Dualities in the Functional Representations of a Production Technology

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Abstract: Given the production technology of a multiproduct firm, economists usually try to represent this technology by functions, namely the cost function, the revenue function and the input and output distance functions. In doing so the analysis directs the attention to the (dual) matching of quantities and prices. Here, the duality scheme is based on Mahler's inequality and stresses dual aspects of associated functions, whereas the underlying optimization problems are not dual programs. Nevertheless, the discussion of shadow pricing reveals the similarities which exist with respect to some appropriately chosen dual programs.

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1 Preliminaries

1.1 Characteristics of the Technology

In order to describe the production technology (of a firm or an economy) we make use of two families of sets, which describe the technological relationship between the space of factors of production $V = \mathbb{R}^m_+$ and the commodity space $X = \mathbb{R}^n_+$. On the one hand, each member $P(\mathbf{v})$ of the family of production possibility sets $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$ includes all commodity bundles $\mathbf{x} \in X$, which are producible by an input vector $\mathbf{v} \in V$. On the other hand, the family $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$ consists of input requirement sets, i.e. $L(\mathbf{x})$ includes all input vectors $\mathbf{v}$ permitting the production of $\mathbf{x}$. Both families are equivalent, i.e.

$$\mathbf{x} \in P(\mathbf{v}) \iff \mathbf{v} \in L(\mathbf{x}), \tag{1.1}$$

and they satisfy certain regularity conditions by assumption. In particular each member of both families is a nonempty closed convex set. The main difference regarding analytical aspects becomes apparent as follows: whereas an (aureoled) input requirement set $L(\mathbf{x})$ does not contain the origin $\mathbf{v} = \mathbf{0}$ for any commodity bundle $\mathbf{x} \in X \setminus \{\mathbf{0}\}$, the possibility of inaction guarantees the origin $\mathbf{x} = \mathbf{0}$ to be an element of the (star-shaped) production possibility set $P(\mathbf{v})$. See Färe (1988) or Bobzin (1998) for details.

In the next step economists try to extract activities, which are technologically efficient, cost minimizing, revenue maximizing and last but not least profit maximizing. In order to value vectors of outputs or vectors of inputs we need prices with the spaces of output prices and input prices being $P_{\mathbf{p}} = \mathbb{R}^n_+$ and $Q = \mathbb{R}^m_+$, respectively.[1]

This papers deals with four aspects of duality theory. The first approach is concerned with the relationship of outputs $\mathbf{x}$ and output prices $\mathbf{p}$ given an input vector $\mathbf{v}$. The second aspect concentrates on the opposite case, i.e. the relationship of inputs $\mathbf{v}$ and input prices $\mathbf{q}$ holding the output vector $\mathbf{x}$ fixed. In the third step the attention is directed to the case where the factor endowment $\mathbf{v}$ and the commodity prices $\mathbf{p}$ are given. Finally, the commodity bundle $\mathbf{x}$ and the vector of input prices $\mathbf{q}$ are assumed to be known. While the first two cases make use of the duality of polar gauges (see Newman (1987) for details), the third and the fourth case are known from linear programming as shadow pricing.

[1] Formally, the sets $P_{\mathbf{p}}$ and Q correspond to the polar cones of X and V, respectively. Take for example $Q = \{\mathbf{q} \in \mathbb{R}^m|\ \mathbf{q}^\mathsf{T}\mathbf{v} \geqq 0\ \ \forall\, \mathbf{v} \in V\}$. This concept is to be distinguished from that of polar sets which will be introduced at a later stage.

In what follows we firstly define functions in order to describe the members of the family $\big(P(\mathbf{v})|\ \mathbf{v} \in V\big)$. In the second step the same is done with respect to the family $\big(L(\mathbf{x})|\ \mathbf{x} \in X\big)$. For a given input vector $\mathbf{v}$ the *revenue function* is defined as the (convex) support function of the production possibility set $P(\mathbf{v})$ holding the price vector $\mathbf{p} \in P_\mathbf{p}$ fixed.

$$r(\mathbf{p}, \mathbf{v}) \coloneqq \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}|\ \mathbf{x} \in P(\mathbf{v})\right\} \tag{1.2}$$

On the contrary the (dimensionless[2]) *output distance function* is defined by

$$t_O(\mathbf{x}, \mathbf{v}) \coloneqq \inf\{\lambda \geqq 0|\ \mathbf{x} \in \lambda P(\mathbf{v})\}. \tag{1.3}$$

This function may be interpreted as the inverse of Farrell's output efficiency measure. As $P(\mathbf{v})$ is a closed convex set containing the origin it is

$$\mathbf{x} \in P(\mathbf{v}) \iff t_O(\mathbf{x}, \mathbf{v}) \leqq 1. \tag{1.4}$$

In order to express the preceding two functions as polar gauges it is useful to describe $P(\mathbf{v})$ by a system of hyperplanes tangent to $P(\mathbf{v})$. In doing so we get

$$P^\circ(\mathbf{v}) = \left\{\mathbf{p} \in P_\mathbf{p}|\ \mathbf{p}^\mathsf{T}\mathbf{x} \leqq 1\ \ \forall\, \mathbf{x} \in P(\mathbf{v})\right\}$$

which is called the *polar production possibility set*. The roles of the two functions $r(\cdot, \mathbf{v})$ and $t_O(\cdot, \mathbf{v})$ regarding this set are interchanged. On the one hand, the output distance function is the (convex) support function[3] of $P^\circ(\mathbf{v})$, i.e. $t_O(\mathbf{x}, \mathbf{v}) = \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x}|\ \mathbf{p} \in P^\circ(\mathbf{v})\right\}$. On the other hand, the revenue function corresponds to the distance function of $P^\circ(\mathbf{v})$, i.e. $r(\mathbf{p}, \mathbf{v}) = \inf\{\mu \geqq 0|\ \mathbf{p} \in \mu P^\circ(\mathbf{v})\}$. Similar to (1.4) we have

$$\mathbf{p} \in P^\circ(\mathbf{v}) \iff r(\mathbf{p}, \mathbf{v}) \leqq 1. \tag{1.5}$$

In terms of convex analysis $r(\cdot, \mathbf{v})$ and $t_O(\cdot, \mathbf{v})$ may be seen as polar gauges which satisfy the following

Proposition 1.1 *Let $P(\mathbf{v})$ be a nonempty closed convex production possibility set containing the origin $\mathbf{x} = \mathbf{0}$. Then the output distance function $t_O(\cdot, \mathbf{x})$ and the revenue function $r(\cdot, \mathbf{x})$ are polar to each other and fulfil Mahler's inequality*[4]

$$\mathbf{p}^\mathsf{T}\mathbf{x} \leqq r(\mathbf{p}, \mathbf{v})\, t_O(\mathbf{x}, \mathbf{v}) \qquad \forall\, \mathbf{p} \in P_\mathbf{p},\ \ \forall\, \mathbf{x} \in X. \tag{1.6}$$

[2] Presumably, this point shows most obviously that we cannot speak of dual programs holding a statement of the type $\sup_X \mu(\mathbf{x}) \leqq \inf_P \nu(\mathbf{p})$.

[3] In order to preserve the dimensionless character of the distance function, $\mathbf{p}^\mathsf{T}\mathbf{x}$ has to be divided by 1$.

[4] Mahler's inequality in general deals with the problem of finding "best" pairs of function (f, g) fulfilling the inequality $f(\mathbf{x}) \cdot g(\mathbf{y}) \geqq \mathbf{x}^\mathsf{T}\mathbf{y}\ \ \forall\mathbf{x},\ \forall\mathbf{y}$.

Proof: Starting with

$$r(\mathbf{p},\mathbf{v}) = \inf\{\mu \geqq 0 |\, \mathbf{p} \in \mu P^{\circ}(\mathbf{v})\} \qquad \forall\, \mathbf{p} \in P_{\mathbf{p}}$$

we can assume $\mu > 0$, because $\mu = 0$ implies $\mathbf{p} = \mathbf{0}$ so that (1.6) is satisfied. Now $\mathbf{p}/\mu \in P^{\circ}(\mathbf{v})$ can be rewritten by the definition of $P^{\circ}(\mathbf{v})$:

$$(\mathbf{p}/\mu)^{\mathsf{T}}(\lambda\mathbf{x}) \leqq \lambda \quad \forall\, \mathbf{x} \in P(\mathbf{v}),\ \forall\, \lambda \geqq 0$$

If $\lambda\mathbf{x}$ is mapped to $\tilde{\mathbf{x}}$, then

$$(\mathbf{p}/\mu)^{\mathsf{T}}\tilde{\mathbf{x}} \leqq \lambda \quad \forall\, \tilde{\mathbf{x}} \in \lambda P(\mathbf{v}),\ \forall\, \lambda \geqq 0$$

where the minimum λ on the right hand side is equivalent to $t_O(\mathbf{x},\mathbf{v})$ for all $\mathbf{x} \in X$. The equation

$$r(\mathbf{p},\mathbf{v}) = \inf\left\{\mu \geqq 0 |\, \mathbf{p}^{\mathsf{T}}\mathbf{x} \leqq \mu\, t_O(\mathbf{x},\mathbf{v}) \ \ \forall \mathbf{x} \in X\right\} \qquad \forall\, \mathbf{p} \in P_{\mathbf{p}}$$

is usually taken to define the polar of the gauge $t_O(\mathbf{x},\mathbf{v})$ and implies (1.6). ■

The analogue definition of the *cost function* $c(\mathbf{q},\mathbf{x}) \coloneqq \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} |\ \mathbf{v} \in L(\mathbf{x})\right\}$ and the *input distance function* $t_I(\mathbf{v},\mathbf{x}) \coloneqq \sup\{\lambda \geqq 0 |\, \mathbf{v} \in \lambda L(\mathbf{x})\}$ yield similar results when defining the reciprocally polar input requirement set by $L_{\circ}(\mathbf{x}) \coloneqq \{\mathbf{q} \in Q |\, \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1\ \forall\, \mathbf{v} \in L(\mathbf{x})\}$. Provided $L(\mathbf{x})$ is a nonempty aureoled closed convex input requirement set not containing the origin then (see Bobzin (1998, Proposition III.17)

$$\mathbf{v} \in L(\mathbf{x}) \iff t_I(\mathbf{v},\mathbf{x}) \geqq 1 \tag{1.7}$$

Proposition 1.2[5] *Let $L(\mathbf{x})$ be an input requirement set satisfying the assumptions of (1.7). Then the input distance function $t_I(\cdot,\mathbf{x})$ and the cost function $c(\cdot,\mathbf{x})$ are polar to each other and fulfil Mahler's inequality*[6]

$$\mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q},\mathbf{x})\, t_I(\mathbf{v},\mathbf{x}) \qquad \forall\, \mathbf{q} \in K(L_{\circ}(\mathbf{x})),\ \ \forall\, \mathbf{v} \in K(L(\mathbf{x})). \tag{1.8}$$

[5] Cf. McFadden (1978, Lemma 5) or Bobzin (1998, Corollary III.18.1).

