Quantified Combinatorial Optimization

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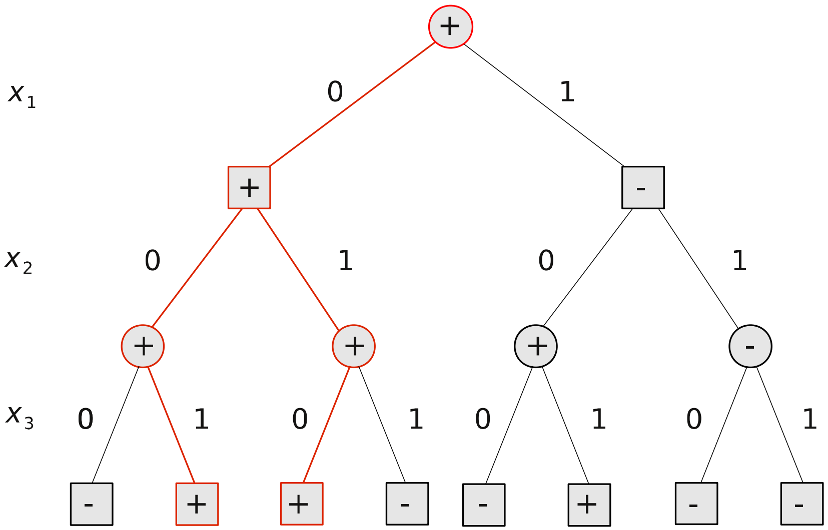
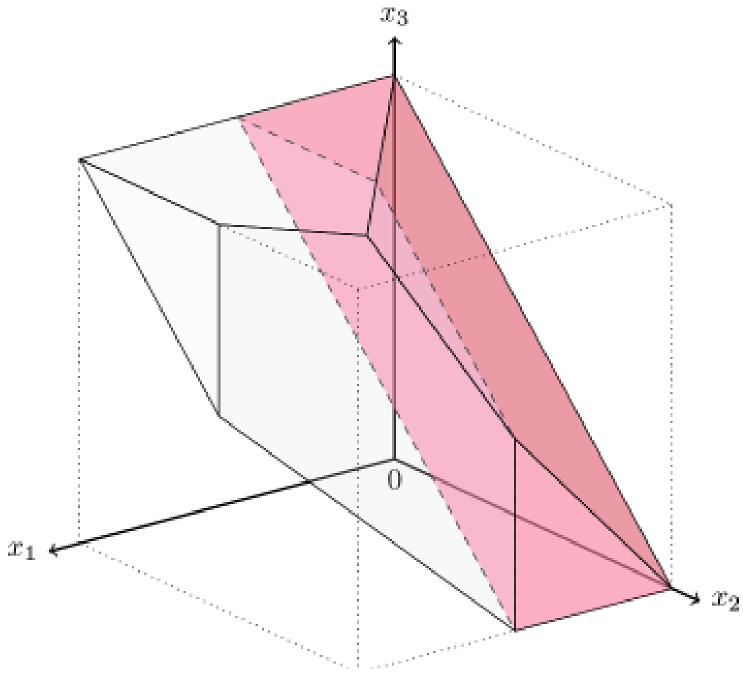
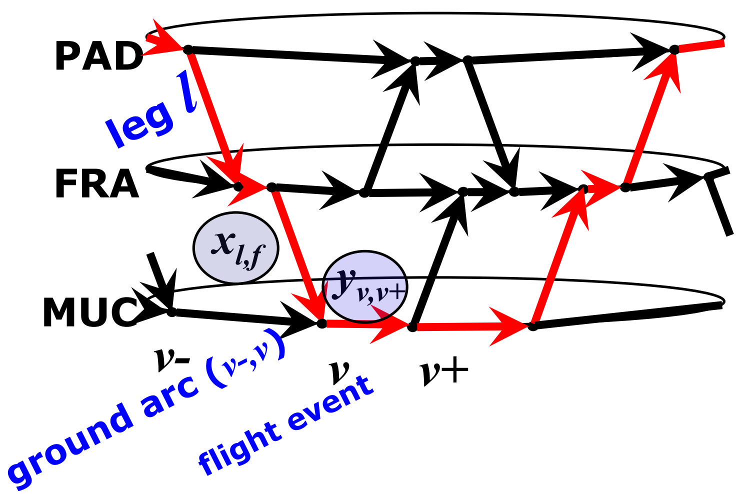




Motivation

Some years ago: Fleet Assignment and disruptions



$x_{l,f}=1$ ⇔ leg l is assigned to subfleet f

$y_{v,v+}$: aircraft on ground

maximize $\sum_{l\in\mathcal{L}}\sum_{f\in\mathcal{F}} p_{l,f}\cdot x_{l,f}$

subject to

$$\sum_{f\in\mathcal{F}} x_{l,f} = 1 \quad \forall\, l\in\mathcal{L}$$

$$\sum_{l_f^{arr}=v} x_{l,f} - \sum_{l_f^{dep}=v} x_{l,f} + y_{v^-,v} - y_{v,v^+} = 0 \quad \forall\, v\in V$$

$$\sum_{l_f\in E_{F0}^f} x_{l,f} + \sum_{(v,v^+)\in E_{G0}^f} y_{v,v^+} \leq size_f \quad \forall f\in\mathcal{F}$$

$$x_{l,f} \in \{0,1\} \quad \forall\, l\in\mathcal{L}; \forall\, f\in\mathcal{F}$$

