in the following. We may assume that appropriate choice of coordinates has led to *balanced growth* right away. We may even assume that all positive growth rates coincide! A simple nonlinear transformation will yield this property.[6]

[5] We will demonstrate below, that unbounded growth rates in directions perpendicular to a balanced growth path occur asymptotically in the Jones-Manuelli-model.

[6] Recall that $(x_i)^{\alpha_i}$ grows with α times the growth rate of x_i.

Beyond these considerations, convention (1) is not yet a satisfying definition, because it lacks some necessary further specifications. Convention (1) does not refer to requirement B and to our understanding there are further requirements one has to impose on *balanced growth.* Numerous authors use the term *balanced growth* exactly with the meaning of convention (1). Lucas (1988, p. 9) gives a hint that this term is nothing but a name which is "as good as any". We refrain from the question of (re-)naming the concept and turn to a slight extension.

The constant growth rate assumption for a particular trajectory does not impose much structure in the neighborhood of this path. It is easy to think of formal examples of dynamical systems with unbounded growth rates in direction and in arbitrary small neighborhoods of the path.[7] To our understanding this purely technical assumption is plausible. But we are not sure whether all standard examples of New Growth Theory satisfy the assumption!

To emphasize the importance of this kind of regularity we make it a formal assumption

Assumption R): Growth rates in direction of a balanced growth path should be finite in a neighborhood of the path even asymptotically.

In terms of empirical analysis a *growth path* is a single time series of observations. The path we observe hardly ever is really balanced - regardless of the additional requirements which are still to come. But we may well observe a convergence towards *balance.* Either due to underlying economic forces or due to appropriate - not necessary unique - policy measures a *balanced growth path* must be attainable or the concept is of almost no importance.

Requirement D): A *balanced growth path* of a model must be attainable. It will possibly be reached only after a long period of transition.

We may call a path gaining *balance* over time and reaching it in infinite future *asymptotically balanced.* Again we want to give a more formal representation of this idea: A formal version of this requirement may be given by

[7] A simple formal example is $\dot{x} = \gamma x + x^2 y$ and $\dot{y} = ay$. The x-axis is a balanced growth path with growth rate γ. However, for any $\epsilon \neq 0$ and $y = \epsilon$ the growth rate $\dot{x}/x = \gamma + \epsilon x$ goes to infinity with increasing x.

Convention 3 (Asymptotically Balanced Growth) *Let $\dot{x} = F(x)$ be an ordinary differential equation on an open set $W \subset R^n_+$. An asymptotically balanced growth path of F with asymptotic growth rates $\tilde{g} = (g_1, \dots g_n)$ is a curve C invariant under F with the property*

$$\lim_{t \mapsto \infty} F_i(x(t)) = g_i x_i(t) \; \forall i$$

Obviously, this convention includes the former one of *balanced growth*. Actually both types of growth paths may appear jointly in a single model. Dependent on the initial state of an economy there may be a collection of *asymptotically balanced growth paths* converging to a particular *balanced growth path*. In a nice and smooth model this collection will have the structure of a differentiable manifold, the stable manifold of the *balanced growth path*. This is the case in many models of optimal growth (e.g. Cass-Koopmans type of optimal growth models which go back to Ramsey (1928), Cass (1965) and Koopmans (1965)). The per capita version of this model has a saddlepoint which corresponds to a balanced growth path of the full model. The (one-dimensional) stable manifold of the saddle corresponds to a two-dimensional stable manifold of the balanced growth path.[8]

In general a model may have a single or even a higher-dimensional manifold of *asymptotically balanced growth paths* but no *balanced growth path* at all. Probably the best known such example with only a single *asymptotically balanced growth path* is the (per capita version of) Jones and Manuelli (1990). In this model the unique path of asymptotically balanced growth is the only optimal path. In such a model of endogenous growth, there is no simple reduction to a model with a rest point![9]

Beyond obtainability immunity against disturbances is a further aspect. A reasonable model should take shocks and marginal policy changes into account. Marginal appropriate policy changes as a reaction to minor shocks may put the economy in a position to reach the former path again. As a consequence, the economy may be thrown back by some amount of time

[8]In this model stability analysis of per capita model is sufficient. I.e. it can be shown that a trajectory of the full model converges to the balanced growth path if an arbitrary point of this trajectory is projected to a point of the stable manifold of the saddlepoint in the reduced model.

