Quantitative methods in macroeconomics

The Solow model

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Overview

1. Model assumptions

2. The dynamics of the model

Model assumptions

- **Production function:**

$$Y_t = F\left(K_t, A_t L_t\right), \tag{1}$$

with

- Y: output
- K: capital input
- L: labor input
- A: level of technology, knowledge, efficiency of work
- AL: effective labor

Model assumptions

- **Assumptions concerning the production function:**
  - Constant returns to scale:

$$F(cK, cAL) = cF(K, AL) \quad \text{for all } c \geq 0. \tag{2}$$

  - Positive, but declining marginal products of capital and labor

$$\frac{\partial F(\bullet)}{\partial K} > 0 \text{ and } \frac{\partial^2 F(\bullet)}{\partial K \partial K} < 0 \tag{3}$$

and

$$\frac{\partial F(\bullet)}{\partial L} > 0 \text{ and } \frac{\partial^2 F(\bullet)}{\partial L \partial L} < 0 \tag{4}$$

Model assumptions

- **Assumptions concerning the production function:**
  - Both production factors are necessary

$$F(0, AL) = 0 \text{ and } F(K, A0) = 0 \tag{5}$$

  - Inada conditions are satisfied

$$\lim_{K \to 0} \frac{\partial F(\bullet)}{\partial K} \to \infty \text{ and } \lim_{K \to \infty} \frac{\partial F(\bullet)}{\partial K} = 0 \tag{6}$$

and

$$\lim_{L \to 0} \frac{\partial F(\bullet)}{\partial L} \to \infty \text{ and } \lim_{L \to \infty} \frac{\partial F(\bullet)}{\partial L} = 0 \tag{7}$$

Model assumptions

- The production function in intensive form:
  - Define $y = \frac{Y}{AL}$ as output per unit of effective labor and $k = \frac{K}{AL}$ as capital per unit of effective labor.
  - Then the production function can be transformed as follows:

$$y = \frac{Y}{AL} = \frac{F(K, AL)}{AL} = f(k), \tag{8}$$

  where $f(k) = F(k, 1)$.

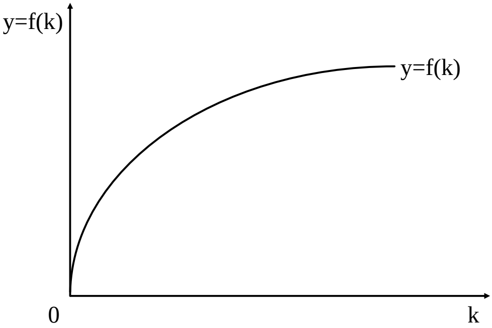
- Special case: Cobb-Douglas production

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha}, \tag{9}$$

where $0 < \alpha < 1$.

Model assumptions

- Graphical illustration of a CD production function in intensive form:



Model assumptions

- The evolution of the production input factors
  - Labor

$$\dot{L}_t = nL_t, \tag{10}$$

where $\dot{L}_t = \frac{\partial L_t}{\partial t}$ (in other words: n is the growth rate of labor).

  - Technology

$$\dot{A}_t = gA_t. \tag{11}$$

- Further assumptions:
  - Saving rate, s, is exogenous.
  - Depreciation rate, δ, is exogenous.
  - Economy is closed, i.e., aggregate savings are equal to aggregate investemt ($S = I$).

The dynamics of the model

- **The dynamics of K**

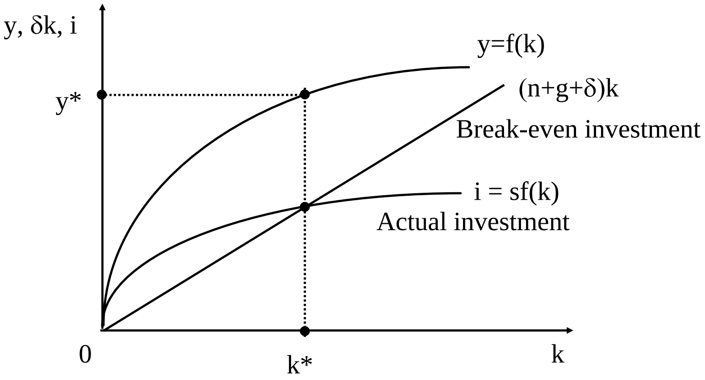
$$\dot{K}_t = sY_t - \delta K_t \tag{12}$$

- **The dynamics of k**
  - Remember: k_t is defined as $\frac{K_t}{A_t L_t}$.
  - Differentiating this expression with respect to t, observing that K_t, A_t and L_t all depend on time, and using the product, quotient and chain rule of differentiation, one obtains (please check):

$$\dot{k}_t = sf(k_t) - (n + g + \delta)k_t \tag{13}$$

  - The above equation states that the rate of change of k_t is the difference between
    - **actual investment per unit of effective labor** ($sf(k_t)$ and
    - **break-even investment** ($(n + g + \delta)k_t$) (amount of investment necessary to keep k at its existing level).