$$y_{v,v^+} \geq 0 \quad \forall\,(v,v^+)\in E_G$$

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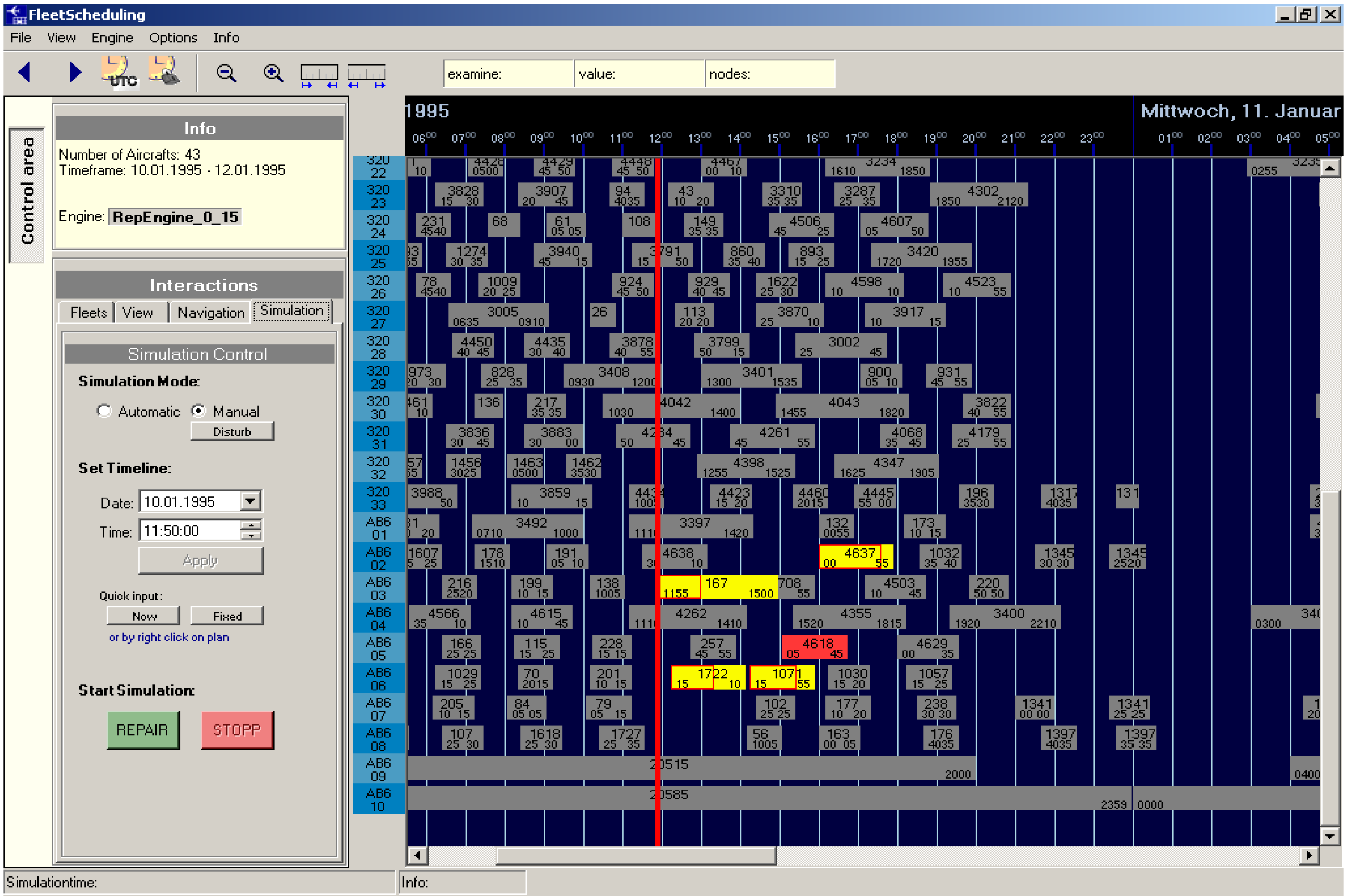
Lufthansa Systems