[6] Given the commodity bundle $\mathbf{x} \in X$, the cones $K(L(\mathbf{x})) \coloneqq \{\lambda\mathbf{v} |\ \mathbf{v} \in L(\mathbf{x}),\ \lambda > 0\}$ and $K(L_{\circ}(\mathbf{x})) \coloneqq \{\lambda\mathbf{q} |\ \mathbf{q} \in L_{\circ}(\mathbf{x}),\ \lambda > 0\}$ ensure $c(\cdot,\mathbf{x})$ and $t_I(\cdot,\mathbf{x})$ to be positive. It is important to note that the two cones do not necessarily include the entire boundary of V or Q, respectively.

$$\boxed{r(\mathbf{p},\mathbf{v}) = \sup_{\mathbf{x}} \left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} | \ \mathbf{x} \in P(\mathbf{v})\right\}} \qquad \boxed{t_O(\mathbf{x},\mathbf{v}) = \sup_{\mathbf{p}} \left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} | \ \mathbf{p} \in P^{\circ}(\mathbf{v})\right\}}$$

$$\boxed{c(\mathbf{q},\mathbf{x}) = \inf_{\mathbf{v}} \left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} | \ \mathbf{v} \in L(\mathbf{x})\right\}} \qquad \boxed{t_I(\mathbf{v},\mathbf{x}) = \inf_{\mathbf{q}} \left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} | \ \mathbf{q} \in L_{\circ}(\mathbf{x})\right\}}$$

Figure 1: Basic programs of the paper

While the preceding two propositions reflect the horizontal parts of Figure 1, the vertical arrow between the two distance functions is an immediate consequence of (1.1), (1.4), and (1.7).

$$t_I(\mathbf{v},\mathbf{x}) \geqq 1 \iff t_O(\mathbf{x},\mathbf{v}) \leqq 1 \tag{1.9}$$

To be more concrete each feasible activity $(\mathbf{v},\mathbf{x})$ implies $t_I(\mathbf{v},\mathbf{x}) \geqq t_O(\mathbf{x},\mathbf{v})$. Similarly, the inequality $c(\mathbf{q},\mathbf{x}) \geqq r(\mathbf{p},\mathbf{v})$ will characterize the relationship of the cost and the revenue function.

Bear in mind that all of the above mentioned functions $r(\cdot,\mathbf{v})$, $t_O(\cdot,\mathbf{v})$, $c(\cdot,\mathbf{x})$, and $t_I(\cdot,\mathbf{x})$ are homogeneous of degree +1. Moreover, one can show that the output correspondence P is homogeneous of degree h if and only if the inverse input correspondence L is homogeneous of degree $1/h$. In this case the output distance function $t_O(\mathbf{x},\cdot)$ and the input distance function $t_I(\mathbf{v},\cdot)$ are homogeneous of degree $-h$ and $-1/h$, respectively. Similarly, the revenue function $r(\mathbf{q},\cdot)$ and the cost function $c(\mathbf{q},\cdot)$ are homogeneous of degree h and $1/h$, respectively.

In the case of a homogeneous output correspondence with $\mathbf{x} \neq \mathbf{0}$ we have

$$t_I(\mathbf{v},\mathbf{x}) = [t_O(\mathbf{x},\mathbf{v})]^{-1/h} \tag{1.10}$$

For the proof rewrite the definition of $t_I(\mathbf{v},\mathbf{x})$ where $\mathbf{x} \neq \mathbf{0}$ implies $\lambda > 0$.

$$\mathbf{v} \in \lambda L(\mathbf{x}) \iff \mathbf{v}/\lambda \in L(\mathbf{x}) \iff \mathbf{x} \in P(\mathbf{v}/\lambda) \iff \mathbf{x} \in \lambda^{-h} P(\mathbf{v})$$

Now taking $\mu = \lambda^{-h}$ yields (1.10) since

$$\sup\left\{\lambda \geqq 0 | \ \mathbf{x} \in \lambda^{-h} P(\mathbf{v})\right\} = \left[\inf\{\mu \geqq 0 | \ \mathbf{x} \in \mu P(\mathbf{v})\}\right]^{-1/h}$$

Comparing (1.10) to (1.9) gives $t_I(\mathbf{v},\mathbf{x}) = 1 \iff t_O(\mathbf{x},\mathbf{v}) = 1$.

1.2 Basics of Differential Theory

Before going into further details of the economic analysis, the widely used concepts of subgradients, supergradients, and gradients are introduced.

A vector $\mathbf{y}$ is said to be a *subgradient* of the function f at point $\hat{\mathbf{x}} \in X$ if

$$f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in X \tag{1.11}$$

The set of all subgradients of f at point $\hat{\mathbf{x}}$ is called the *subdifferential* of f at point $\hat{\mathbf{x}}$ and is denoted by $\partial f(\hat{\mathbf{x}})$.
A vector $\mathbf{y}$ is a *supergradient* of the function g at $\hat{\mathbf{x}} \in X$ if

$$g(\mathbf{x}) \leqq g(\hat{\mathbf{x}}) + \mathbf{y}^{\mathsf{T}}(\mathbf{x} - \hat{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in X\,. \tag{1.12}$$

The set of all supergradients of g at point $\hat{\mathbf{x}}$ is called the *superdifferential* of g at point $\hat{\mathbf{x}}$ and is denoted by $\Delta g(\hat{\mathbf{x}})$.

It is immediate from these definitions that a convex function f attains its minimum at $\hat{\mathbf{x}}$ if and only if $\mathbf{0} \in \partial f(\hat{\mathbf{x}})$, i.e.

$$f(\mathbf{x}) \geqq f(\hat{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in X$$

Conversely, a concave function g reaches its maximum at $\hat{\mathbf{x}}$ if and only if $\mathbf{0} \in \Delta g(\hat{\mathbf{x}})$.

For a convex function f the subdifferential $\partial f(\mathbf{x})$ is a closed (possibly empty) convex set. If $\partial f(\hat{\mathbf{x}}) \neq \emptyset$, then f is said to be *subdifferentiable* at point $\hat{\mathbf{x}}$. Moreover, for a proper[7] convex function f Rockafellar (1972) proves[8]

$$\begin{aligned}
&\mathbf{x} \notin \mathrm{Dom}\, f && \Longrightarrow \quad \partial f(\mathbf{x}) = \emptyset\,, \\
&\mathbf{x} \in \mathrm{rint}(\mathrm{Dom}\, f) && \Longrightarrow \quad \partial f(\mathbf{x}) \neq \emptyset\,, \\
&\mathbf{x} \in \mathrm{int}(\mathrm{Dom}\, f) && \Longleftrightarrow \quad \partial f(\mathbf{x}) \neq \emptyset \quad \text{and bounded.}
\end{aligned}$$

The concept of gradients is taken from Blum, Öttli (1975). It is slightly different from the usual definition but more appropriate regarding the Kuhn-Tucker conditions, which will be used at a later stage. Given a convex set

[7] A function $f\colon X \to [-\infty, +\infty]$ is said to be proper if $f(\mathbf{x}) < +\infty$ for at least one $\mathbf{x}$ and $f(\mathbf{x}) > -\infty$ for every $\mathbf{x}$. The effective domain is defined by $\mathrm{Dom}\, f = \{\mathbf{x} \in X |\ f(\mathbf{x}) < +\infty\}$. Sometimes the set $\text{n-Dom}\, f = \{\mathbf{x} \in X |\ f(\mathbf{x}) > -\infty\}$ is also needed.

[8] The relative interior of a convex set $C \subset \mathbb{R}^n$ is denoted by $\mathrm{rint}\, C$. For example, if C is a line connecting two distinct points $\mathbf{x}^1, \mathbf{x}^2$ in $\mathbb{R}^3$, then $\mathrm{rint}\, C = C \setminus \{\mathbf{x}^1, \mathbf{x}^2\}$. Bear in mind that there is no need to distinguish the relative interior of a set $C \subset \mathbb{R}^n$ from its interior as long as C is n-dimensional, $\mathrm{rint}\, C = \mathrm{int}\, C$.