[9]This is not to say that there are no ways to reduce the asymptotically balanced growth path to a trajectory which converges to a rest point. E.g. Barro and Sala-I-Martin (1995, pp.161) and section 3 of this paper.

but on the same track again. In other words, adapted policy measures place the economy on the stable manifold of the former *asymptotically balanced growth path.*

Requirement E): A *(locally) stable asymptotically balanced growth path* is a growth path the economy may return to even after digression due to a minor economic shock and possibly after appropriate changes in economic policy.

Model builders have to derive from economic theory the forces which eventually push the economy back to the former path - rather than to any other growth path. The forces may be intrinsic like market forces or they may be due to more or less explicit and direct control through institutions whitch apply instruments of economic policy.

At the latest these considerations call for a comprehensive incorporation of many ideas of modern macro- and microeconomic theory. In this respect New Growth Theory is much more adequate than "traditional theory". However, the way general theory enters the arena of New Growth models is not generic and hence leads to less robust models. By not generic we mean that the strucural variety of the new elements is very limited. Functional form's are very restrictive and only along a thin line of new developments in growth theory implications of these new elements are or perhaps can be analyzed. Consequently, by less robust we mean that the resulting collection of New Growth models is a thin set. Slight changes in the structure of models throw us out of the world we know and we can understand.

Requirement F): A model of economic growth should be structurally robust.

We avoid using the term structurally stable to emphasize that this is an informal statement. The property we probably want is reasonable perturbations of the formal framework should not change the qualitative outcome of the analysis. The mathematics of structural stability of dynamical systems is well developed, but quite complicated to apply.

A couple of fundamental questions have to be answered before a formal approach is possible. And it becomes more intricate if the stability analysis is supposed to be carried through within a restrictive class of functional forms etc. How restrictive the framework should be is a deep economic question! The state of the art in modelling economic growth is a collection of very narrow models rather than a general theory. In particular this holds for the so called 'New Growth Theory'. Clearly we do not want to consider robustness in a class of arbitrary functions. Sound economic reasoning leads to restrictions we should not disregard. On the other hand, many limitations seem to be caused by expected complications in the analysis of more general models. I am sure that the analysis of growth paths viewed as trajecories rather than rest points of a reduced model will eventually relax some limitations. But it calls for an investment into new techniques at the beginning.

3 Well Known Examples

A well known and equally well understood example is the Cobb-Douglas version of the Ak-model of Jones and Manuelli (1990). The model leads to a differential equation[10]

$$\begin{aligned} \dot{k} &= A_1 k + A_2 k^{\alpha} - c - \delta k - nk \\ \dot{c} &= \sigma(A_1 - \alpha A_2 k^{\alpha-1} - \delta - \rho - n)c \end{aligned} \tag{JM}$$

which is a variation of the linear Ak-model.

$$\begin{aligned} \dot{k} &= B_1 k - c \\ \dot{c} &= B_2 c \\ &\quad \text{with } B_1 = A_1 - (\delta + n) \\ &\quad \text{and } B_2 = \sigma(A_1 - (\rho + \delta + n)) \end{aligned} \tag{AK}$$

The Ak-model has an unstable balanced growth path with slope $c/k = B := B_1 - B_2$ with growth rate $\gamma = B_2$. The intuition is that the Jones-Manuelli-model should have an unstable asymptotically balanced growth path with the same asymptotic slope B.

One method to analyze the model is a transformation proposed by Barro and Sala-I-Martin (1995, pp.161). Define new variables $\chi \equiv c/k$ and $z \equiv (Ak + Bk^{\alpha})/k$ and get a differential equation in terms of χ and z with a saddlepoint. This procedure is not generic. It is rather designed for this particular example and here it gives the correct answer. The optimal (consumption-) policy follows the stable manifold of this system towards

[10]The model is presented in standard notation, which is so much standard that I think I can do without explaining the symbols.

the saddlepoint. Transformed back into original variables this yields the unique asymptotically balanced growth path.