Motivation

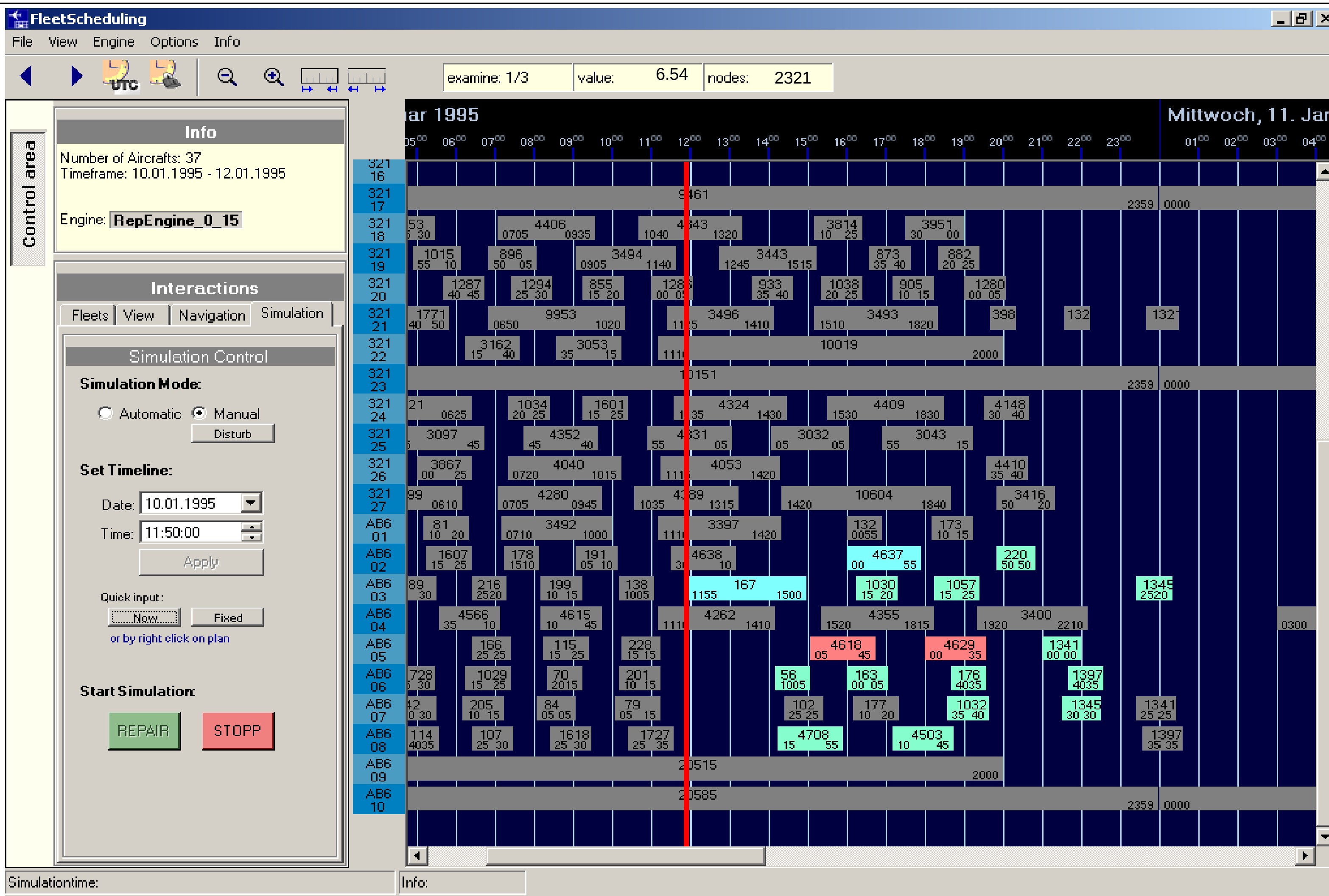
TECHNISCHE UNIVERSITÄT DARMSTADT



FLUID SYSTEM TECHNIK

Motivation

TECHNISCHE UNIVERSITÄT DARMSTADT



FLUID SYSTEM TECHNIK

Motivation

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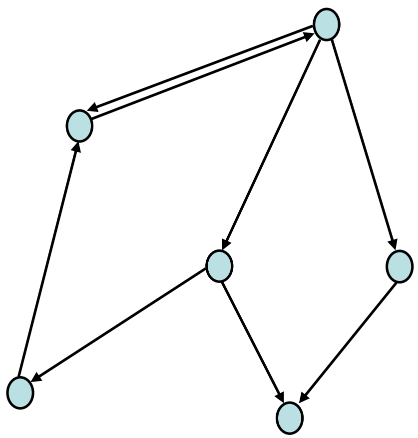
Some years ago: Railway delay management and robust planning

given: **plan** **nodes (stations), arcs (tracks)**

12:45 Hamburg – Berlin 15:45
13:00 Hamburg – Kiel 14:30
...

12:45 Berlin – München 15:45
13:00 Berlin – Darmstadt 16:30
...



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further attributes of the model:
arcs have capacities, FIFO ...
- passengers follow routes (paths), weights and desired start times
- online decision problem: Should we wait for the connecting train?
- minimal travel times at arcs are uncertain

wanted: plan and policy which minimizes the sum of all travel times of all passengers.

FLUID SYSTEM TECHNIK

Motivation

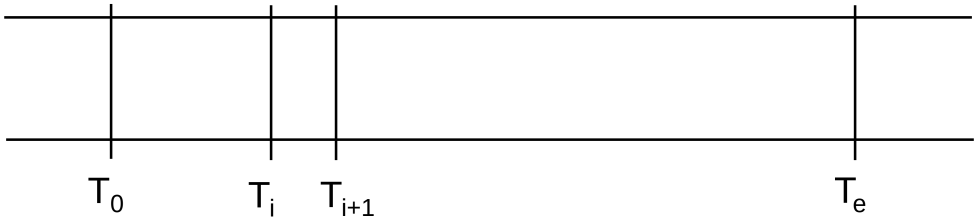
Optimization model „Game against Nature":

1. Player 1: The company — optimizes its plan.

2. Player 2: Nature — „rolls dice".

3. Game: from time T_0 to T_e. After each time step T_i, Nature determines some data. Then the company reacts with a recovery action.

4. The game is a multi-level decision process.

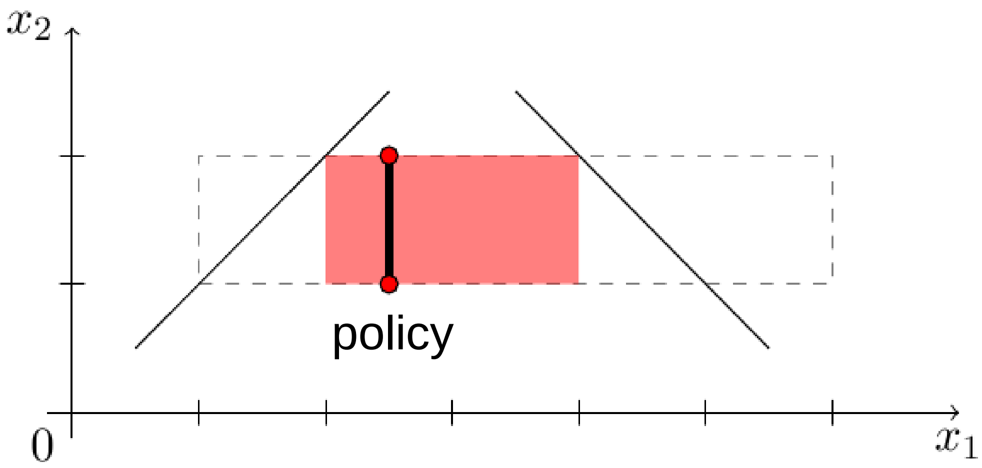


Quantified Linear Programs [1)]