$C \subset \mathbb{R}^n$ the function $f\colon C \to \mathbb{R}$ is differentiable at a point $\hat{\mathbf{x}} \in C$ "regarding C" if there is a vector $\nabla f(\hat{\mathbf{x}})$ such that

$$(1.13)\quad f(\mathbf{x}) = f(\hat{\mathbf{x}}) + \nabla f(\hat{\mathbf{x}})^{\mathsf{T}}(\mathbf{x}-\hat{\mathbf{x}}) + \frac{\mathbf{r}(\mathbf{x}-\hat{\mathbf{x}})}{\|\mathbf{x}-\hat{\mathbf{x}}\|} \qquad \forall\, \mathbf{x} \in C$$
$$\text{and}\quad \lim_{\mathbf{x}\to\hat{\mathbf{x}}} \frac{\mathbf{r}(\mathbf{x}-\hat{\mathbf{x}})}{\|\mathbf{x}-\hat{\mathbf{x}}\|} = \mathbf{0}$$

In general $\nabla f(\hat{\mathbf{x}})$ is not uniquely determined. However, if $\hat{\mathbf{x}} \in \operatorname{int} C$ and (1.13) holds good, then f is *differentiable* at $\hat{\mathbf{x}}$ in the usual sense, i.e. the *gradient* $\nabla f(\hat{\mathbf{x}})$ is the vector of partial derivatives evaluated at $\hat{\mathbf{x}}$. Moreover, the gradient – if it exists – is uniquely determined, provided C is a convex set with $\operatorname{int} C \neq \emptyset$.

The relationship of subgradients and gradients turns out to be very simple. Let $\mathbf{x}$ be a point at which the convex function f is finite. If f is differentiable at point $\mathbf{x}$, then the gradient $\nabla f(\mathbf{x})$ is the unique subgradient of f at $\mathbf{x}$, $\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$. Conversely, if a convex function f has a unique subgradient $\mathbf{y}(\mathbf{x})$ at point $\mathbf{x}$, then f is differentiable at $\mathbf{x}$ and $\mathbf{y}(\mathbf{x}) = \nabla f(\mathbf{x})$.

2 Revenue Maximization

According to the relationships (1.2) to (1.5) the two programs

$$\text{(P1)}\qquad r(\mathbf{p},\mathbf{v}) = \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} \mid t_O(\mathbf{x},\mathbf{v}) \leqq 1\right\} \qquad \forall\, \mathbf{p} \in P_{\mathbf{p}}$$

$$\text{(D1)}\qquad t_O(\mathbf{x},\mathbf{v}) = \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} \mid r(\mathbf{p},\mathbf{v}) \leqq 1\right\} \qquad \forall\, \mathbf{x} \in X$$

are related to each other by Mahler's inequality (1.6). In order to put the relationship of a commodity bundle $\mathbf{x}$ and its prices $\mathbf{p}$ in concrete form the following proposition is stated:

Proposition 2.1 *For a pair of polar points* $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$ *to satisfy (1.6) for a given input vector* $\mathbf{v} > \mathbf{0}$ *as an equation, it is necessary and sufficient that the output vector* $\hat{\mathbf{x}}$ *solves the problem of revenue maximization* (P1) *given* $\hat{\mathbf{p}} \in P_{\mathbf{p}} \setminus \{\mathbf{0}\}$ *or dually that* $\hat{\mathbf{p}}$ *is an optimal solution to* (D1) *given* $\hat{\mathbf{x}} \in X \setminus \{\mathbf{0}\}$.

Provided the functions are differentiable, the Kuhn-Tucker conditions yield two systems of equations which are dual to each other for an optimal pair of polar points $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$ *with* $\hat{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{x}} = 1$:

$$\text{(2.1a)}\qquad \nabla_{\mathbf{p}} r(\hat{\mathbf{p}},\mathbf{v}) = \hat{\mathbf{x}}$$

$$\text{(2.1b)}\qquad \nabla_{\mathbf{x}} t_O(\hat{\mathbf{x}},\mathbf{v}) = \hat{\mathbf{p}}$$

$$\boxed{\hat{\mathbf{x}} \in P(\mathbf{v})} \qquad \boxed{\hat{\mathbf{p}} \in P^{\circ}(\mathbf{v})}$$

$$\boxed{t_O(\hat{\mathbf{x}}, \mathbf{v}) = 1} \qquad \boxed{r(\hat{\mathbf{p}}, \mathbf{v}) = 1}$$

Figure 2: Dual relationships of a pair of polar points $(\hat{\mathbf{p}}, \hat{\mathbf{x}})$

The duality results of this proposition inherit a perfect symmetry for the case $\hat{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{x}} = 1$ which is shown in Figure 2. In view of dimensional aspects it may be useful to write

$$\nabla_{\mathbf{x}} t_O(\hat{\mathbf{x}}, \mathbf{v}) = \frac{\hat{\mathbf{p}}}{\hat{\mathbf{p}}^{\mathsf{T}}\hat{\mathbf{x}}} \qquad \left[\frac{1}{\text{units of } \mathbf{x}}\right]$$

The **proof** of Proposition 2.1 consists of two parts regarding (P1) and (D1). As the first part parallels the second one, we can ignore one of them. Here, the Lagrange function for (P1) is chosen

$$\mathcal{L}_1(\mathbf{x}, \lambda_1) = \hat{\mathbf{p}}^{\mathsf{T}}\mathbf{x} + \lambda_1(1 - t_O(\mathbf{x}, \mathbf{v}))\,,$$

where the Lagrange multiplier λ_1 is measured in \$. The Lagrange multiplier of the second Lagrange function would have no dimension. Presuming an input vector $\mathbf{v} > \mathbf{0}$, the output possibility set $P(\mathbf{v})$ has a relatively interior point $\tilde{\mathbf{x}}$ satisfying $t_O(\tilde{\mathbf{x}}, \mathbf{v}) < 1$. Thus, $\tilde{\mathbf{x}}$ fulfils Slater's condition such that the Kuhn-Tucker conditions for (P1)

$$[\text{a}] \qquad \mathcal{L}_1(\hat{\mathbf{x}}, \hat{\lambda}_1) \geqq \mathcal{L}_1(\mathbf{x}, \hat{\lambda}_1) \qquad \forall\, \mathbf{x} \in X$$

$$[\text{b}] \qquad \hat{\lambda}_1 \geqq 0, \quad 1 - t_O(\hat{\mathbf{x}}, \mathbf{v}) \geqq 0, \quad \hat{\lambda}_1(1 - t_O(\hat{\mathbf{x}}, \mathbf{v})) = 0$$

are necessary and sufficient for $(\hat{\mathbf{x}}, \hat{\lambda}_1)$ to be a saddle point of the concave Lagrange function $\mathcal{L}_1$ or, equivalently, for $\hat{\mathbf{x}}$ to solve (P1) for $\mathbf{p} = \hat{\mathbf{p}}$.

Regarding the supergradient inequality (1.12), the Lagrange function $\mathcal{L}_1(\cdot, \hat{\lambda}_1)$ attains its maximum at $\hat{\mathbf{x}}$ – see condition [a] – if and only if $\mathbf{y} = \mathbf{0}$ is a supergradient of $\mathcal{L}_1(\cdot, \hat{\lambda}_1)$ at $\hat{\mathbf{x}}$, i.e. $\mathbf{0} \in \Delta_{\mathbf{x}}\mathcal{L}_1(\hat{\mathbf{x}}, \hat{\lambda}_1)$.

Since the concave effective domains of the n-proper concave objective function $f_0(\mathbf{x}) = \hat{\mathbf{p}}^{\mathsf{T}}\mathbf{x}$ and the n-proper concave restriction $f_1(\mathbf{x}) = 1 - t_O(\mathbf{x}, \mathbf{v})$ have a relatively interior point in common, i.e. $\mathbb{R}^n \cap \operatorname{rint} \mathbb{R}^n_+ \neq \emptyset$, it follows[9]

$$\begin{aligned} \mathbf{0} \in \Delta_{\mathbf{x}}\mathcal{L}_1(\hat{\mathbf{x}}, \hat{\lambda}_1) \quad &\Longleftrightarrow \quad \mathbf{0} \in \left[\Delta_{\mathbf{x}} f_0(\hat{\mathbf{x}}) + \hat{\lambda}_1\, \Delta_{\mathbf{x}} f_1(\hat{\mathbf{x}})\right] \\ &\Longleftrightarrow \quad \mathbf{0} \in \left[\{\hat{\mathbf{p}}\} + \hat{\lambda}_1\, \Delta_{\mathbf{x}}(-t_O(\hat{\mathbf{x}}, \mathbf{v}))\right] \\ &\Longleftrightarrow \quad \hat{\mathbf{p}} \in \hat{\lambda}_1\, \partial_{\mathbf{x}} t_O(\hat{\mathbf{x}}, \mathbf{v}) \end{aligned} \tag{2.2}$$

[9] Cf. Rockafellar (1972, Theorem 23.8).
The subdifferential $\partial f(\mathbf{x})$ of a convex function f at point $\mathbf{x}$ and the superdifferential $\Delta(-f(\mathbf{x}))$ of the concave function $-f$ at point $\mathbf{x}$ satisfy $-\partial f(\mathbf{x}) = \Delta(-f(\mathbf{x}))$.

The assumption $\hat{\mathbf{p}} \neq \mathbf{0}$ requires $\hat{\lambda}_1 > 0$ and, therefore,

$$t_O(\hat{\mathbf{x}}, \mathbf{v}) = 1 \quad \text{and} \quad \mathcal{L}_1(\hat{\mathbf{x}}, \hat{\lambda}_1) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} = r(\hat{\mathbf{p}}, \mathbf{v})$$

or

$$r(\hat{\mathbf{p}}, \mathbf{v})\, t_O(\hat{\mathbf{x}}, \mathbf{v}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}}$$

as stated by Proposition 2.1. Hence, the linear function $\hat{\mathbf{p}}^\mathsf{T}\mathbf{x}$ achieves its maximum at $\hat{\mathbf{x}}$ over the convex set $P(\mathbf{v})$, which is equivalent to $\hat{\mathbf{x}} \in \partial_\mathbf{p}\, r(\hat{\mathbf{p}}, \mathbf{v})$ (see Bobzin (1998, Proposition III.8)). Under the assumption of differentiability this yields (2.1a).