Whereas the structure of the model is considerably robust and not hybrid at all this method of analysis does not generalize to structural variations. Consider a model introduced by Steger (2000). It is an Ak-model of economic development with productive consumption. Let $\Psi(c)$ measure the net cost of consumption (ncc) and suppose this is the concave function of form $c - c^{\beta}$. Let $\eta(c)$ denote the elasticity of the marginal net cost of consumption.

$$\dot{k} = B_1 k - \Psi(c)\,, \quad \dot{c} = B_2 \frac{c}{1 + \theta\eta(c)}$$

Asymptotically the productive consumption effect vanishes and the model coincides with the basic Ak-model. I.e. there is an asymptotically balanced growth path approaching the line $c/k = B_2 - B_1$. But productive consumption spoils the possibility of an elementary transformation.

However, the asymptotic correspondence with the Ak-model preserves a property useful for the analysis. It can be shown that the asymptotically balanced growth path is normally hyperbolic, i.e. its stability can be studied by linearization of the dynamics along the path. This will eventually confirm the intuition that the qualitative behavior of the model coincides with the (linear) Ak model.

Another well known example is Romer's model of endogenous technological change (Romer 1990). A final product is produced from unskilled labor, human capital and an aggregate of intermediate products. Capital is used to manufacture intermediates and the productivity of capital depends on the degree of differentiation A. A can be increased by $R\&D$ activities at a rate proportional to the use of human capital in this sector. Output which is not consumed increases the stock of capital, and the optimal consumption path is derived by intertemporal optimization. The total amount of labor and human capital is constant.

Under certain parameter restrictions this model has a (unique) uneven balanced growth path. It can be shown, that this path has a two-dimensional stable manifold which covers the state space, i.e. for any initial condition there is an optimal path approching the unique balanced growth path. Such a path is asymptotically balanced. The collection of these asymptotically balanced solutions form the stable manifold of the balanced growth path.

More complicated is the situation in Lucas' human capital model (Lucas

1988). Human capital may either be used in the production of a physical good or in the "production" of new human capital. In this model parameter bifurcations occur. Stability depends on the size of parameters. Under weak conditions, the model has a balanced growth path. If the initial conditions of the economy do not yield balanced growth from the beginning, the question of existence of asymptotically balanced solutions arises.

Among other authors Benhabib and Perli (1994) discuss the saddlepoint stability of a reduced version of this model. But a careful examination of the full model shows that reasonable parameter values may yield a saddlepoint in the reduced model that corresponds to an unstable balanced growth path! In other words, only a proper analysis of the balanced growth path of the full model leads to the conditions under which this model is economically meaningful.

4 Mathematical Tools

In this section we want to introduce two procedures to analyze stability of an (asymptotically) balanced growth path. The initial consideration is similar in both cases. Mathematical methods to analyse the behavior near trajectories in bounded sets are well developed in the literature. They may be difficult to apply, but they exist. The key idea of both procedures is to associate the unbounded system with a particular bounded system. This has to be done in such a way that statements about stability correspond properly. The reductions we have seen in the last section do not satisfy this crucial requirement.

4.1 A Prototype Model of Balanced Growth

Computational efforts for both procedures are much smaller for the case of even balanced growth along a coordinate axis. This is only a question of choice of coordinates. Simple transformations can always turn a model into this form. In the following we therefore restrict the analysis to a prototype, a C^1 autonomous ordinary differential equation on an open subset of R^n of form

$$\begin{aligned} \dot{x} &= G(x,y) = \gamma x + \tilde{G}(x,y) \\ \dot{y} &= F(x,y) \end{aligned} \tag{PT}$$

with $x \epsilon R$, $y \epsilon R^{n-1}$, $\gamma > 0$.