$$\exists x_1 \in [0,1]\ \forall x_2 \in [0,1]\ \exists x_3 \in [0,1]:$$

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

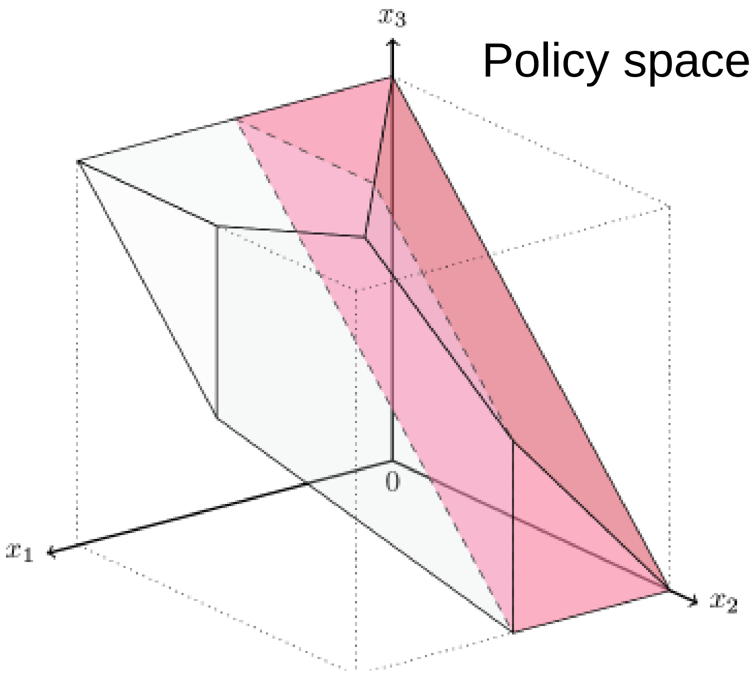
- The union of all games of all winning policies for the existential player forms a polytope.



$$\exists x_1 \in [1,6]\ \forall x_2 \in [1,2] : x_1 + x_2 \leq 6 \ \wedge \ x_2 - x_1 \leq 0$$

1) K. Subramani. Analyzing selected quantified integer programs. Springer, LNAI 3097, pages 342–356, 2004.

Quantified Linear Programs



Quantified Linear Program (QLP)

- continuous variables
- polyhedral solution space
- relaxation of QIP

- There is a winning policy for the existential player against the universal player
  if and only if
  there is a wining policy against the universal player when he is limited to {0,1}
- If the number of quantifier-alternations is constant the vertices of the polytope of winning policies can be described with polynomially many bits.

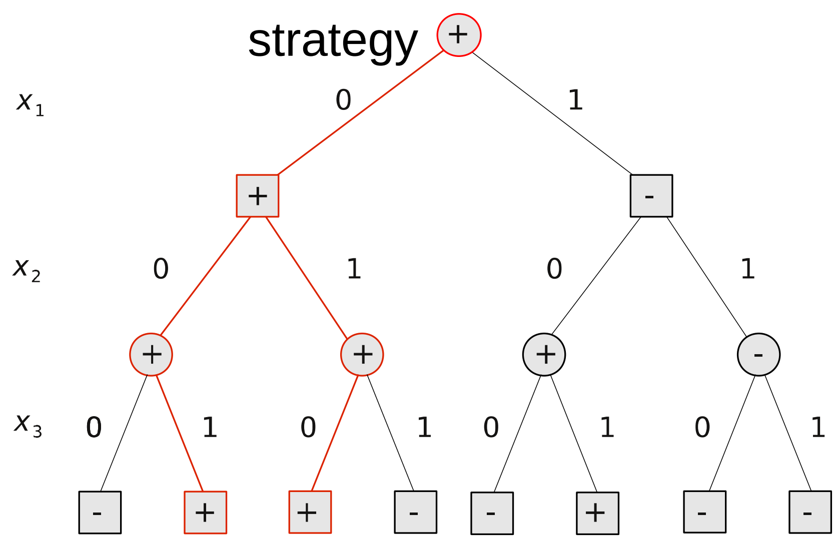
Quantified Linear Integer Programs

$$\exists x_1 \in [0,1]\ \forall x_2 \in [0,1]\ \exists x_3 \in [0,1]\ \text{, x integer:}$$

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

Quantified Integer Program (QIP)

- integral variables
- can be visualized as a game-tree, strategy
- PSPACE-complete



A minmax-objective can be added:
E.g.: The existential player tries to minimize $3x_1 + 2x_2 - x_4$ against all odds.

Quantified Jobshop

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Machines
Papers 1, 2, 3

Table 1: Jobshop model notation.

J	set of jobs
M	set of machines
T	set of tasks, $T \subseteq J \times M$
O	taskorder, $O \subseteq T \times T$
$s_{j,m}$	start time (integer) of task (j,m)
$d_{j,m}$	duration of task (j,m)
$\delta_{j,m}$	additional duration of task (j,m) in case of delay
$e_{j,m}$	earliness of task (j,m), i.e., $e = \max\{d^1 - d^2, 0\}$
$\overline{e}$	mean earliness
m	makespan
$r_{u,j,m}$	indicator of unary encoding of retarded task
$\tilde{r}_b$	indicator of binary encoding retarded task
w	wrapping indicator for binary to unary translation

Table 2: Jobshop tasks.

job	machine	duration	extra
Paper1	Blue	45	5
Paper1	Yellow	10	0
Paper2	Blue	20	5
Paper2	Green	10	10
Paper2	Yellow	34	0
Paper3	Blue	12	0
Paper3	Green	17	0
Paper3	Yellow	28	20

Table 3: Jobshop order.

prior task	later task
Paper1 Blue	Paper1 Yellow
Paper2 Green	Paper2 Blue
Paper2 Blue	Paper2 Yellow
Paper3 Yellow	Paper3 Blue
Paper3 Blue	Paper3 Green

Quantified Jobshop

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$$\min \quad m^2 + k \cdot \bar{e} + \frac{1}{M} \cdot m^1 \qquad \text{s.t.} \qquad \exists\, s^1\, y^1\, m^1\ \forall\, \tilde{r}\ \exists\, r\, w,\, s^2\, y^2\, m^2,\, e : \tag{1}$$