Moreover, $\hat{\mathbf{p}}$ is an optimal solution to

$$\begin{aligned} 1 = t_O(\hat{\mathbf{x}}, \mathbf{v}) &= \sup\left\{\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} |\ r(\mathbf{p}, \mathbf{v}) \leqq 1\right\} \\ &= \sup\left\{\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} |\ \mathbf{p} \in P^\circ(\mathbf{v})\right\} \\ &= \sup\left\{\mathbf{p}^\mathsf{T}\hat{\mathbf{x}} |\ \mathbf{p}^\mathsf{T}\mathbf{x} \leqq 1 \ \ \forall\, \mathbf{x} \in P(\mathbf{v})\right\} \end{aligned}$$

because each alternative price vector $\mathbf{p}$ with $t_O(\hat{\mathbf{x}}, \mathbf{v}) > 1$ implies $\hat{\mathbf{x}} \notin P(\mathbf{v})$. Again this statement is equivalent to $\hat{\mathbf{p}} \in \partial_\mathbf{x} t_O(\hat{\mathbf{x}}, \mathbf{v})$ and the assumption of differentiability results in (2.1b). Now, in comparison to (2.2) $\hat{\lambda}_1 = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} = 1$ holds good.

Finally, we turn over to a parametric variation of the given input vector $\mathbf{v}$. Assuming differentiability, the the envelope theorem yields

$$\nabla_\mathbf{v} r(\hat{\mathbf{p}}, \mathbf{v}) = -r(\hat{\mathbf{p}}, \mathbf{v})\, \nabla_\mathbf{v} t_O(\hat{\mathbf{x}}, \mathbf{v}) \tag{2.3}$$

For a homogeneous technology the most recent gradient can be substituted by the relationship given in (1.10).

$$\nabla_\mathbf{v} t_I(\mathbf{v}, \hat{\mathbf{x}}) = -\frac{1}{h}\,[t_O(\hat{\mathbf{x}}, \mathbf{v})]^{-\frac{1}{h}-1}\, \nabla_\mathbf{v} t_O(\hat{\mathbf{x}}, \mathbf{v}) = -\frac{1}{h}\, \nabla_\mathbf{v} t_O(\hat{\mathbf{x}}, \mathbf{v})$$

As will be proved in the subsequent section a cost minimum pair $(\tilde{\mathbf{q}}, \tilde{\mathbf{v}})$ with $\tilde{\mathbf{q}}^\mathsf{T}\tilde{\mathbf{v}} = 1$ holds $\nabla_\mathbf{v} t_I(\tilde{\mathbf{v}}, \hat{\mathbf{x}}) = \tilde{\mathbf{q}}$ so that (2.3) becomes[10]

$$\nabla_\mathbf{v} r(\hat{\mathbf{p}}, \tilde{\mathbf{v}}) = h\, \tilde{\mathbf{q}}\, r(\hat{\mathbf{p}}, \tilde{\mathbf{v}}) \qquad \left[\frac{\$}{\text{units of } \mathbf{v}}\right] \tag{2.4}$$

For the sake of clarity in the linear homogeneous case with $r(\hat{\mathbf{p}}, \tilde{\mathbf{v}}) = \hat{\mathbf{p}}^\mathsf{T}\hat{\mathbf{x}} = 1$ the factor prices have to agree with their respective marginal revenue.

$$\nabla_\mathbf{v} r(\hat{\mathbf{p}}, \tilde{\mathbf{v}}) = \tilde{\mathbf{q}}$$

There is only little surprise that Euler's theorem results in

$$\nabla_\mathbf{v} r(\hat{\mathbf{p}}, \tilde{\mathbf{v}})^\mathsf{T}\tilde{\mathbf{v}} = h\, \tilde{\mathbf{q}}^\mathsf{T}\tilde{\mathbf{v}}\, r(\hat{\mathbf{p}}, \tilde{\mathbf{v}}) = h\, r(\hat{\mathbf{p}}, \tilde{\mathbf{v}})$$

[10] Writing out in full we have with respect to dimensions $\nabla_\mathbf{v} t_I(\tilde{\mathbf{v}}, \hat{\mathbf{x}}) = \tilde{\mathbf{q}}/\tilde{\mathbf{q}}^\mathsf{T}\tilde{\mathbf{v}} = \tilde{\mathbf{q}}$.

3 Cost Minimization

While (P1) and (D1) are concerned with polar gauges, the following programs reflect the relationship of reciprocally polar gauges. The cost function and the input distance function are given in the form of (P2) and (D2), respectively.

$$\text{(P2)} \qquad c(\mathbf{q},\mathbf{x}) = \inf\left\{\mathbf{q}^\top\mathbf{v} \mid t_I(\mathbf{v},\mathbf{x}) \geqq 1\right\} \qquad \forall\, \mathbf{q} \in K(L_\circ(\mathbf{x}))$$

$$\text{(D2)} \qquad t_I(\mathbf{v},\mathbf{x}) = \inf\left\{\mathbf{q}^\top\mathbf{v} \mid c(\mathbf{q},\mathbf{x}) \geqq 1\right\} \qquad \forall\, \mathbf{v} \in K(L(\mathbf{x}))$$

Their relationship is discussed in full in Shephard (1953). In particular equation (3.1a) of the following proposition is frequently referred to as Shephard's theorem or Shephard's lemma. Accordingly, (3.1b) is called Hotelling's theorem. However, at this point the two equations are not the result of some kind of Lagrange duality or of the envelope theorem.[11]

Proposition 3.1 (Shephard's Theorem) *For a pair of polar points* $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ *to satisfy (1.8) for a given commodity bundle* $\mathbf{x} \in X \setminus \{\mathbf{0}\}$ *as an equation, it is necessary and sufficient that the input vector* $\hat{\mathbf{v}}$ *solves the problem of cost minimization* (P2) *given* $\hat{\mathbf{q}} \in K(L_\circ(\mathbf{x}))$ *or dually that* $\hat{\mathbf{q}}$ *is an optimal solution to* (D2) *given* $\hat{\mathbf{v}} \in K(L(\mathbf{x}))$.

Provided the functions are differentiable, the Kuhn-Tucker conditions yield two systems of equations which are dual to each other for an optimal pair of polar points $(\hat{\mathbf{q}}, \hat{\mathbf{v}})$ *with* $\hat{\mathbf{q}}^\top\hat{\mathbf{v}} = 1$:

$$\text{(3.1a)} \qquad \nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\mathbf{x}) = \hat{\mathbf{v}}$$

$$\text{(3.1b)} \qquad \nabla_{\mathbf{v}} t_I(\hat{\mathbf{v}},\mathbf{x}) = \hat{\mathbf{q}}.$$

The proof corresponds to that of Proposition 2.1 and is ignored. Once more the duality shows a perfect symmetry for $\hat{\mathbf{q}}^\top\mathbf{v} = 1$, which is similar to that of Figure 2. Provided the Lagrange function of (P2)[12]

$$\mathcal{L}_2(\mathbf{v},\lambda_2;\mathbf{x}) = \hat{\mathbf{q}}^\top\mathbf{v} + \lambda_2(1 - t_I(\mathbf{v},\mathbf{x}))$$

is differentiable at $(\hat{\mathbf{v}}, \hat{\lambda}_2, \mathbf{x})$, the effects of a parametric variation of $\mathbf{x}$ can be studied. The envelope theorem yields

$$\text{(3.2)} \qquad \nabla_{\mathbf{x}} c(\hat{\mathbf{q}},\mathbf{x}) = -c(\hat{\mathbf{q}},\mathbf{x})\, \nabla_{\mathbf{x}} t_I(\hat{\mathbf{v}},\mathbf{x}).$$

[11] In addition to that the result is independent of any assumption on homogeneity as suggested by Diewert (1974, p. 112). A correct notion with respect to consumer preferences can be taken from Blackorby, Primont, Russel (1978, p. 34).

[12] Notice that $t_I(\cdot,\mathbf{x})$ is concave for each nonempty convex input requirement set $L(\mathbf{x})$, i.e. $1 - t_I(\cdot,\mathbf{x})$ is convex.

Now, observe (1.10) for a homogeneous technology.

$$\nabla_{\mathbf{x}} t_O(\mathbf{x}, \hat{\mathbf{v}}) = -h\,[t_I(\hat{\mathbf{v}}, \mathbf{x})]^{-h-1}\,\nabla_{\mathbf{x}} t_I(\hat{\mathbf{v}}, \mathbf{x}) = -h\,\nabla_{\mathbf{x}} t_I(\hat{\mathbf{v}}, \mathbf{x})$$

By Proposition 2.1 each pair $(\mathbf{p}, \mathbf{x})$, which satisfies (1.6) as an equation (without normalizing $\mathbf{p}^\mathsf{T}\mathbf{x} = 1$), yields

$$\nabla_{\mathbf{x}} t_I(\hat{\mathbf{v}}, \mathbf{x}) = -\frac{\mathbf{p}}{h\,\mathbf{p}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \mathbf{x}) = \frac{\mathbf{p}\,c(\hat{\mathbf{q}}, \mathbf{x})}{h\,\mathbf{p}^\mathsf{T}\mathbf{x}} \tag{3.3}$$

In the linear homogeneous case the last equation becomes the rule of marginal cost pricing

$$\nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \mathbf{x}) = \mathbf{p}\,,$$

given $c(\hat{\mathbf{q}}, \mathbf{x}) = \hat{\mathbf{q}}^\mathsf{T}\hat{\mathbf{v}} = \mathbf{p}^\mathsf{T}\mathbf{x}$. This case will be dealt with in the next section.