Definition 1 (Balanced Growth) *A differential equation of form* (PT) *is a model of balanced growth, if*

(a) $\tilde{G}(x,0) = 0$ *for all* x

(b) $F(x,0) = 0$ *for all* x

We say the i-th normal growth rate is bounded along $y = 0$*, if* $F_i(x,0)/y_i$ *is bounded.*

A prototype model (PT) of balanced growth has a solution $x(t) = x_0 e^{\gamma t}$, $y_i(t) = 0$ for all i. Notice that we do not require the normal growth rates $F_i(x,0)/y_i$ to be bounded.

Asymptotic balanced growth is a weaker property and the conditions can be relaxed to:

Definition 2 (Asymptotic Balanced Growth) *A differential equation of form* (PT) *is a model of asymptotic balanced growth, if*

(c) $\lim_{x \mapsto \infty} \tilde{G}(x,0) = 0$

(d) $\lim_{x \mapsto \infty} F(x,0)/G(x,0) = 0$

Condition (d) requires the slope of the vector field to approach zero along $y = 0$. Due to (c) condition (d) reduces to $\lim_{x \mapsto \infty} F(x,0)/x = 0$. In other words, in the limit x grows with rate γ and all other variables are stationary.

(d) does not imply asymptotically bounded normal growth rates! Such a requirement would be too restrictive.

Recall the Jones-Manuelli-model (JM). A rotation

$$x := \frac{1}{\sqrt{1+B^2}}(k + Bc)\,, \quad y := \frac{1}{\sqrt{1+B^2}}(c - Bk)$$

yields a prototype model of asymptotic balanced growth:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} &= \begin{pmatrix} B_2 & -1 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\quad + \frac{1}{\sqrt{D}} \begin{pmatrix} 1 & B \\ -B & 1 \end{pmatrix} \begin{pmatrix} f\left(\frac{x-By}{\sqrt{D}}\right) \\ -\sigma f'\left(\frac{x-By}{\sqrt{D}}\right) \cdot \left(\frac{Bx+y}{\sqrt{D}}\right) \end{pmatrix} \end{aligned}$$

Clearly, the model has an asymptotic balanced growth path and it approaches $y = 0$. However, in the limit y has unbounded growth rate. The unbounded normal growth rate is the implication of the nonlinear component in the production function. Only the (linear) Ak-model has a path which is balanced from the beginning. In that case B_1 is the normal growth rate.

From the economic point of view possible unboundedness of normal growth rates certainly is an issue one should be aware of. Stability analysis without careful consideration of the possible implications is necessary. Reduced model analysis always involves the danger of missing this phenomenon. The methods we are going to introduce in the next section resume the phenomenon.

4.2 Towards the Analysis of Asymptotic Balanced Growth

First we demonstrate the ideas behind the methods by means of an example we already considered before. Recall the prototype form of the Ak-model

$$\dot{x} = B_2 x - y\,, \quad \dot{y} = B_1 y$$

The solution of the generic initial value problem is

$$x(t) = -\frac{y_0}{B}\, e^{B_1 t} + \left(x_0 - \frac{y_0}{B}\right) e^{B_2 t} \quad \text{and} \quad y(t) = y_0\, e^{B_1 t}$$

with $B = B_1 - B_2$ as before. For the sake of a general analysis we drop the assumptions on the signs of B_1, B_2 and B, respectively.

Obviously, the solution converges exponentially towards the x-axis if $B_1 < 0$. It diverges if $B_1 > 0$. For negative B_1 the whole plane is the stable manifold of the x-axis.

The first approach is to convert the system into a time dependent system. This can be done in such a way that the motion along the balanced growth path is identified with true time running. The motion transversal to the balanced growth path then is transformed into a changing motion around zero.

To derive a first time dependent version of the system we take ratios

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{B_1 y}{B_2 x - y}$$

If we consider x to be the exogenous variable and get a time dependent system in standard notation

$$\dot{y} = \frac{B_1 y}{B_2 t - y}$$

But the derivative of $\dot{y}$ with respect to y vanishes for large t at $y = 0$. This inconvenience can be eliminated by using lnx as exogenous variable instead. We get

$$\dot{y} = \frac{B_1 y}{B_2 - ye^{-t}}$$

with derivative

$$\frac{d\dot{y}}{dy} = \frac{B_1(B_2 - ye^{-t}) + B_1 ye^{-t}}{(B_2 - ye^{-t})^2}$$

For small y and $t \mapsto \infty$ this derivative converges to B_1/B_2. Here the sign of B_2 matters for stability because a negative B_2 reverses the direction of 'time'. If there is no time reversal we confirm the result, that $y = 0$ is unstable if and only if B_2 is positive.

