Dynamic Macroeconomics: Problem Set 5

Universität Siegen

Contents

1. Problem 1 (Budget constraints)
2. Problem 2 (Euler Equation)
3. Problem 3 (Permanent income hypothesis)

From period to lifetime budget constraint

- The period budget constraint is given by

$$\underbrace{c_t + a_{t+1}}_{\text{Expenditure}} = \underbrace{(1+r)a_t + y_t}_{\text{Resources}}, \quad \forall t = 1, 2, \cdots \tag{PB}$$

- Solve for a_t:

$$a_t = \frac{1}{1+r}\left(c_t - y_t + a_{t+1}\right). \tag{1}$$

- Forwarding this expression one period yields

$$a_{t+1} = \frac{1}{1+r}\left(c_{t+1} - y_{t+1} + a_{t+2}\right).$$

- We now plug this equation into (1)

$$a_t = \frac{1}{1+r}\left(c_t - y_t + \frac{1}{1+r}\left(c_{t+1} - y_{t+1} + a_{t+2}\right)\right).$$

Iterated substitution

- Rewriting this gives

$$a_t = \frac{1}{1+r}(c_t - y_t) + \left(\frac{1}{1+r}\right)^2 (c_{t+1} - y_{t+1} + a_{t+2}). \qquad (2)$$

- We can then forward (1) one more period to substitute a_{t+2}.

$$a_{t+2} = \frac{1}{1+r}(c_{t+2} - y_{t+2} + a_{t+3}).$$

- Substituting this relationship in (2) again gives

$$a_t = \frac{1}{1+r}(c_t - y_t) + \left(\frac{1}{1+r}\right)^2 \times$$
$$\times \left(c_{t+1} - y_{t+1} + \frac{1}{1+r}(c_{t+2} - y_{t+2} + a_{t+3})\right). \qquad (3)$$

Iterated substitution

- This we can write as

$$a_t = \frac{1}{1+r}(c_t - y_t) + \left(\frac{1}{1+r}\right)^2 (c_{t+1} - y_{t+1}) + \left(\frac{1}{1+r}\right)^3 (c_{t+2} - y_{t+2} + a_{t+3}) .$$

- We could do this substitution for a_{t+3} one more time but we can already see how the expression evolves when we repeat this procedure an infinite number of times, the result is

$$a_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^{s+1} (c_{t+s} - y_{t+s}) + \underbrace{\lim_{s\to\infty} \left(\frac{1}{1+r}\right)^{s+1} a_{t+s+1}}_{=0 \text{ (by assumption)}} .$$

No-ponzi game condition

- The condition

$$\lim_{s\to\infty}\left(\frac{1}{1+r}\right)^{s+1} a_{t+s+1} = 0$$

  is called *no-ponzi game condition*.
- Multiplying by $1+r$ and rearranging yields the lifetime budget constraint

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s} = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} y_{t+s}. \qquad \text{(LB1)}$$

The lifetime budget constraint

- Consider the lifetime budget constraint LB1.

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s} = (1+r)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} y_{t+s}. \qquad \text{(LB1)}$$

- The left hand side represents the present discounted value of consumption expenditures over the whole lifecycle of the agent.
- The right hand side consists of the present discounted value of income over the whole lifecycle of the agent his inital wealth.
- We can interpret the right hand side of this equation as household‘s (lifetime) wealth.

More on the no-ponzi condition

- Consider an household facing the constraint

$$c_t + a_{t+1} = (1+r)a_t, \quad \forall t \geq 0$$

  and we do not impose the no-ponzi game condition.
- Suppose solution to maximisation problem is given by $\{c_t^*\}_{t=0}^{\infty}$ with $c_t^* \leq \bar{c}, \forall t$.
- Agent could improve upon $\{c_t^*\}$ as follows: Set $\bar{c}_0 = c_0^* + 1$ and set $\bar{a}_1 = a_1^* - 1$. (Consume one unit more by borrowing). For $t \geq 1$ leave $\bar{c}_t = c_t^*$ and set $\bar{a}_{t+1} = a_{t+1}^* - (1+r)^t \cdot 1$. (Borrow more to pay for increased consumption at date 0).
- Agent will have more utility if he follows this path. Hence $\{c_t^*\}_{t=0}^{\infty}$ can't be optimal.
- However, agents debt is increasing under the new path without bound and is never repaid.
- To rule this out, need the no-ponzi condition

$$\lim_{s\to\infty} \left(\frac{1}{1+r}\right)^{s+1} a_{t+s+1} = 0$$

Contents

1 Problem 1 (Budget constraints)

2 Problem 2 (Euler Equation)

3 Problem 3 (Permanent income hypothesis)

2.a. Derive the Euler equation for the consumers optimization problem.