4 Shadow Pricing

4.1 Given Factor Endowment

In accordance with the theory of international trade (e.g. Dixit, Norman (1980)) it is now assumed that the factor supplies $\mathbf{v}$ of an economy are given. Holding the vector of output prices $\mathbf{p}$ fixed we reuse (P1)[13]

$$r(\mathbf{p}, \mathbf{v}) = \sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x} \mid t_O(\mathbf{x}, \mathbf{v}) \leqq 1\right\}. \tag{P3}$$

In the theory of international trade the constraint $t_O(\mathbf{x}, \mathbf{v}) \leqq 1$ is usually given in the slightly different form of a transformation function $t(\mathbf{x}, \mathbf{v}) \leqq 0$. Concerning (1.9) for a feasible activity $(\mathbf{v}, \mathbf{x})$ – as a first attempt – the program (P3) may now be viewed as being opposite to

$$t_I(\mathbf{v}, \hat{\mathbf{x}}) = \inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v} \mid c(\mathbf{q}, \hat{\mathbf{x}}) \geqq 1\right\}.$$

Following the recommendation of Färe (1988), the linear homogeneity of the cost function $c(\cdot, \hat{\mathbf{x}})$ gives

$$t_O(\hat{\mathbf{x}}, \mathbf{v}) \leqq 1 \iff t_I(\mathbf{v}, \hat{\mathbf{x}}) = \frac{1}{\alpha}\inf\left\{\mathbf{q}^\mathsf{T}\mathbf{v} \mid c(\mathbf{q}, \hat{\mathbf{x}}) \geqq \alpha\right\} \geqq 1$$

[13] Recall that $t_O(\cdot, \mathbf{v})$ is convex. However, $t_I(\mathbf{v}, \cdot)$ needs not to be concave. In fact, $t_I(\mathbf{v}, \cdot)$ is quasi-concave iff L is quasi-concave. Hence $\sup\left\{\mathbf{p}^\mathsf{T}\mathbf{x} \mid t_I(\mathbf{v}, \mathbf{x}) \geqq 1\right\}$ is omitted because it lacks the property of concavity.

for each $\alpha > 0$. Up until now α is a pure scalar which has no dimension. At the same time $t_I(\cdot, \hat{\mathbf{x}})$ has no dimension by construction. In the next step $\hat{\mathbf{x}}$ is supposed to be an optimal solution to (P3) such that $r(\mathbf{p}, \mathbf{v}) = \mathbf{p}^\mathsf{T}\hat{\mathbf{x}} > 0$. Taking $\alpha \cdot 1\$ = r(\mathbf{p}, \mathbf{v})$ then

$$t_I(\mathbf{v}, \hat{\mathbf{x}})\, r(\mathbf{p}, \mathbf{v}) = \inf_{\mathbf{q}} \left\{\mathbf{q}^\mathsf{T}\mathbf{v} \mid c(\mathbf{q}, \hat{\mathbf{x}}) \geqq r(\mathbf{p}, \mathbf{v})\right\} \geqq r(\mathbf{p}, \mathbf{v}) \tag{4.1}$$

is given in \$. Besides that the constraint of (P3) yields via (1.9) a weak duality of the type

$$\inf_{\mathbf{q}} \left\{\mathbf{q}^\mathsf{T}\mathbf{v} \mid c(\mathbf{q}, \hat{\mathbf{x}}) \geqq r(\mathbf{p}, \mathbf{v})\right\} \geqq \sup_{\mathbf{x}} \left\{\mathbf{p}^\mathsf{T}\mathbf{x} \mid t_O(\mathbf{x}, \mathbf{v}) \leqq 1\right\} \tag{4.2}$$

However, the latter problem does not depend on any factor price vector $\mathbf{q}$ and especially the optimal solution of the former problem has no importance. This relationship will be picked up in the next section concerning linear programming.

The solution of (P3) has been discussed in section 2. Thus, the attention is directed to the left hand side of (4.2). If $\hat{\mathbf{q}}$ solves

$$\inf_{\mathbf{q}} \left\{\mathbf{q}^\mathsf{T}\mathbf{v} \mid c(\mathbf{q}, \hat{\mathbf{x}}) \geqq r(\mathbf{p}, \mathbf{v})\right\} \tag{D3}$$

then $\hat{\mathbf{q}}$ is called a cost minimal shadow price vector for $\mathbf{v}$. Assuming a feasible activity $(\mathbf{v}, \hat{\mathbf{x}})$, i.e. $\mathbf{v} \in L(\hat{\mathbf{x}})$, (D3) is a convex program provided the input requirement set is convex. Notice that the constraint $f_1(\mathbf{q}) = r(\mathbf{p}, \mathbf{v}) - c(\mathbf{q}, \hat{\mathbf{x}})$ is defined on the convex set Dom $f_1 = Q$. As Q has at least one relatively interior point $\tilde{\mathbf{q}}$ satisfying $f_1(\tilde{\mathbf{q}}) < 0$ or $c(\tilde{\mathbf{q}}, \hat{\mathbf{x}}) > r(\mathbf{p}, \mathbf{v})$, the vector of shadow prices $\tilde{\mathbf{q}}$ fulfils Slater's conditions so that the following Kuhn-Tucker conditions are necessary and sufficient for $(\hat{\mathbf{q}}, \hat{\lambda}_3)$ to be a saddle point of the Lagrange function $\mathcal{L}_3$ or, equivalently, for $\hat{\mathbf{q}}$ to solve problem (D3) for the given input vector $\mathbf{v}$.

With regard to the Lagrange function including a dimensionless Lagrange multiplier λ_3

$$\mathcal{L}_3(\mathbf{q}, \lambda_3) = \mathbf{q}^\mathsf{T}\mathbf{v} + \lambda_3\big(r(\mathbf{p}, \mathbf{v}) - c(\mathbf{q}, \hat{\mathbf{x}})\big)$$

the Kuhn-Tucker conditions[14]

$$\text{[a]} \qquad \mathcal{L}_3(\hat{\mathbf{q}}, \hat{\lambda}_3) \leqq \mathcal{L}_3(\mathbf{q}, \hat{\lambda}_3) \qquad \forall\, \mathbf{q} \in Q$$

$$\text{[b]} \qquad \hat{\lambda}_3 \geqq 0, \quad r(\mathbf{p}, \mathbf{v}) - c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \leqq 0, \quad \hat{\lambda}_3\big(r(\mathbf{p}, \mathbf{v}) - c(\hat{\mathbf{q}}, \hat{\mathbf{x}})\big) = 0$$

[14] The inequality $0 < r(\mathbf{p}, \mathbf{v}) \leqq c(\mathbf{q}, \hat{\mathbf{x}})$ requires $\mathbf{q} \in K(L_\circ(\hat{\mathbf{x}}))$ so that $c(\mathbf{q}, \hat{\mathbf{x}}) > 0$.

ensue the existence of a saddle point $(\hat{\mathbf{q}}, \hat{\lambda}_3)$.

In accordance with (1.11) the convex Lagrange function $\mathcal{L}_3(\cdot, \hat{\lambda}_3)$ attains its minimum at $\hat{\mathbf{q}}$ (see [a]) if and only if $\mathbf{y} = \mathbf{0}$ is a subgradient of $\mathcal{L}_3(\cdot, \hat{\lambda}_3)$ at $\hat{\mathbf{q}}$, i.e. $\mathbf{0} \in \partial_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}}, \hat{\lambda}_3)$. However each vector $\mathbf{y} \neq \mathbf{0}$ in (1.11) yields also [a] provided

$$\mathbf{y}^{\mathsf{T}}(\mathbf{q} - \hat{\mathbf{q}}) \geqq 0 \qquad \forall\, \mathbf{q} \in Q.$$

With $Q = \mathbb{R}^m_+$ it is not too hard to prove that this is equivalent to

$$\mathbf{y} \geqq \mathbf{0}, \quad \hat{\mathbf{q}} \geqq \mathbf{0}, \quad \mathbf{y}^{\mathsf{T}}\hat{\mathbf{q}} = 0. \tag{4.3}$$

These conditions will be picked up in (4.8), where the factor supply $\mathbf{v}$ may differ from the inputs demanded $\mathbf{v}^*$, i.e. $\mathbf{y} = \mathbf{v} - \mathbf{v}^* \neq \mathbf{0}$.

Define for the sake of brevity $f_0(\mathbf{q}) = \mathbf{q}^{\mathsf{T}}\mathbf{v}$. Both functions – the objective function f_0 and the constraint f_1 – are proper and convex. Thus, regarding the above mentioned Slater's condition, [a] is equivalent to[15]

$$\begin{aligned}
\mathbf{0} \in \partial_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}}, \hat{\lambda}_3) \quad &\Longleftrightarrow \quad \mathbf{0} \in \left[\partial_{\mathbf{q}} f_0(\hat{\mathbf{q}}) + \hat{\lambda}_3 \partial_{\mathbf{q}} f_1(\hat{\mathbf{q}})\right] \\
&\Longleftrightarrow \quad \mathbf{0} \in \left[\{\mathbf{v}\} + \hat{\lambda}_3 \partial_{\mathbf{q}}(-c(\hat{\mathbf{q}}, \hat{\mathbf{x}}))\right] \\
&\Longleftrightarrow \quad \mathbf{v} \in \hat{\lambda}_3 \Delta_{\mathbf{q}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}})
\end{aligned}$$

Assuming $\mathbf{v} > \mathbf{0}$ requires $\hat{\lambda}_3 > 0$ and $\hat{\mathbf{q}} \in Q$. Besides that [b] ensues

$$\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} = r(\mathbf{p}, \mathbf{v}) = c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \quad \text{and} \tag{4.4}$$

$$\mathcal{L}_3(\hat{\mathbf{q}}, \hat{\lambda}_3) = \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v} = t_I(\mathbf{v}, \hat{\mathbf{x}})\, r(\mathbf{p}, \mathbf{v}). \tag{4.5}$$

At this point it is important to know that the superdifferential $\hat{\lambda}_3 \Delta_{\mathbf{q}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}})$ does not include any input vector with $\mathbf{0} < \mathbf{v}^* < \mathbf{v}$.[16] In this case $\mathbf{y} = \mathbf{v} - \mathbf{v}^* > \mathbf{0}$ could be used to produce more of a good j which has a positive price p_j. But this contradicts the assumption of a revenue maximizing vector of outputs $\hat{\mathbf{x}}$. Therefore, in line with the outcomes in Ruys, Weddepohl (1979) regarding linear programming and duality, we have

$$t_I(\mathbf{v}, \hat{\mathbf{x}}) = 1, \tag{4.6}$$

$$\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} = r(\mathbf{p}, \mathbf{v}) = c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v} \tag{4.7}$$

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[15] If $\mathbf{q}$ lies in the boundary of Q then $\Delta_{\mathbf{q}} c(\mathbf{q}, \hat{\mathbf{x}})$ may be empty or even unbounded. In the case of a Cobb-Douglas production function we have $\Delta_{\mathbf{q}} c(\mathbf{q}, x) = \emptyset$ for every price vector $\mathbf{q}$ including a zero price. The reason is that the corresponding factor demand goes to infinity.