The dynamics of the model: Graphical illustration



The dynamics of the model: Graphical illustration

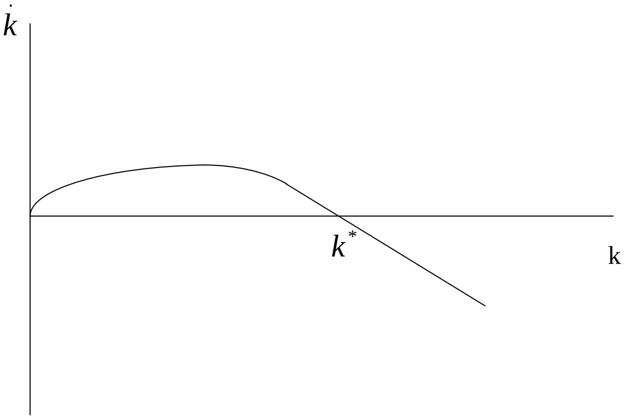


Figure 1.3 The phase diagram for *k* in the Solow model

The balanced growth path

- Balanced growth path (steady state): A situation where each variable is growing at a constant rate.
- In equilibrium:
    - L grows at rate n.
    - A grows at rate g
    - K grows at rate $n + g$
    - Y grows at rate $n + g$
    - $\frac{K}{L}$ grows at rate g
    - $\frac{Y}{L}$ grows at rate g

The impact of a change in the saving rate: Graphical solution

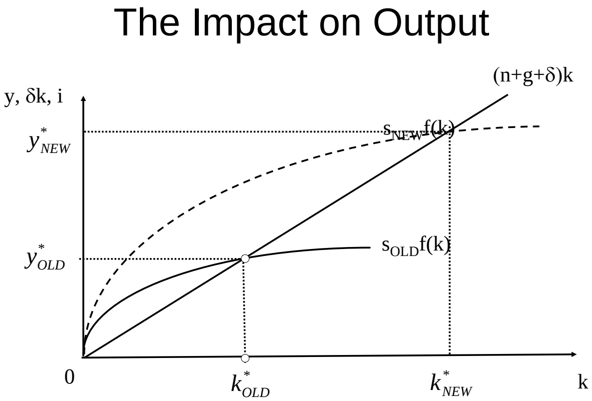


Figure 1.4 The effects of an increase in the saving rate on investment

The impact of a change in the saving rate: Impact on consumption

- An increase in the saving rate has the following two effects on consumption (Remember: Steady-state consumption is given by $c^* = (1-s)f(k^*)$, where an * denotes steady-state values.):
    - Initially, the increase in s lowers consumption.
    - However, in the medium- and long-run the initial effect is mitigated through the increase in k. In the new steady state consumption might even be higher.
- Algebraically, the impact of a change in s can be computed as follows:
    - In the steady state we have:

$$c^* = f(k^*) - sf(k^*) = f(k^*) - (n+g+\delta)k^*, \tag{14}$$

with $k^* = k^*(s, n, g, \delta)$.

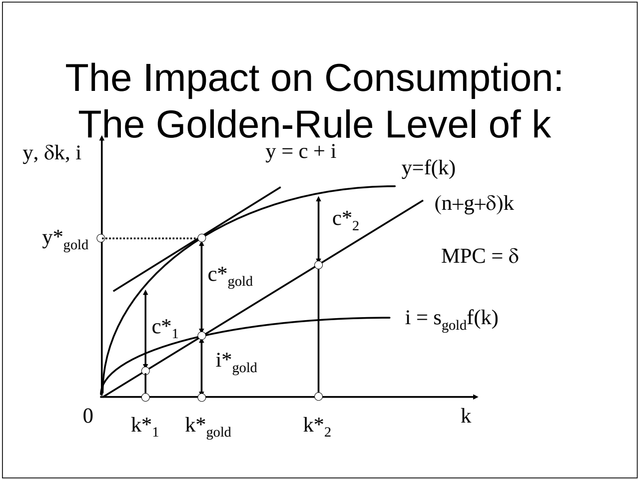
The impact of a change in the saving rate: Impact on consumption

