Quantitative methods in macroeconomics

Difference and differential equations

Guenter W. Beck
University of Siegen

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Overview

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2. Linear differential equations with constant coefficients: Inhomogenous equations

3. Linear difference equations: Homogenous equations

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Linear differential equations with constant coefficients: Homogenous equations

- Assume we have the following differential equation:

$$\dot{x}_t = b + a x_t, \quad t \geq 0 \tag{1}$$

  where a, b and an initial condition, x_0, are given (and are not time dependent).

- If $b = 0$ then we have the homogenous equation

$$\dot{x}_t - a x_t = 0. \tag{2}$$

- Rewriting yields:

$$\frac{\dot{x}_t}{x_t} = a, \tag{3}$$

  i.e., the variable x_t grows ($a > 0$) or decays ($a < 0$) exponentially at rate a.

Linear differential equations with constant coefficients: Homogenous equations

- The solution to this differential equation is:

$$x_t = e^{at} x_0. \tag{4}$$

(To check that this is indeed the solution for equation (2) plug the obtained expression for x_t into (2) and solve the resulting expression.)

- **Stability properties** of the solution. The solution
    - is unstable, i.e., diverges to $\pm\infty$ (depending on the sign of x_0) if $a > 0$.
    - is stable, i.e., decays to zero if $a < 0$.

Linear differential equations with constant coefficients: Inhomogenous equations

- Assume we have the following differential equation:

$$\dot{x}_t - ax_t = b, \quad t \geq 0 \tag{5}$$

  where a, b and an initial condition, x_0, are given (and are not time dependent) and $b \neq 0$.

- To solve this inhomogenous differential equation we transform it into a homogenous equation by a change of variables.
- In the steady state we have $\dot{x}_t = 0$. Then

$$x^{stst} = x^* = -\frac{b}{a}, \tag{6}$$

  where x^* is called steady-state value of x_t.

Linear differential equations with constant coefficients: Inhomogenous equations

- Now define a new variable, y_t, as the difference between the actual and steady-state value of x_t, i.e.,

$$y_t \equiv x_t - x^*. \tag{7}$$

- Then we can write:

$$\begin{aligned} \dot{y}_t = \dot{x}_t &= ax_t + b \\ &= a\left(x_t - x^* + x^*\right) + b \\ &= ay_t - b + b \\ &= ay_t. \end{aligned} \tag{8}$$

Linear differential equations with constant coefficients: Inhomogenous equations

- The new variable follows a homogenous differential equation which can easily be solved. The solution is:

$$y_t = e^{at} y_0. \tag{9}$$

- Plugging back in the definition of y_t yields:

$$\left(x_t - x^*\right) = e^{at}\left(x_0 - x^*\right) \tag{10}$$

which can be simplified to:

$$x_t = \left(1 - e^{at}\right) x^* + e^{at} x_0. \tag{11}$$

Linear differential equations with constant coefficients: Inhomogenous equations

- **Stability properties**. The solution
  - is globally stable if $a < 0$. In this case $e^{at} \to 0$ as $t \to \infty$ and $x_t \to x^*$ irrespective of the value of the initial condition.
  - diverges to plus or minus infinity (depending on whether the initial value is below or above its steady-state value), if $a > 0$ and $x_0 \neq x^*$.
  - does not move if $x_0 = x^*$.
  - is given by $x_t = tb + x_0$ if $a = 0$.

Linear difference equations: Homogenous equations

- Assume we have the following linear difference equation

$$x_{t+1} - ax_t = b, \;\; t = 0, 1, 2, \ldots \tag{12}$$

given scalars a, b and an initial condition x_0.

- In the homogenous case ($b = 0$) we have:

$$x_{t+1} - ax_t = 0 \Leftrightarrow x_{t+1} = ax_t. \tag{13}$$

- Backward substitution delivers the following solution:

$$x_t = a^t x_0. \tag{14}$$

- **Stability properties** of the solution. The solution
  - is unstable, i.e., diverges to $\pm\infty$ (depending on the sign of x_0) if $|a| \geq 1$.
  - is stable, i.e., decays to zero if $|a| < 1$.

Linear difference equations: Inhomogenous equations

- If $b \neq 0$ we proceed as in the case with continuous time. We thus start by defining a new variable z_t which is defined as the deviation of x_t from its steady state x^*, i.e.

$$z_t = x_t - x^*. \tag{15}$$

- Before we continue it is useful to compute x^*. In the steady state we have $x_{t+1} = x_t = x^*$. Then:

$$x_{t+1} - ax_t = b \Leftrightarrow x^* - ax^* = b \Rightarrow x^* = \frac{b}{1-a}. \tag{16}$$

which is well defined as long as $a \neq 1$ (otherwise there is no steady state).

Linear difference equations: Inhomogenous equations

- Using the new-defined variable the original difference equation can be rewritten as follows:

$$\begin{aligned} x_{t+1} - a x_t = b &\Leftrightarrow \\ z_{t+1} + x^* - a\left(x_t + x^*\right) = b &\Leftrightarrow \\ z_{t+1} + \frac{b}{1-a} - a z_t - \frac{ab}{1-a} = b &\Leftrightarrow \\ z_{t+1} - a z_t = 0. & \end{aligned} \tag{17}$$

- This is an homogenous difference equation in z_t and has the following solution:

$$z_t = a^t z_0. \tag{18}$$

Linear difference equations: Inhomogenous equations

- Redoing the performed change-of-variable yields:

$$(x_t - x^*) = a^t (x_0 - x^*) \tag{19}$$

which can be rewritten as:

$$x_t = (1 - a^t)x^* + a^t x_0. \tag{20}$$

- **Stability properties**. The system is
  - stable if $|a| < 1$ irrespective of the value of the initial condition.
  - unstable if $|a| > 1$.