[16] For the comparison of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we use the following notation: $\mathbf{x} > \mathbf{y} \iff x_j > y_j \;\; j = 1, \ldots, n$; $\mathbf{x} \geqq \mathbf{y} \iff x_j \geqq y_j \;\; j = 1, \ldots, n$; $\mathbf{x} \geq \mathbf{y} \iff [\mathbf{x} \geqq \mathbf{y} \wedge \mathbf{x} \neq \mathbf{y}]$.

In comparence to (4.2) this outcome represents the strong duality regarding the *shadow price vector* $\hat{\mathbf{q}}$.

Condition (4.5) states that the linear function $\mathbf{q}^{\mathsf{T}}\mathbf{v}$ attains its minimum over the convex set $L_\circ(\hat{\mathbf{x}})$ at the point $\hat{\mathbf{q}}/r(\mathbf{p},\mathbf{v})$. Hence, in contrast to (2.4) the vector of shadow prices yiels

$$\hat{\mathbf{q}}/r(\mathbf{p},\mathbf{v}) \in \Delta_{\mathbf{v}} t_I(\mathbf{v},\hat{\mathbf{x}})$$

even for $h \neq 1$.
Although $\hat{\lambda}_3 \Delta_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}})$ does not include any input vector $\mathbf{v}^*$ with $\mathbf{0} < \mathbf{v}^* < \mathbf{v}$, this is perfectly possible for an input vector satisfying $\mathbf{0} < \mathbf{v}^* \leq \mathbf{v}$. Taking again $\mathbf{y} = \mathbf{v} - \mathbf{v}^* \geq \mathbf{0}$ results in $\mathbf{y} \in \partial_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}},\hat{\lambda}_3)$. Regarding the price vector $\hat{\mathbf{q}}$ this relation is fulfilled if and only if (4.3) holds true:

$$\text{(4.8)} \qquad \mathbf{v} - \mathbf{v}^* \geqq \mathbf{0}, \quad \hat{\mathbf{q}} \geqq \mathbf{0}, \quad (\mathbf{v} - \mathbf{v}^*)^{\mathsf{T}}\hat{\mathbf{q}} = 0$$

On the one hand, a positive shadow price $\hat{q}_j$ implies $v_j = v_j^*$ and, on the other hand, $v_j > v_j^*$ requires explicitely $\hat{q}_j = 0$.

Remark (Homogeneity): An output correspondence P which is homogeneous of degree $+h$ yields

$$\begin{aligned} \hat{\mathbf{x}} \text{ solves (P3)} \iff \mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} = r(\mathbf{p},\mathbf{v}) &\iff t_O(\hat{\mathbf{x}},\mathbf{v}) = 1 \\ \iff t_I(\mathbf{v},\hat{\mathbf{x}}) = 1 &\iff c(\hat{\mathbf{q}},\hat{\mathbf{x}}) = \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v} \end{aligned}$$

Although h can differ from 1, (4.7) states that revenue equals cost. This result is now discussed in more detail under the assumption of differentiability.

Remark (Differentiability): If $\mathcal{L}_3(\cdot,\lambda_3)$ is differentiable at $\hat{\mathbf{q}} \in Q$ “regarding Q”, then [a] can be substituted by the linearised Kuhn-Tucker condition

$$\text{[c]} \qquad \nabla_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}},\hat{\lambda}_3)^{\mathsf{T}}(\mathbf{q} - \hat{\mathbf{q}}) \geqq 0 \qquad \forall\, \mathbf{q} \in Q.$$

As $Q = \mathbb{R}^m_+$ the linearised Kuhn-Tucker condition [c] is equivalent to both of the following systems

$$\text{[c']} \qquad \nabla_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}},\hat{\lambda}_3) \geqq \mathbf{0}, \quad \hat{\mathbf{q}} \geqq \mathbf{0}, \quad \nabla_{\mathbf{q}} \mathcal{L}_3(\hat{\mathbf{q}},\hat{\lambda}_3)^{\mathsf{T}}\hat{\mathbf{q}} = 0$$

$$\text{[c”]} \qquad \mathbf{v} - \hat{\lambda}_3 \nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}}) \geqq \mathbf{0}, \quad \hat{\mathbf{q}} \geqq \mathbf{0}, \quad \big(\mathbf{v} - \hat{\lambda}_3 \nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}})\big)^{\mathsf{T}}\hat{\mathbf{q}} = 0$$

Regarding Euler’s theorem, [c”] yields

$$t_I(\mathbf{v},\hat{\mathbf{x}})\, r(\mathbf{p},\mathbf{v}) = \hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v} = \hat{\lambda}_3\, \hat{\mathbf{q}}^{\mathsf{T}} \nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}}) = \hat{\lambda}_3\, c(\hat{\mathbf{q}},\hat{\mathbf{x}}) -$$

Therefore, by (4.7), $\hat{\lambda}_3 = t_I(\mathbf{v}, \hat{\mathbf{x}}) = 1$. The effects of a parametric variation of the commodity bundle $\hat{\mathbf{x}}$ can be studied by applying the envelope theorem:

$$\begin{aligned}
&\nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}})\, \mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} + t_I(\mathbf{v}, \hat{\mathbf{x}})\mathbf{p} = t_I(\mathbf{v}, \hat{\mathbf{x}})\big(\mathbf{p} - \nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}})\big) \\
\iff\ &\nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = -\frac{\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}}}{t_I(\mathbf{v}, \hat{\mathbf{x}})}\, \nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}}) \qquad \text{cf. (3.2)} \\
\iff\ &\nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = \frac{\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}}}{h\, \nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}})^{\mathsf{T}}\hat{\mathbf{x}}}\, \nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}})
\end{aligned}$$

According to (1.10) it appears

$$\nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}}) = -\frac{1}{h}\, [t_O(\hat{\mathbf{x}}, \mathbf{v})]^{-\frac{1}{h}-1}\, \nabla_{\mathbf{x}} t_O(\hat{\mathbf{x}}, \mathbf{v})$$

where the revenue maximizing $\hat{\mathbf{x}}$ results in $t_O(\hat{\mathbf{x}}, \mathbf{v}) = 1$. Using (2.1b) without normalizing $\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} = 1$ implies

$$\nabla_{\mathbf{x}} t_O(\hat{\mathbf{x}}, \mathbf{v}) = \frac{\mathbf{p}}{\mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}}} \iff \nabla_{\mathbf{x}} t_I(\mathbf{v}, \hat{\mathbf{x}}) = -\frac{\mathbf{p}}{h\, \mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}}} \iff \nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = \frac{\mathbf{p}}{h}$$

This result confirms (3.3) and leads to marginal cost pricing for $h = 1$. In the special case of two commodities (even for $h \neq 1$) the marginal rate of transformation takes the form

$$\frac{\frac{\partial c}{\partial x_1}(\hat{\mathbf{q}}, \hat{\mathbf{x}})}{\frac{\partial c}{\partial x_2}(\hat{\mathbf{q}}, \hat{\mathbf{x}})} = \frac{p_1}{p_2} = \frac{\frac{\partial t_I}{\partial x_1}(\mathbf{v}, \hat{\mathbf{x}})}{\frac{\partial t_I}{\partial x_2}(\mathbf{v}, \hat{\mathbf{x}})} = \frac{\frac{\partial t_O}{\partial x_1}(\hat{\mathbf{x}}, \mathbf{v})}{\frac{\partial t_O}{\partial x_2}(\hat{\mathbf{x}}, \mathbf{v})}$$

Moreover, Euler's theorem reveals (4.7), i.e. zero profit for every $h > 0$.

$$\frac{1}{h}\, \mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}} = \nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}})^{\mathsf{T}}\hat{\mathbf{x}} = \frac{1}{h}\, c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \tag{4.9}$$

This result is important to know, because a profit maximizing pricing rule $\nabla_{\mathbf{x}} c(\mathbf{q}, \hat{\mathbf{x}}) = \mathbf{p}$ is consistent with the shadow prices $\hat{\mathbf{q}}$ in (4.7) if and only if $h = 1$. The relationship of (4.9) states that

$$\nabla_{\mathbf{x}} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \gtreqless \mathbf{p} \quad \text{if} \quad h \lesseqgtr 1.$$

A similar pricing rule regarding inputs results from applying the envelope theorem with respect to the factor endowment $\mathbf{v}$.

$$\begin{aligned}
&\nabla_{\mathbf{v}} t_I(\mathbf{v}, \hat{\mathbf{x}})\, r(\mathbf{p}, \mathbf{v}) + t_I(\mathbf{v}, \hat{\mathbf{x}})\, \nabla_{\mathbf{v}} r(\mathbf{p}, \mathbf{v}) = \hat{\mathbf{q}} + t_I(\mathbf{v}, \hat{\mathbf{x}})\, \nabla_{\mathbf{v}} r(\mathbf{p}, \mathbf{v}) \\
\iff\ &\nabla_{\mathbf{v}} t_I(\mathbf{v}, \hat{\mathbf{x}}) = \hat{\mathbf{q}}/r(\mathbf{p}, \mathbf{v}) \qquad \text{cf. (3.1b)}
\end{aligned}$$

Again, Euler's theorem yields (4.5) or similarly (4.4).