- The representative household maximizes

$$\max_{\{c_{t+s}\}_{s=0}^{\infty}} \sum_{s=0}^{\infty} \beta^s u(c_{t+s})$$

subject to

$$c_t + a_{t+1} = (1+r)a_t + y_t. \tag{PB}$$

with period utility function

$$u(c_{t+s}) = c_{t+s} - \frac{\alpha}{2} c_{t+s}^2.$$

The Lagrange function for the consumers optimization problem is given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [c_t + a_{t+1} - (1+r)a_t - y_t] \tag{4}$$

or, written out a bit more explicitly

$$\mathcal{L} = ... + \beta^t u(c_t) + \lambda_t [c_t + a_{t+1} - (1+r)a_t - y_t]$$

2.a. Derive the Euler equation for the consumers optimization problem.

The first-order conditions with respect to c_t, c_{t+1}, a_{t+1} are given by

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Leftrightarrow \beta^t u'(c_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = 0 \Leftrightarrow \beta^{t+1} u'(c_{t+1}) = \lambda_{t+1}$$
$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Leftrightarrow \lambda_t = (1+r)\lambda_{t+1}$$

Combining the conditions, we get out usual Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

and for the given functional form, with $u'(c_t) = 1 - \alpha c_t$, the Euler equation becomes

$$1 - \alpha c_t = \beta(1+r)(1 - \alpha c_{t+1}). \tag{5}$$

2.b. Discuss the determinants of the growth rate of consumption

- Writing the expression for period t yields the Euler equation

$$1 - \alpha c_t = \beta(1 + r)(1 - \alpha c_{t+1}). \quad (6)$$

- When does expected consumption rise, i.e. when is the gross growth rate $c_{t+1}/c_t > 1$?

$$\frac{c_{t+1}}{c_t} = (1 + g_c) > 1 \text{ if } g_c > 0$$
$$\frac{c_{t+1} - c_t}{c_t} = \frac{c_{t+1}}{c_t} - 1 = g_c,$$

  where g_c is the growth rate and $1 + g$ is the gross growth rate of consumption.

- g_c is positive if $c_{t+1} > c_t$.

2.b. Discuss the determinants of the growth rate of consumption

- The Euler equation in rewritten form is

$$\frac{1-\alpha c_{t+1}}{1-\alpha c_t} = \frac{1}{(1+r)\beta} \lesseqgtr 1.$$

- The expected growth rate is positive if $(1+r)\beta > 1$.
- The expected growth rate is negative if $(1+r)\beta < 1$.
- The expected growth rate is zero if $(1+r)\beta = 1$.
- Note that this result is valid for concave utility functions in general.

Contents

1. Problem 1 (Budget constraints)
2. Problem 2 (Euler Equation)
3. Problem 3 (Permanent income hypothesis)

3.a. Discuss the economic meaning of the assumption $\beta(1 + r) = 1$.

- If $(1 + r)\beta > 1$ – A consumer will gain from transfering money to the next period – Savings
- If $(1 + r)\beta < 1$ – A conmuser will loss form transfering money to the next period, so she would like to transfer money from the future period to the current period – Loans.
- If $(1 + r)\beta = 1$ – A consumer is indifferent between periods – Indifference.

To see how the economy can generate savings we need to netralize aforementioned incentives to lend or save, therefore $(1 + r)\beta = 1$.

3.b. Use the Euler equation derived above and the lifetime budget constraint to show how consumption depends on lifetime wealth.

- Since we assumed that $(1+r)\beta = 1$, the Euler equation as of period t is

$$c_t = c_{t+1}.$$

- Iterating forward (and using the *LIE*) we have

$$c_t = c_{t+1} = c_{t+2} = \cdots = c_t = \cdots .$$

- Using this in the life-time budget constraint we get

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c = (1+r)a_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t \tag{7}$$

3.b. Use the Euler equation derived above ...

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c = (1+r)a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

$$c\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t = (1+r)a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

$$c\frac{1}{1-\frac{1}{1+r}} = (1+r)a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

$$c\frac{1}{\frac{r}{1+r}} = (1+r)a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

$$c\frac{1+r}{r} = (1+r)a_0 + \sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

$$c = ra_0 + \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t$$

3.b. Use the Euler equation derived above ...

$$c_t = \underbrace{ra_0 + \underbrace{\frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t y_t}_{\text{annuity value of lifetime income}}}_{\text{lifetime wealth}} .$$

- Note that $r/(1+r)$ is the *marginal propensity to consume*.
- It tells us by how much current consumption is increased when lifetime wealth changes.