In higher dimensional models stability of a time varying system is more complicated to analyze. We will go into some of the related technical details in section 4.3.

The second approach is a compactification of an (asymptotically) balanced growth path by a simple projection. We call it directional compactification because the dynamics in directions perpendicular to the (asymptotically) balanced growth path remain unchanged.

The projection $z = x/(1+x)$ takes the infinite horizon of the balanced growth path $x = \infty$ to $z = 1$. This yields

$$\dot{z} = (1-z)\left(B_2 z - (1-z)y\right), \quad \dot{y} = B_1 y$$

We are interested in the behavior of this system near the rest point $(1,0)$. The Jacobian matrix of the compactified system is

$$J = \begin{pmatrix} B_2(1-2z) + 2(1-z)y & -(1-z)^2 \\ 0 & B_1 \end{pmatrix}$$

At $(1,0)$ the Jacobian reduces to

$$J = \begin{pmatrix} -B_2 & 0 \\ 0 & B_1 \end{pmatrix}$$

Again we confirm the stability condition: the infinite horizon is a sink if $B_1 < 0$ and a saddle otherwise.

Although the idea behind both tools seems quite simple there are pitfalls to avoid in more general applications. In the following subsections we go into more details and point out what the possible problems are.

4.3 Slowly Varying Systems

Recall the prototype system (PT). If there is balanced growth along $y = 0$ we may consider x itself as exogenous variable and eliminate real time t. The first step to do so is to take ratios

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{F(x,y)}{\gamma x + \tilde{G}(x,y)}$$

Formally the system is an $(n-1)$-dimensional time dependent differential equation

$$\begin{aligned} \dot{y} &= \frac{e^{\tau} F(e^{\tau}, y)}{B_2 e^{\tau} + \tilde{G}(e^{\tau}, y)} \\ &= \frac{F(e^{\tau}, y)}{B_2 + \tilde{G}(e^{\tau}, y)/e^{\tau}} \end{aligned}$$

In case of balanced growth the term $\tilde{G}(e^{\tau}, 0)/e^{\tau}$ is equal to zero for all τ due to condition (a). If growth is only asymptotically balanced this term vanishes asymptotically. Hence, the procedure is still valid, provided τ is large enough.[11] Using lnx instead of x as exogenous variable is only a question of convenience. There is no deeper reason behind this choice.

The goal now is to analyze this system for $\tau \mapsto \infty$ near $y = 0$. The system is well defined and in principle suitable for a proper analysis. However, linearization with respect to y is not always an appropriate way to continue. The eigenvalues of the linear part of such a system may be misleading compared to the decisive role they play in autonomous systems. Markus and Yamabe (1960) give an example of a nonautonomous system where all eigenvalues have negative real part but the system has saddle structure. On the

[11] For finite time there may be reversals in the direction of $\dot{x}$. In such a case we cannot use x or $ln(x)$ as exogenous variable for too small values of x.

other hand, Skoog and Lau (1972) provide an example of a uniformly exponentially stable system with a positive eigenvalue for all τ. Slowly varying systems are those where the linear part tells the whole stability story. The crucial point is that the 'tendency' to converge or diverge at some point of time according to the eigenvalues may be dominated by the change of the system itself. [12] By definition of the term slowly varying systems are those system where eigenvalues determine stability. However, the conditions are very difficult to check.

Appendix: Slowly Varying Nonlinear Systems

An extended treatment of slowly varying nonlinear systems can be found in Barman (1973) and Vidyasagar (1993). The latter author provides a fairly general theorem which represents the state of the art.