According to (2.4), in the special case of two inputs the marginal rate of substitution becomes

$$\frac{\dfrac{\partial r}{\partial v_1}(\mathbf{p},\mathbf{v})}{\dfrac{\partial r}{\partial v_2}(\mathbf{p},\mathbf{v})} = \frac{\hat{q}_1}{\hat{q}_2} = \frac{\dfrac{\partial t_O}{\partial v_1}(\hat{\mathbf{x}},\mathbf{v})}{\dfrac{\partial t_O}{\partial v_2}(\hat{\mathbf{x}},\mathbf{v})} = \frac{\dfrac{\partial t_I}{\partial v_1}(\mathbf{v},\hat{\mathbf{x}})}{\dfrac{\partial t_I}{\partial v_2}(\mathbf{v},\hat{\mathbf{x}})}$$

Summarizing the main results for $h = 1$ we note

$$\begin{aligned}
&\nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}}) = \mathbf{v},\\
&\nabla_{\mathbf{v}} t_I(\mathbf{v},\hat{\mathbf{x}}) = -\nabla_{\mathbf{v}} t_O(\hat{\mathbf{x}},\mathbf{v}) = \hat{\mathbf{q}}/r(\mathbf{p},\mathbf{v}),\\
&\nabla_{\mathbf{x}} t_I(\mathbf{v},\hat{\mathbf{x}}) = -\nabla_{\mathbf{x}} t_O(\hat{\mathbf{x}},\mathbf{v}) = -\mathbf{p} \quad\Longrightarrow\quad \nabla_{\mathbf{x}} c(\mathbf{q},\hat{\mathbf{x}}) = \mathbf{p}
\end{aligned}$$

In a more condensed form we have

$$\hat{\mathbf{q}}^\top \mathbf{v} = \hat{\mathbf{q}}^\top \nabla_{\mathbf{q}} c(\hat{\mathbf{q}},\hat{\mathbf{x}}) = c(\hat{\mathbf{q}},\hat{\mathbf{x}}) = \nabla_{\mathbf{x}} c(\hat{\mathbf{q}},\hat{\mathbf{x}})^\top \hat{\mathbf{x}} = \mathbf{p}^\top \hat{\mathbf{x}}$$

A point of further research is the question as to how far this result can be transferred to an economy which consists of many sectors. Although the answer will be given in another paper, the idea may be pointed out by a few steps. Following the theory of international trade, suppose that each commodity x_j is produced by a different sector with all production functions being homogeneous of degree +1.

$$c(\mathbf{q},\mathbf{x}) = \sum_{j=1}^{n} c_j(\mathbf{q},x_j) = \sum_{j=1}^{n} \frac{\partial c_j}{\partial x_j}(\mathbf{q},x_j)\,x_j = \sum_{j=1}^{n} b_j(\mathbf{q})\,x_j$$

where $b_j(\mathbf{q})$ denotes the marginal and average cost of sector j. Moreover,

$$\nabla_{\mathbf{q}} c_j(\mathbf{q},x_j) = \nabla_{\mathbf{q}} b_j(\mathbf{q})\,x_j = \mathbf{v}_j^*$$

is the cost minimal factor demand of sector j. Noting the restriction of (D3) it is natural to impose $b_j(\mathbf{q}) \geqq p_j$ on all sectors so that their behavior is characterized by

$$\begin{aligned}
&(4.7) \quad\Longrightarrow\quad (b_j(\hat{\mathbf{q}}) - p_j)\hat{x}_j = 0 \qquad j = 1,\ldots,n\\
&(4.8) \quad\Longrightarrow\quad (\mathbf{v} - \textstyle\sum_j \mathbf{v}_j^*)^\top \hat{\mathbf{q}} = 0
\end{aligned}$$

No sector is allowed to achieve a positive profit and their common factor demand must not exceed the given factor supply. Similar results can be found in Proposition 5.2.

4.2 Given Amounts of Outputs

In the opposite case of section 4.1 the vector of factor prices $\mathbf{q}$ and the commodity bundle $\mathbf{x}$ are given instead of $\mathbf{p}$ and $\mathbf{v}$. Observing (1.9)

$$\inf_{\mathbf{q}} \left\{ \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid t_I(\mathbf{x}, \mathbf{v}) \geqq 1 \right\}$$

the constraint yields together with (1.4)

$$t_I(\mathbf{x}, \mathbf{v}) \geqq 1 \iff t_O(\mathbf{x}, \mathbf{v}) = \frac{1}{\alpha} \sup_{\mathbf{p}} \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} \mid r(\mathbf{p}, \mathbf{v}) \leqq \alpha \right\} \leqq 1$$

Again for an optimal solution $\hat{\mathbf{v}}$ with $\mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} > 0$ we have two associated programs

(P4) $$c(\mathbf{q}, \mathbf{x}) = \inf_{\mathbf{v}} \left\{ \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid t_I(\mathbf{x}, \mathbf{v}) \geqq 1 \right\}$$

(D4) $$t_O(\mathbf{x}, \mathbf{v})\, c(\mathbf{q}, \mathbf{x}) = \sup_{\mathbf{p}} \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} \mid r(\mathbf{p}, \mathbf{v}) \leqq c(\mathbf{q}, \mathbf{x}) \right\}$$

Analogue to (4.2) the weak duality appears to be

$$\inf \left\{ \mathbf{q}^{\mathsf{T}}\mathbf{v} \mid t_I(\mathbf{x}, \mathbf{v}) \geqq 1 \right\} = \mathbf{q}^{\mathsf{T}}\hat{\mathbf{v}} \geqq \hat{\mathbf{p}}^{\mathsf{T}}\mathbf{x} = \sup \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} \mid r(\mathbf{p}, \hat{\mathbf{v}}) \leqq c(\mathbf{q}, \mathbf{x}) \right\}$$

Because a more detailed analysis of this inequality reveals no further insights of major importance we now turn over to the case of a linear production technology.

5 Shadow Pricing with a Linear Technology

In order to discuss (4.2) in terms of linear programming it will be useful to go one step further ahead in duality theory. Starting with the restriction $t_O(\mathbf{x}, \mathbf{v}) \leqq 1$ on the right hand side of (4.2), we know that this relation is equivalent to $\mathbf{v} \in L(\mathbf{x})$, see (1.1), (1.7), and (1.9). Under the conditions of (1.7) one can show that the polar set of $L_\circ(\mathbf{x})$ satisfies

$$\begin{aligned} L(\mathbf{x}) = L_{\circ\circ}(\mathbf{x}) &:= \left\{ \mathbf{v} \in V \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq 1 \ \ \forall\, \mathbf{q} \in L_\circ(\mathbf{x}) \right\} \\ &= \left\{ \mathbf{v} \in V \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q}, \mathbf{x}) \ \ \forall\, \mathbf{q} \in Q \right\} \end{aligned}$$

Now the revenue maximization problem becomes a problem with infinitely many constraints whose structure is discussed in Blum, Öttli (1975).

(5.1) $$r(\mathbf{p}, \mathbf{v}) = \sup \left\{ \mathbf{p}^{\mathsf{T}}\mathbf{x} \mid \mathbf{q}^{\mathsf{T}}\mathbf{v} \geqq c(\mathbf{q}, \mathbf{x}) \ \ \forall\, \mathbf{q} \in Q \right\}$$

With that (4.2) can be discussed in terms of linear programming.

A linear production technology is characterized by an $m \times n$-matrix $\mathbf{A}$ where the constant elements a_{ij} determine the input coefficients of the input $i = 1, \dots, m$ in the production of good $j = 1, \dots, m$. Now the constraint of (P3) takes the form $\mathbf{Ax} \leqq \mathbf{v}$. From the theory of linear programming it is well known that the following programs

$$\text{(P4)} \qquad \sup_{\mathbf{x}} \left\{ \mathbf{p}^\mathsf{T}\mathbf{x} \,|\, \mathbf{Ax} \leqq \mathbf{v},\ \mathbf{x} \in X \right\}$$

$$\text{(D4)} \qquad \inf_{\mathbf{q}} \left\{ \mathbf{q}^\mathsf{T}\mathbf{v} \,|\, \mathbf{A}^\mathsf{T}\mathbf{q} \geqq \mathbf{p},\ \mathbf{q} \in Q \right\}$$

are dual to each other where (4.2) corresponds to

$$\mathbf{q}^\mathsf{T}\mathbf{v} \geqq \mathbf{q}^\mathsf{T}\mathbf{Ax} \geqq \mathbf{p}^\mathsf{T}\mathbf{x} \qquad \forall\, (\mathbf{x}, \mathbf{q}) \in X \times Q.$$

However, in view of (5.1) the constraints of (P4) may be seen in the form $\mathbf{q}^\mathsf{T}(\mathbf{Ax}) \leqq \mathbf{q}^\mathsf{T}\mathbf{v}$ for all $\mathbf{q} \in Q$. Actually, this system of inequalities can be found in the Kuhn-Tucker conditions of (P4). To show this the Lagrangean form, which is introduced in Walk (1989), is taken as a substitute for the Lagrange functions of (P4) and (D4):

$$\varphi(\mathbf{q}, \mathbf{x}) := \mathbf{p}^\mathsf{T}\mathbf{x} + \mathbf{q}^\mathsf{T}\mathbf{v} - \mathbf{q}^\mathsf{T}\mathbf{Ax}$$