- Impact of a change in s: Analytical derivation (continued):
  - To see what impact a change in s has on c^* this expression has to be differentiated with respect to s. The result is:

$$\frac{\partial c^*}{\partial s} = \left[f'(k^*(s,n,g,\delta)) - (n+g+\delta)\right]\frac{\partial k^*(s,n,g,\delta)}{\partial s}. \qquad (15)$$

  - The second term of this expression (derivative of k with respect to s) is always positive. (Why?)
  - Thus, a change in s increases steady-state consumption (per unit of effective labor) if the expression in the squared brackets is larger than zero, given the old steady-state value of k.
  - Maximum steady-state consumption is reached when $f'(k^*(s,n,g,\delta)) = (n+g+\delta)$. In this case, k has reached its "golden-rule level".

Golden-rule level of *k*



The speed of convergence

- Objective: Want to know how fast k converges to steady state.
- Solution: Obtain solution of differential equation for k derived above (see equation (13)).
- Assume that production function is Cobb-Douglas. Then:

$$\dot{k}_t = sk_t^{\alpha} - (n + g + \delta)k_t. \tag{16}$$

- Problem: Equation is nonlinear.
- Trick: Define the capital/output ratio, denoted by x_t, as follows:

$$x_t = \frac{k_t}{y_t} = \frac{k_t}{k_t^{\alpha}} = k_t^{1-\alpha}. \tag{17}$$

- Then (please check):

$$\dot{x}_t = (1 - \alpha)k_t^{-\alpha}\dot{k}_t \tag{18}$$

and

$$\frac{\dot{x}_t}{x_t} = \frac{(1 - \alpha)k_t^{-\alpha}\dot{k}_t}{k_t^{1-\alpha}} = (1 - \alpha)\frac{\dot{k}_t}{k_t} \tag{19}$$

The speed of convergence

- Transform the differential equation for $\dot{k}$ as follows:

$$\frac{\dot{k}_t}{k_t} = s k_t^{\alpha-1} - (n + g + \delta) \tag{20}$$

- Solving equation (**??**) for $\frac{\dot{k}_t}{k_t}$ and plugging the resulting expression into (20) yields:

$$\frac{1}{1-\alpha} \frac{\dot{x}_t}{x_t} = \frac{s}{x_t} - (n + g + \delta). \tag{21}$$

- This can be rewritten as:

$$\dot{x}_t = (1-\alpha)s - (1-\alpha)(n + g + \delta)x_t. \tag{22}$$

The speed of convergence

- The just derived equation represents a linear differential equations with constant coefficients and a constant forcing term. To solve it we proceed as follows:
  - First, we compute the steady state as follows
$$x^* = \frac{s}{n+g+\delta} \tag{23}$$
  - Secondly, we the transform the differential equation into a homogenous equation. To do so introduce a new variable, z_t, which is defined as the difference between the actual capital/output ratio and its steady-state value, i.e.,
$$z_t \equiv x_t - x^*. \tag{24}$$
  - Then

$$\begin{aligned} \dot{z}_t = \dot{x}_t &= (1-\alpha)s - (1-\alpha)(n+g+\delta)x_t \qquad (25)\\ &= (1-\alpha)s - (1-\alpha)(n+g+\delta)\left[z_t + x^*\right] \\ &= (1-\alpha)s - (1-\alpha)(n+g+\delta)\left[z_t + \frac{s}{n+g+\delta}\right] \end{aligned}$$

The speed of convergence

- - Thirdly, solve the transformed homogenous differential equation as follows:
$$z_t = e^{-\lambda t} z_0. \tag{27}$$
  - Plugging back in the definition of z_t and rearranging gives:
$$x_t = \left(1 - e^{-\lambda t}\right) x^* + e^{-\lambda t} x_0. \tag{28}$$
    The actual capital/output ratio is a weighted average of its steady-state value.
  - Note that:
$$\lim_{t \to \infty} e^{-\lambda t} = 0. \tag{29}$$
  - The speed at which convergence to the steady state occurs is measured by $\lambda = (1 - \alpha)(n + g + \delta)$. Higher values of α, e.g., imply lower speeds of convergence.
  - Since $x_t = k_t^{1-\alpha}$ we obtain for k_t:
$$k_t = \left[\left(1 - e^{-\lambda t}\right) \frac{s}{n + g + \delta} + e^{-\lambda t} k_0^{1-\alpha}\right]^{\frac{1}{1-\alpha}}. \tag{30}$$