Consider nonautonomous systems of form

$$\dot{x}(t) = f(t, x(t)), \quad \forall t \geq 0 \qquad (SV)$$

Let $s(\tau, t, x)$ be the solution of (SV) started at time t with initial condition x and evaluated at time τ. Furthermore, consider the autonomous system which results from freezing (SV) at time r, i.e. $\dot{x}(t) = f(r, x(t))$. Let $s_r(\tau, t, x)$ for $r \geq 0$ be the corresponding solution of the frozen system.

With this terminology Vidyasagar's theorem (1993, p. 248) reads

Theorem (Vidyasagar, p. 248)
Suppose

(i) f is C^1,
(ii) $\sup_{x \epsilon R^n} \sup_{t \geq 0} \|D_2(f(t,x)\| =: \lambda < \infty$,
(iii) there exist constants μ, $\delta > 0$ such that

$$\|s_r(\tau, t, x)\| \leq \mu \|x\| e^{-\delta(\tau - t)} \quad \forall \tau \geq t \geq 0, \forall x \epsilon R^n, \forall r \epsilon R_+$$

and finally
(iv) there exists a constant $\varepsilon > 0$ such that

$$\|D_1 f(t,x)\| \leq \varepsilon \|x\| \quad \forall t \geq 0, \forall x \epsilon R^n$$

[12] The article by Ilchmann, Owens and Pratzel-Wolters (1987) is a recent survey of the relevant mathematical literature. By the way, the theory is mainly developed for the field of electrical engeneering.

Then the nonautonomous system (SV) is globally exponentially stable provided

$$\varepsilon < \frac{\delta((p-1)\delta - \lambda)}{p\,\mu^p}$$

where $p > 1$ is any number such that $(p-1)\delta > \lambda$.

Further Remarks on Slowly Varying Systems

Slowly varying linear systems are fairly well understood. Though, the list of sufficient conditions for stability is and probably will remain open for extension.

Among the more general questions is the one concerning the existence and respective dimension of stable and unstable manifolds of hyperbolic critical points. At least in the generic case the technique for autonomous systems should generalize. (I have not check yet whether this has been dealt with in the literature! Most of the literature I have seen stems from the field of electrical engeneering.)

Apart from problems in the theory of non-linear dynamical systems which are related to the main issue adressed in the analysis of slowly varying linear systems. Non-linear systems are *changing systems* in the sense that their linear part is not constant. Along a trajectory this change may be translated into a time change of the system. Attractiveness of a trajectory then becomes the analogon to stability of an isolated critical point. Again existence and dimension of stable and unstable manifolds are important issues.

4.4 Directional Compactification

Figure 1 illustrates the geometry of the approach. The necessary calculations are straight forward. However, some extra considerations and a specific assumption will turn out to be necessary.

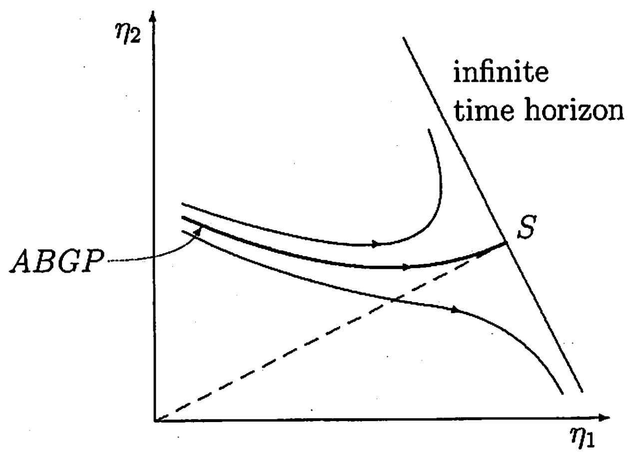
Recall the system (ABG)

$$\begin{aligned}\dot{x} &= G(x,y) = \gamma x + \tilde{G}(x,y)\\ \dot{y} &= F(x,y)\end{aligned}$$