A point $(\hat{\mathbf{q}}, \hat{\mathbf{x}})$ is said to be a saddle point of φ with respect to $Q \times X$ if

$$\text{(5.2)} \qquad \varphi(\hat{\mathbf{q}}, \mathbf{x}) \geqq \varphi(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \geqq \varphi(\mathbf{q}, \hat{\mathbf{x}}) \qquad \forall\, (\mathbf{q}, \mathbf{x}) \in Q \times X.$$

or, equivalently,

$$\text{(5.3a)} \qquad \hat{\mathbf{q}}^\mathsf{T}(\mathbf{v} - \mathbf{A}\hat{\mathbf{x}}) \leqq \mathbf{q}^\mathsf{T}(\mathbf{v} - \mathbf{A}\hat{\mathbf{x}}) \qquad \forall\, \mathbf{q} \in Q$$

$$\text{(5.3b)} \qquad (\mathbf{p} - \mathbf{A}^\mathsf{T}\hat{\mathbf{q}})^\mathsf{T}\mathbf{x} \leqq (\mathbf{p} - \mathbf{A}^\mathsf{T}\hat{\mathbf{q}})^\mathsf{T}\hat{\mathbf{x}} \qquad \forall\, \mathbf{x} \in X$$

As $Q \times X = \mathbb{R}_+^{m+n}$ the equivalent Kuhn-Tucker conditions describe here the property of complementary slackness (e.g. Vanderbei (1998)).

$$\text{(5.4a)} \qquad \mathbf{A}\hat{\mathbf{x}} \leqq \mathbf{v}\,, \qquad \hat{\mathbf{x}} \geqq \mathbf{0}\,, \qquad \hat{\mathbf{q}}^\mathsf{T}\big(\mathbf{v} - \mathbf{A}\hat{\mathbf{x}}\big) = 0\,,$$

$$\text{(5.4b)} \qquad \mathbf{A}^\mathsf{T}\hat{\mathbf{q}} \geqq \mathbf{p}\,, \qquad \hat{\mathbf{q}} \geqq \mathbf{0}\,, \qquad \big(\mathbf{A}^\mathsf{T}\hat{\mathbf{q}} - \mathbf{p}\big)^\mathsf{T}\hat{\mathbf{x}} = 0$$

They implicitly include the conditions $0 \leqq \mathbf{q}^\mathsf{T}(\mathbf{v} - \mathbf{A}\hat{\mathbf{x}})$ for all $\mathbf{q} \in Q$ and $(\mathbf{p} - \mathbf{A}^\mathsf{T}\hat{\mathbf{q}})^\mathsf{T}\mathbf{x} \leqq 0$ for all $\mathbf{x} \in X$. Transferred to convex programming (5.3) corresponds to

$$\mathbf{q}^\mathsf{T}\mathbf{v} - c(\mathbf{q}, \hat{\mathbf{x}}) \geqq \hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \qquad \forall\, \mathbf{q} \in Q$$

$$\mathbf{p}^\mathsf{T}\mathbf{x} - c(\hat{\mathbf{q}}, \mathbf{x}) \leqq \mathbf{p}^\mathsf{T}\hat{\mathbf{x}} - c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) \qquad \forall\, \mathbf{x} \in X$$

where, by (4.7), $\hat{\mathbf{q}}^{\mathsf{T}}\mathbf{v} = c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = \mathbf{p}^{\mathsf{T}}\hat{\mathbf{x}}$. In the next step two generalized conjugate functions[17] are defined, where c_* corresponds to the restriction of (5.1) and c^* is similar to the constraint of (4.1).

$$c_*(\mathbf{v}, \mathbf{x}) \coloneqq \inf\left\{\mathbf{q}^{\mathsf{T}}\mathbf{v} - c(\mathbf{q}, \mathbf{x}) \mid \mathbf{q} \in Q\right\} \qquad \text{with} \quad c_*(\mathbf{v}, \hat{\mathbf{x}}) = 0$$
$$c^*(\mathbf{p}, \mathbf{q}) \coloneqq \sup\left\{\mathbf{p}^{\mathsf{T}}\mathbf{x} - c(\mathbf{q}, \mathbf{x}) \mid \mathbf{x} \in X\right\} \qquad \text{with} \quad c^*(\mathbf{p}, \hat{\mathbf{q}}) = 0$$

The latter problem determines a profit maximum commodity bundle $\mathbf{x}$. On the other hand, the former problem seeks for a price vector such that the difference between the value of the given factor endowment $\mathbf{v}$ and the minimum cost in the production of $\mathbf{x}$ is minimal.

The subsequent analysis is based on the Lagrangean form

$$\varphi(\mathbf{q}, \mathbf{x}) \coloneqq \mathbf{p}^{\mathsf{T}}\mathbf{x} + \mathbf{q}^{\mathsf{T}}\mathbf{v} - c(\mathbf{q}, \mathbf{x}) \,.$$

By the following definitions

$$M(\mathbf{q}) \coloneqq \sup_{\mathbf{x}\in X}\{\varphi(\mathbf{q}, \mathbf{x})\} = \mathbf{q}^{\mathsf{T}}\mathbf{v} + c^*(\mathbf{p}, \mathbf{q}) \qquad Q^\circ \coloneqq \{\mathbf{q} \in Q \mid M(\mathbf{q}) < +\infty\}$$
$$m(\mathbf{x}) \coloneqq \inf_{\mathbf{q}\in Q}\{\varphi(\mathbf{q}, \mathbf{x})\} = \mathbf{p}^{\mathsf{T}}\mathbf{x} + c_*(\mathbf{v}, \mathbf{x}) \qquad X^\circ \coloneqq \{\mathbf{x} \in X \mid m(\mathbf{x}) > -\infty\}$$

the functions $M\colon Q^\circ \to \mathbb{R}$ and $m\colon X^\circ \to \mathbb{R}$ constitute a pair of dual programs.

(P5) $\qquad \inf\{M(\mathbf{q}) \mid \mathbf{q} \in Q^\circ\}$

(D5) $\qquad \sup\{m(\mathbf{x}) \mid \mathbf{x} \in X^\circ\}$

Here, (D5) is a problem of revenue maximization, where the revenue $\mathbf{p}^{\mathsf{T}}\mathbf{x}$ is corrected by the cost term $c_*(\mathbf{v}, \mathbf{x})$. The opposite problem (P5) seeks to minimize cost $\mathbf{q}^{\mathsf{T}}\mathbf{v}$ plus a term, which has been interpreted as maximum profit.

Regarding these two problems, we have the following

Proposition 5.1 *The subsequent three statements are equivalent:*

1. *The function φ has a saddle point $(\hat{\mathbf{x}}, \hat{\mathbf{q}}) \in X \times Q$ such that (5.2) holds good.*

[17] Analogue to Mahler's inequality the duality scheme of (convex) conjugate functions deals with the problem of finding "best" pairs of function (f, g) fulfilling the Young-Fenchel inequality $f(\mathbf{x}) + g(\mathbf{y}) \geqq \mathbf{x}^{\mathsf{T}}\mathbf{y}\ \ \forall\mathbf{x},\ \forall\mathbf{y}$.

2. *$X^\circ \neq \emptyset$ and $Q^\circ \neq \emptyset$. The problems (P5) and (D5) are realizable and*

$$\min\{M(\mathbf{q}) \mid \mathbf{q} \in Q^\circ\} = \varphi(\hat{\mathbf{q}}, \hat{\mathbf{x}}) = \max\{m(\mathbf{x}) \mid \mathbf{x} \in X^\circ\}$$

3. *There is a pair $(\hat{\mathbf{x}}, \hat{\mathbf{q}}) \in X^\circ \times Q^\circ$ such that*

$$\begin{aligned} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) &= \mathbf{p}^\mathsf{T}\hat{\mathbf{x}} - c^*(\mathbf{p}, \hat{\mathbf{q}}) \\ c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) &= \hat{\mathbf{q}}^\mathsf{T}\mathbf{v} - c_*(\mathbf{v}, \hat{\mathbf{x}}) \end{aligned}$$

The proof is straight forward and in line with Walk (1989) it is recommended to go through the following sequence: 1. $\Longrightarrow$ 3. $\Longrightarrow$ 2. $\Longrightarrow$ 1.

Proposition 5.2 (Kuhn-Tucker conditions) *Let $c(\mathbf{q}, \mathbf{x})$ be concave-convex. Suppose c to be differentiable at $(\hat{\mathbf{q}}, \hat{\mathbf{x}})$ "regarding $Q \times X$", where $Q = \mathbb{R}^m_+$ and $X = \mathbb{R}^n_+$. Then the statements of Proposition 5.1 are satisfied if and only if*

$$\begin{aligned} &\nabla_\mathbf{x} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) - \mathbf{p} \geqq \mathbf{0}\,, \qquad \hat{\mathbf{x}} \in X\,, \qquad \hat{\mathbf{x}}^\mathsf{T}\big(\nabla_\mathbf{x} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) - \mathbf{p}\big) = 0\,, \\ &\nabla_\mathbf{q} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) - \mathbf{v} \leqq \mathbf{0}\,, \qquad \hat{\mathbf{q}} \in Q\,, \qquad \hat{\mathbf{q}}^\mathsf{T}\big(\nabla_\mathbf{q} c(\hat{\mathbf{q}}, \hat{\mathbf{x}}) - \mathbf{v}\big) = 0 \end{aligned}$$

Here, the assumption that $c(\mathbf{q}, \cdot)$ is convex, rules out increasing returns to scale. The Kuhn-Tucker conditions correspond to the system (5.4). As it is common practice, the two preceding equalities can be interpreted as follows: if the marginal cost of good j exceeds the price p_j, nothing will be produced ($\hat{x}_j = 0$). On the other hand, $\hat{x}_j > 0$ implies marginal cost pricing. Moreover, a positive factor price $\hat{q}_i$ induces that the factor supply corresponds to the factor demand of input i. Finally, an excess supply of input i requires a zero price $\hat{q}_i$.

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