$\forall\, \tilde{r}$ → bin. encoding of scenario index

4 → 01000

$$s^1_{j,m} + d_{j,m} \le m^1 \qquad \forall\, (j,m) \in T \tag{2}$$

$$s^1_{i,m} + d_{i,m} \le s^1_{j,n} \qquad \forall\, (i,m,j,n) \in O \tag{3}$$

$$s^1_{i,m} + d_{i,m} \le s^1_{j,m} + M \cdot (1 - y^1_{i,m,j}) \qquad \forall\, (i,m) \in T,\ (j,m) \in T \tag{4}$$

$$s^1_{j,m} + d_{j,m} \le s^1_{i,m} + M \cdot y^1_{i,m,j} \qquad \forall\, (i,m) \in T,\ (j,m) \in T \tag{5}$$

$$\sum_{(u,j,m) \in U} u \cdot r_{j,m} = \sum_{b \in B} 2^b \cdot \tilde{r}_b - |T| \cdot w \quad \wedge \quad \sum_{\substack{u \in U \\ (j,m) = T_u}} r_{j,m} \le 1 \tag{6}$$

$$s^2_{j,m} + d_{j,m} + \delta_{j,m} \cdot r_{j,m} \le m^2 \qquad \forall\, (j,m) \in T \tag{7}$$

$$s^2_{i,m} + d_{i,m} + \delta_{i,m} \cdot r_{i,m} \le s^2_{j,n} \qquad \forall\, (i,m,j,n) \in O \tag{8}$$

$$s^2_{i,m} + d_{i,m} + \delta_{i,m} \cdot r_{i,m} \le s^2_{j,m} + M \cdot (1 - y^2_{i,m,j}) \qquad \forall\, (i,m) \in T,\ (j,m) \in T \tag{9}$$

$$s^2_{j,m} + d_{j,m} + \delta_{i,m} \cdot r_{i,m} \le s^2_{i,m} + M \cdot y^2_{i,m,j} \qquad \forall\, (i,m) \in T,\ (j,m) \in T \tag{10}$$

$$e_{i,m} \ge s^1_{i,m} - s^2_{i,m} \qquad \forall\, (i,m) \in T \tag{11}$$

$$e_{i,m} \ge 0 \tag{12}$$

$$\bar{e} = \frac{1}{|T|} \cdot \sum_{(i,m) \in T} e_{i,m} \tag{13}$$

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Quantified Jobshop

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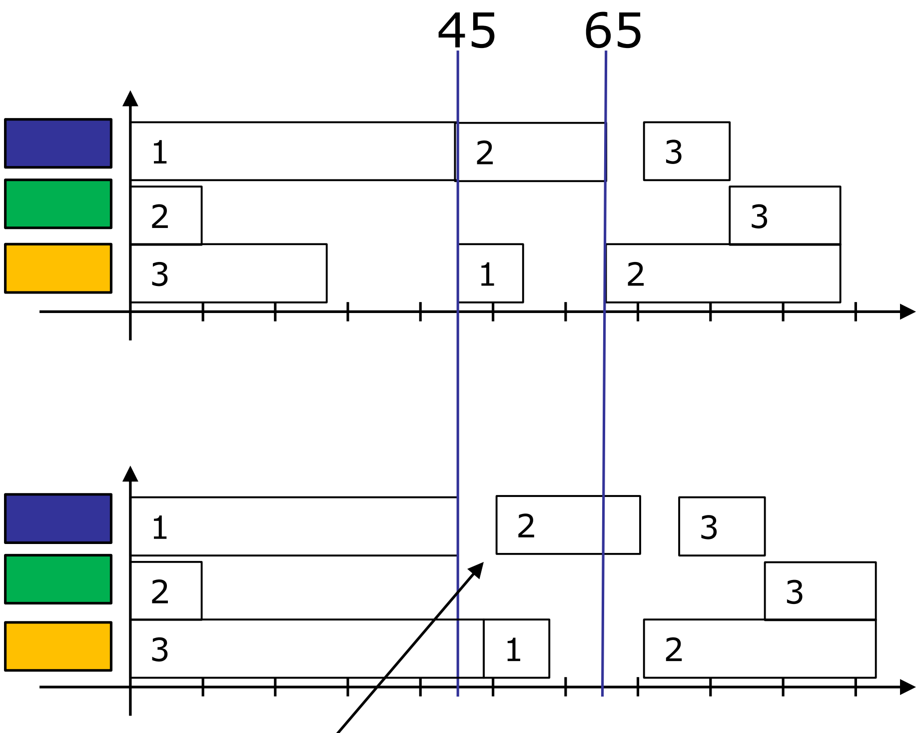
Table 4: Solution of the Jobshop Example.

sc.	1B	1Y	2G	2B	2Y	3Y	3B	3G	m.s.
	first stage solution								
	0	45	0	45	65	0	70	82	99
1	0	50	0	50	70	0	70	82	104
2	0	45	0	45	65	0	65	82	99
3	0	45	0	45	70	0	70	82	104
4	0	45	0	45	65	0	65	82	99
5	0	45	0	45	65	0	65	82	99
6	0	45	0	45	65	0	65	82	99
7	0	45	0	45	65	0	70	82	99
8	0	48	0	50	70	0	75	87	104



Some space, i.e. this is not
The worst case scenario.