3.c.i First suppose $y_0 = \bar{y}$ and $u_t = 0, t = 0, 1, \ldots$. Solve for savings.

Income follows a random walk

$$y_{t+1} = (1-\rho)\bar{y} + \rho y_t + u_{t+1}, \quad 0 < \rho \leq 1$$

y_0 is given, so we start with period 1

$$y_1 = \bar{y} - \rho\bar{y} + \rho y_0 + u_0 = \bar{y} - \rho\bar{y} + \rho\bar{y} = \bar{y}$$

Next, using the expression for lifetime income from resutls form 3.b. we can show that lifetime income is given by

$$\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \bar{y} = \bar{y}\frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r}\bar{y}$$

so that consumption in each period of life is

$$c = \frac{r}{1+r}\frac{1+r}{r}\bar{y} = \bar{y}$$

and thus savings are

$$s_t = y_t - c_t = \bar{y} - \bar{y} = 0$$

3.c.ii Then consider the case $y_0 = \bar{y}$ and $u_0 = 1$ and $u_t = 0, t = 1, 2, \ldots$. Discuss how this affects consumption and saving patterns.

For the income stream in situation ii) we first have to compute its effects on the time-path of income. This is done in the Table 1.

Table: $u_0 = 1$ - Effect on income time-path

Period	u_t	y_t
$y_0 = \bar{y}$	1	since the shock has accured y_0 changes to $y_0 = \bar{y} + 1$
$y_0 = \bar{y} + 1$	0	$y_1 = \bar{y} - \rho\bar{y} + \rho * (\bar{y} + 1) + 0 = \bar{y} + \rho$
$y_1 = \bar{y} + \rho$	0	$y_2 = \bar{y} - \rho\bar{y} + \rho * (\bar{y} + \rho) + 0 = \bar{y} + \rho^2$
$y_2 = \bar{y} + \rho^2$	0	$y_3 = \bar{y} - \rho\bar{y} + \rho * (\bar{y} + \rho^2) + 0 = \bar{y} + \rho^3$
...	...	...

3.c.ii Then consider the case $y_0 = \bar{y}$ and $u_0 = 1$ and $u_t = 0, t = 1, 2, \ldots$. Discuss how this affects consumption and saving patterns.

For the income stream in situation ii) we first have to compute its effects on the time-path of income. This is done in the Table 2.

Table: $u_0 = 1$ - Effect on income time-path

Period	u_t	Effect on y_t	Present Value of The Effect
0	1	1	1
1	0	p	$p\left(\frac{1}{1+r}\right)^1$
2	0	p^2	$p^2\left(\frac{1}{1+r}\right)^2$
3	0	p^3	$p^3\left(\frac{1}{1+r}\right)^3$
...	...	...	...

3.c.ii Then consider the case $y_0 = \bar{y}$ and $u_0 = 1$ and $u_t = 0, t = 1, 2, \ldots$. Discuss how this affects consumption and saving patterns.

To find the effect on life-time wealth, we have to sum up the elements in the *Present Value* column. This gives

$$c_t = \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t(\bar{y}+\rho^t) = \frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\bar{y}+$$

$$\frac{r}{1+r}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t\rho^t = \frac{r}{1+r}\bar{y}\frac{1}{1-\frac{1}{1+r}} + \frac{r}{1+r}\frac{1}{1-\frac{p}{1+r}}$$

$$= \frac{r}{1+r}\frac{1+r}{r}\bar{y} + \frac{r}{1+r} * \frac{1+r}{1+r-p} = \bar{y} + \frac{r}{1+r-p}$$

$$s_t = y_t - c_t = \bar{y} + \rho^t - \bar{y} - \frac{r}{1+r-p} = \rho^t - \frac{r}{1+r-p}$$

3.c.ii Then consider the case $y_0 = \bar{y}$ and $u_0 = 1$ and $u_t = 0, t = 1, 2, \ldots$. Discuss how this affects consumption and saving patterns.

Then the effect on consumption, relative to case i), is given by

$$\Delta c = \frac{r}{1 + r - p}$$

Then the effect on savings, relative to case i), is given by

$$\Delta s = \rho^t - \frac{r}{1 + r - p}$$

Expected change in lifetime income

- Innovations u_0 to income lead to increases in consumption.
- The size of the increase depends upon the size of ρ, i.e. the persistence of the innovations.
- The higher ρ, the higher the change in consumption, because with high ρ the increase in income is very permanent.
- If $\rho = 1$, then consumption changes one for one with innovations to income. In this case, a innovation leads to a permanent increase in income and hence effects consumption.
- Innovations also affect the saving pattern, such that consumers start to transfer income from one period to another.