Directional compactification by $z = x/(1+x)$ with inverse $x = z/(1-z)$ yields

$$\dot{z} = (1-z)^2\, G(\frac{z}{1-z}, y)$$

Figure 1: Asymyptotic balanced growth after directional compactification



$$
\begin{aligned}
&= z(1-z)\left(\gamma + \frac{1-z}{z}\tilde{G}(\frac{z}{1-z}, y)\right) \\
\dot{y} &= F(\frac{z}{1-z}, y)
\end{aligned}
$$

Notice that due to condition (c) for asymptotic balanced growth $\dot{z}$ cannot have a pole at $z = 1$ and $y = 0$. The expression

$$\frac{1-z}{z}\tilde{G}(\frac{z}{1-z}, y)$$

has to converge to zero!

Due to condition (d) the components of $F(\frac{z}{1-z}, y)$ cannot diverge to fast either. The expressions

$$\frac{1-z}{z}F_i(\frac{z}{1-z}, 0)$$

converge to zero. In other words, $F(\frac{z}{1-z}, 0)$ is of order less than 1 in the first variable.

There is a further implication of regularity: After directional compactification $z = 1$ is an invariant manifold. I.e.: In general the flow is tangent to the infinite horizon.

The same kind of problem arises with respect to $F(\frac{z}{1-z}, y)$. Again boundedness of growth rates imposes the property we need: it implies boundedness of all components even at $z = 1$.

From the literature on compactification of planar systems we borough the idea for the solution: we multiply $\dot{z}$ by an apropriate power of $(1-z)$ to guarantee differentiability at the infinite horizon. Of course we have to assume that this is possible.[13]

The Jacobian matrix J of the system at the infinite horizon of the balanced growth path can easily be computed. In the upper left corner we find

$$J_{1,1} = \gamma(1-2z) - 2(1-z)\,G(\frac{z}{1-z}, y) + D_x G(\frac{z}{1-z}, y)$$

The rest of the first line is given by $(1-z)^2 D_y G(\frac{z}{1-z}, y)$. The rest of the first column is $\frac{1}{(1-z)^2} D_x F(\frac{z}{1-z}, y)$. Finally, at the lower right there is the $(n-1)\times(n-1)$ block of partial derivatives of F with respect to y.

$$J = \begin{pmatrix} \gamma(1-2z) - 2(1-z)\,G(\frac{z}{1-z}, y) + D_x G(\frac{z}{1-z}, y) & \\ D_x F(\frac{z}{1-z}, y) & D_y F(\frac{z}{1-z}, y) \end{pmatrix}$$

can easily be computed. It contains an $(n-1)$ block of partial derivatives of F with respect to y. The corresponding $(n-1)$ eigenvalues are eigenvalues both of the compactified system and of the full system. The remaining eigenvalue is the negative of the balanced growth rate γ. Hence the balanced growth path is normally hyperbolic even towards the infinite horizon. The asymptotic eigenvalues in normal direction, i.e. those of F, determine the stability properties of the balanced growth path.[14] To make these arguments work we have to assume

Assumption 1: The eigenvalues of $D_y F(x, 0)$ converge along the path of asymptotically balanced growth.

5 Final Remarks

The constant growth rate property may be the core of any definition of (asymptotically) balanced growth, but it is not a sufficiently precise description. Coordinate dependence reveals that a more detailed definition

[13]We boroughed the idea for this tool from the literature on compactification of planar systems initiated by Poincaré (1882). He used projections of planar vector fields to the sphere. Chicone and Sotomayor (1986) give a modern discussion technique.

[14]C.f. Pugh and Shub (1970) on linearization and normal hyperbolicity.

is necessary. The economic forces perpendicular to the path under consideration deserve more attention. The analysis of the Jones-Manuelli model reveals this clearly.

The proposed methods of analysis (the time varying systems approach and the directional compactification approach) both are useful tools as demonstrated in the paper. However, they need to be sharpend. Their application to more complex models of economic growth is tractable. As tools they probably can be developed up to the point, where no transformation of the differential equation has to be carried out. Within their limits of applicability they then may be used to prove that linearization of the original system answers all relevant question. In other words, if the tools are sharpened we may realize that we do not need them anymore.

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