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Quantified Jobshop

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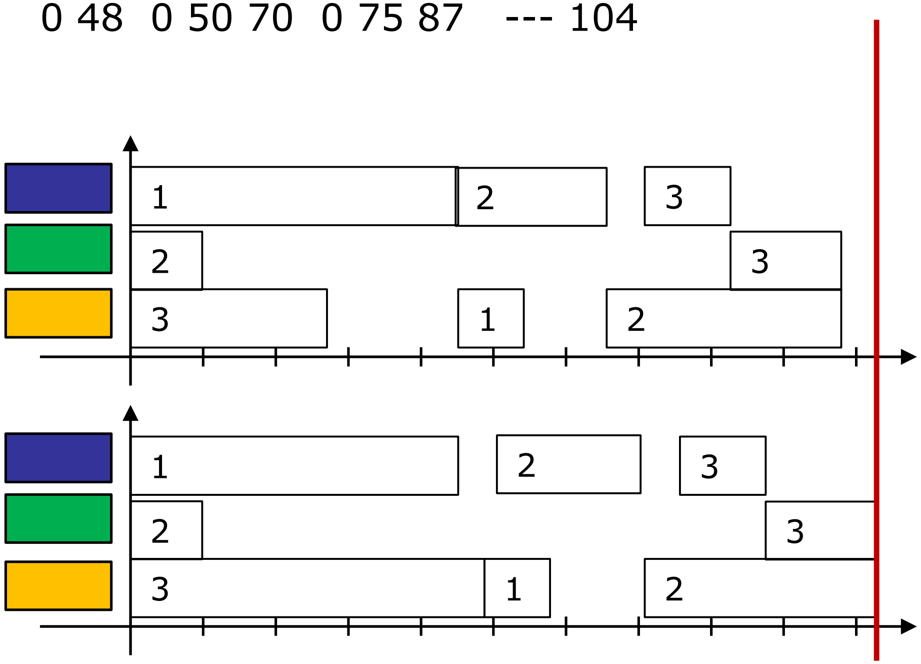
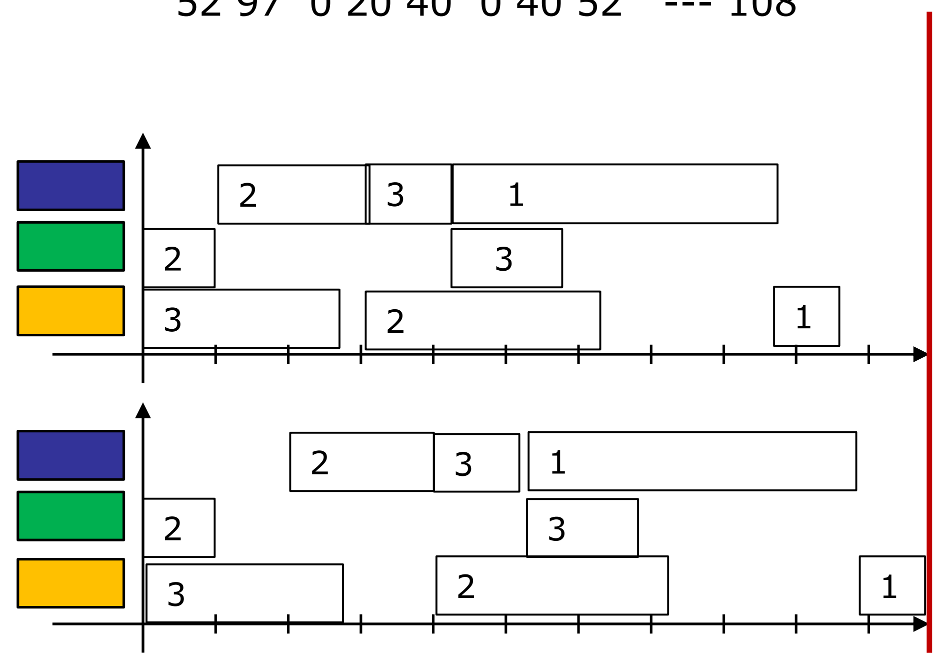
Optimal first-stage solution
0 45 0 45 65 0 70 82 --- 99
Scenario 8:
0 48 0 50 70 0 75 87 --- 104

Optimal deterministic solution
42 87 0 10 30 0 30 42 --- 97
Scenario 4:
52 97 0 20 40 0 40 52 --- 108





No scenario is worse: 104

no awareness: 108

FLUID SYSTEM TECHNIK

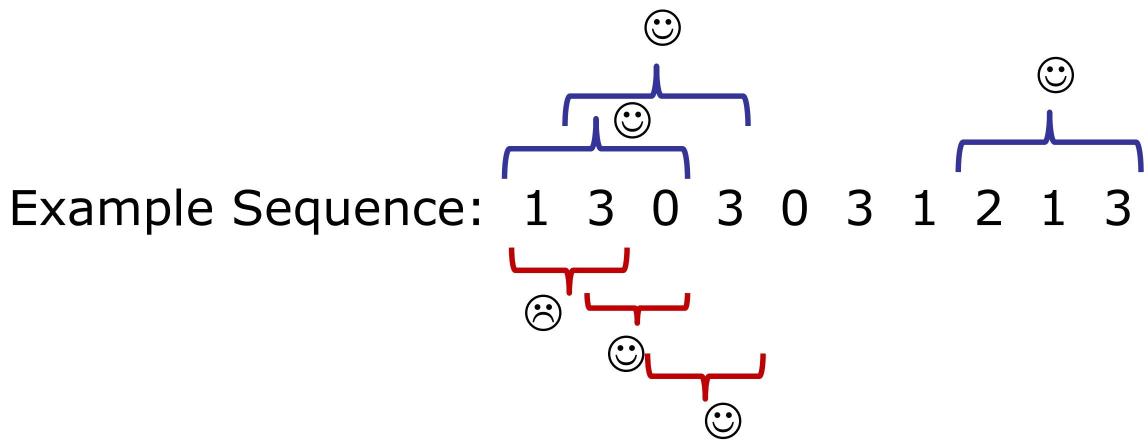
Quantified Carsequencing

Options: 0: basic 1: big wheels 2: no roof

Table 5: Car Instance.

opt	r	s
1	1	2
2	2	3

class	cars	opt 1	opt 2
0	2	0	0
1	3	0	1
2	1	1	0
3	4	1	1



Minimize the violated rules

Quantified Carsequencing

Options: 0: basic 1: big wheels 2: no roof

Table 5: Car Instance.

opt	r	s
1	1	2
2	2	3

class	cars	opt 1	opt 2
0	2	0	0
1	3	0	1
2	1	1	0
3	4	1	1

Example Sequence: 1 3 0 3 0 3 1 2 1 3

Disruption:

Example Sequence: 1 0 3 0 3 1 3 2 1 3

Recover-Action:

Example Sequence: 1 0 3 0 3 1 3 1 3 2

Quantified Carsequencing

Options: 0: basic 1: big wheels 2: no roof

Table 6: Notation of the car sequencing model.

O	set of options
C	set of classes, $C \subseteq \mathcal{P}(O)$
T	set of timesteps ($\lvert T \rvert$ equals number of models)
T^k	set of intervals (by first timestep) for option k , $T^k = \{1, \ldots, \lvert T \rvert - s_k + 1\}$
$r_k : s_k$	at most r_k out of s_k successively sequenced models may require option k
D_c	demand of models of class c
$A_{k,c}$	indicator, if models of class c require option k
$x_{t,c}$	indicator, if a model of class c is produced at timestep t
y_{k,t_0}	indicator, if the sequencing rule $r_k : s_k$ beginning at timestep t_0 is satisfied
S	set of stages (duplicates of the original variables after each move)
σ	stage index: 1 pre-scheduling, 2 state after delay, 3 re-scheduling
F	ordered set of possible delays, $F = \{(t,t') \in T \times T,\, t < t'\}$
U	unary encoding vector of F
m_u	indicator of unary encoding for the delay from timestep t to t', $(t,t') \in U$
B	binary encoding of possible delays
$\widetilde{\mathrm{m}}_b$	indicator of binary encoding for the delay, $b \in B$
w	wrapping indicator for binary to unary translation

Quantified Carsequencing

$$\min \sum_{k \in O} \sum_{t_0 \in T^k} y^1_{k,t_0} \qquad \text{s.t.} \qquad \exists\, x^1\, y^1\, \forall\, \tilde{m}\, \exists\, m\, w,\, x^2\, y^2,\, a,\, x^3\, y^3 : \tag{14}$$

$$\sum_{t \in T} x^s_{t,c} = D_c \qquad \forall\, c \in C,\, s \in S \tag{15}$$

$$\sum_{c \in C} x^s_{t,c} = 1 \qquad \forall\, t \in T,\, s \in S \tag{16}$$

$$\sum_{t=t_0}^{t_0+s_k} \sum_{c \in C} A_{k,c} \cdot x^s_{t,c} \leq r_k + M \cdot y^s_{k,t_0} \qquad \forall\, k \in O,\, t_0 \in T^k,\, s \in S \tag{17}$$

$$\sum_{u \in U} u \cdot m_u = \sum_{b \in B} 2^b \cdot \tilde{m}_b - |F| \cdot w \quad \wedge \quad \sum_{u \in U} m_u \leq 1 \tag{18}$$

$$|x^2_{t,c} - x^1_{t,c}| \leq \sum_{\substack{u \in U \\ (t_i,t_j)=F_u \\ t_i < t < t_j}} m_u \quad \forall\, c \in C,\, t \in T \tag{19}$$

$$\textit{further stage-connecting constraints} \ldots \tag{20}$$

Quantified Carsequencing

Table 7: Solution of the Car Sequencing Example.

Scenario	Mal.	Ans.	Production Sequence	Scenario	Mal.	Ans.	Production Sequence
first stage solution:			1, 3, 0, 3, 0, 3, 1, 2, 1, 3	*first stage solution:*			1, 3, 0, 3, 0, 3, 1, 2, 1, 3
1	–	(4, 5)	1, 3, 0, 3, 3, 0, 1, 2, 1, 3	24	(2, 8)	–	1, 3, 3, 0, 3, 1, 2, 1, 0, 3
2	(0, 1)	(2, 7)	3, 1, 3, 0, 3, 1, 2, 0, 1, 3	25	(2, 9)	–	1, 3, 3, 0, 3, 1, 2, 1, 3, 0
3	(0, 2)	–	3, 0, 1, 3, 0, 3, 1, 2, 1, 3	26	(3, 4)	–	1, 3, 0, 0, 3, 3, 1, 2, 1, 3
4	(0, 3)	–	3, 0, 3, 1, 0, 3, 1, 2, 1, 3	27	(3, 5)	–	1, 3, 0, 0, 3, 3, 1, 2, 1, 3
5	(0, 4)	–	3, 0, 3, 0, 1, 3, 1, 2, 1, 3	28	(3, 6)	–	1, 3, 0, 0, 3, 1, 3, 2, 1, 3
6	(0, 5)	(6, 8)	3, 0, 3, 0, 3, 1, 2, 1, 1, 3	29	(3, 7)	–	1, 3, 0, 0, 3, 1, 2, 3, 1, 3
7	(0, 6)	(7, 8)	3, 0, 3, 0, 3, 1, 1, 1, 2, 3	30	(3, 8)	–	1, 3, 0, 0, 3, 1, 2, 1, 3, 3
8	(0, 7)	(8, 9)	3, 0, 3, 0, 3, 1, 2, 1, 3, 1	31	(3, 9)	–	1, 3, 0, 0, 3, 1, 2, 1, 3, 3
9	(0, 8)	–	3, 0, 3, 0, 3, 1, 2, 1, 1, 3	32	(4, 5)	–	1, 3, 0, 3, 3, 0, 1, 2, 1, 3
10	(0, 9)	–	3, 0, 3, 0, 3, 1, 2, 1, 3, 1	33	(4, 6)	–	1, 3, 0, 3, 3, 1, 0, 2, 1, 3
11	(1, 2)	–	1, 0, 3, 3, 0, 3, 1, 2, 1, 3	34	(4, 7)	(8, 9)	1, 3, 0, 3, 3, 1, 2, 0, 3, 1
12	(1, 3)	–	1, 0, 3, 3, 0, 3, 1, 2, 1, 3	35	(4, 8)	–	1, 3, 0, 3, 3, 1, 2, 1, 0, 3
13	(1, 4)	–	1, 0, 3, 0, 3, 3, 1, 2, 1, 3	36	(4, 9)	–	1, 3, 0, 3, 3, 1, 2, 1, 3, 0
14	(1, 5)	–	1, 0, 3, 0, 3, 3, 1, 2, 1, 3	37	(5, 6)	(7, 9)	1, 3, 0, 3, 0, 1, 3, 1, 3, 2
15	(1, 6)	–	1, 0, 3, 0, 3, 1, 3, 2, 1, 3	38	(5, 7)	–	1, 3, 0, 3, 0, 1, 2, 3, 1, 3
16	(1, 7)	–	1, 0, 3, 0, 3, 1, 2, 3, 1, 3	39	(5, 8)	–	1, 3, 0, 3, 0, 1, 2, 1, 3, 3
17	(1, 8)	–	1, 0, 3, 0, 3, 1, 2, 1, 3, 3	40	(5, 9)	–	1, 3, 0, 3, 0, 1, 2, 1, 3, 3
18	(1, 9)	–	1, 0, 3, 0, 3, 1, 2, 1, 3, 3	41	(6, 7)	(8, 9)	1, 3, 0, 3, 0, 3, 2, 1, 3, 1
19	(2, 3)	(4, 9)	1, 3, 3, 0, 3, 1, 2, 1, 3, 0	42	(6, 8)	–	1, 3, 0, 3, 0, 3, 2, 1, 1, 3
20	(2, 4)	–	1, 3, 3, 0, 0, 3, 1, 2, 1, 3	43	(6, 9)	–	1, 3, 0, 3, 0, 3, 2, 1, 3, 1
21	(2, 5)	–	1, 3, 3, 0, 3, 0, 1, 2, 1, 3	44	(7, 8)	–	1, 3, 0, 3, 0, 3, 1, 1, 2, 3
22	(2, 6)	–	1, 3, 3, 0, 3, 1, 0, 2, 1, 3	45	(7, 9)	–	1, 3, 0, 3, 0, 3, 1, 1, 3, 2
23	(2, 7)	–	1, 3, 3, 0, 3, 1, 2, 0, 1, 3	46	(8, 9)	–	1, 3, 0, 3, 0, 3, 1, 2, 3, 1

P	Set of items
M	Set of machines
I	Set of tasks
T	Number of periods (1…T)
M_i	Machine needed by task i
P_i	Item produced by task i
I_i	$= \{ j \in I : M_j = M_i, j \neq i \}$. Set of tasks that concur with task i for the same machine.
S_p	Nonnegative holding costs for one unit of item p one period in inventory.
B_p	Nonnegative cost for backlogging one unit of item p for one period
O_p	Stockout cost for not fulfilling the (external) demand of one unit of item p
$X_{i,j}$	Nonnegative setup costs for switching from task i to j (on machine $M_i = M_j$)
Q_i	Production cost for one unit of item P_i with task i
$D_{p,t}$	External demand for item p in period t
$A_{p,q}$	Gozinto-factor. Zero if item q is not immediate successor of item p. Otherwise: quantity of item p directly needed to produce one item of q.
$\Delta_{p,i}$	(Integral) number of periods for „transporting" item p to machine M_i (for task i)
$C_{i,t}$	(Maximum) production quantity of item P_i with task i during period t
$T_{i,j}$	(Fractional) number of periods for switching from task i to j
$R_{i,j}$	$= \lfloor T_{i,j} \rfloor$. Number of periods excusively used for switching from task i to j
$R^f_{i,j}$	$= T_{i,j} - R_{i,j}$. Fraction of the $(R_{i,j}+1)$th period after switching from task i to j used for switching

$q_{i,t}$	Production quantity of item P_i by task i in period t
$s_{p,t}$	Inventory for item p at end of period t
$b_{p,t}$	Backlog of item p at end of period t
$o_{p,t}$	Shortfall quantity of item p in period t (no backlog)
$x_{i,j,t}$	Binary variable which indicates whether a setup change from task i to j occurs in (at the beginning of) period t ($x_{i,j,t}$=1) or not ($x_{i,j,t}$=0)
$y_{i,t}$	Binary variable which indicates whether machine M_i is set up for task i at the beginning of (during) period t ($y_{i,t}$=1) or not ($y_{i,t}$=0)

Thank you For your Attention!

minimize $$\sum_{t=1}^{T}\Big(\sum_{p\in P}(S_p s_{p,t}+B_p b_{p,t}+O_p o_{p,t})+\sum_{\substack{i\in I\\ j\in I_i}} X_{i,j} x_{i,j,t}+\sum_{i\in I} Q_i q_{i,t}\Big)$$

$$s_{p,t-1}+\sum_{i:P_i=p} q_{i,t}-D_{p,t}-\sum_{\substack{q,\\ j:P_j=q}} A_{q,p}\, q_{j,t+\Delta_{p,j}}+b_{p,t}+o_{p,t}-b_{p,t-1}=s_{p,t}, \forall p\in P, t=1\ldots T$$

$$b_{p,t}+o_{p,t}-b_{p,t-1}\le D_{p,t}, \forall p\in P, t=1\ldots T$$

$$\sum_{i:M_i=m} y_{i,t}=1, \forall m\in M, t=1\ldots T$$

$$y_{i,t-1}+y_{i,t}-1\le x_{i,j,t}, \forall i\in I, j\in I_i, t=1\ldots T$$

$$C_{i,t}\Big(y_{i,t}-\sum_{\substack{j\in I_i\\ t-R_{j,i}<t'\le t}} x_{j,i,t'}-\sum_{j\in I_i} R^f_{j,i}\, x_{j,i,t-R_{j,i}}\Big)\ge q_{i,t}, \forall i\in I, t=1\ldots T$$

$$q_{i,t}\ge 0; s_{p,t}, b_{p,t}, o_{p,t}\ge 0; x_{i,j,t}, y_{i,t}\in\{0,1\}; \forall i\in I, \forall p\in P, t=1\ldots T$$

	$y_{i,t}=0$	$y_{i,t}=1$
$y_{i,t-1}=1$	$x_{i,j,t}$ darf 0 sein	$x_{i,j,t}$ = 1!
$y_{i,t-1}=0$	$x_{i,j,t}$ darf 0 sein	$x_{i,j,t}$ darf 0 sein