Dynamic Macroeconomics: Problem Set 4

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1.a How long does it take for the GDP of the three countries to double?

- We first have to compute the growth rates for the three countries.
- We are looking for the constant annual growth rate g_y such that

$$y_{2009} = (1 + g_y)^{2009-1980} y_{1980}$$

holds.

- We can compute it as follows

$$\begin{aligned} y_{2009} &= (1 + g_y)^{2009-1980} y_{1980} && |\text{take logarithm} \\ \ln\, y_{2009} &= (2009 - 1980)\ln\,(1 + g_y) + \ln\, y_{1980} \\ \ln\,(1 + g_y) &= \frac{\ln\, y_{2009} - \ln\, y_{1980}}{2009 - 1980} && |\ln\,(1 + g_y) \approx g_y \\ g_y &\approx \frac{\ln\, y_{2009} - \ln\, y_{1980}}{2009 - 1980} \end{aligned}$$

- For China $g_y = 0.06$, for Germany $g_y = 0.014$ and for the US $g_y = 0.017$.

1.a How long does it take for the GDP of the three countries to double?

- The value of GDP in period t can be written as follows

$$y_t = (1 + g_y)^t y_0$$

- We are looking for the t for which $y_t = 2 \cdot y_0$. We can compute this value as follows

$$\begin{aligned} 2y_0 &= (1 + g_y)^t y_0 \\ 2 &= (1 + g_y)^t \\ \ln 2 &= t \ln(1 + g_y) \\ t &= \frac{\ln 2}{\ln(1 + g_y)} \\ &\approx \frac{\ln 2}{g_y} \approx \frac{0.70}{g_y}. \end{aligned}$$

- For China $t = 11.55$, for Germany $t = 49.5$ and for the US $t = 40.77$.

1.b How long will it take (starting in 2009) for the level of GDP in China to catch up with the one in the United States?

- Assuming a constant growth rate for all future periods holds

$$y_t = (1 + g_y)^{t-2009} y_{2009}$$

- We are looking for that t, for which real GDP per capita is equal in the US and China, i.e.

$$\begin{aligned}
y_t^{China} &= y_t^{US} \\
(1+0.06)^{t-2009} y_{2009}^{China} &= (1+0.017)^{t-2009} y_{2009}^{US} \\
(t-2009)\ln\,(1+0.06) + \ln\, y_{2009}^{China} &= (t-2009)\ln\,(1+0.017) + \ln\, y_{2009}^{US} \\
t - 2009(\ln(1+0.06) - \ln(1+0.17)) &= \ln\, y_{2009}^{US} - \ln\, y_{2009}^{China} \\
t - 2009 &= \frac{\ln\, y_{2009}^{US} - \ln\, y_{2009}^{China}}{\ln(1+0.06) - \ln(1+0.17)} \\
t - 2009 &= 41.3 \\
t &= 2050.3
\end{aligned}$$

2.a Find the first-order conditions characterizing optimal firm behavior.

Firms solve the problem

$$\max_{N_t, K_t} AK_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t.$$

The first-order conditions are

$$FOC1 : \alpha A K_t^{\alpha-1} N_t^{1-\alpha} - R_t = 0$$
$$FOC2 : (1-\alpha) A K_t^{\alpha} N_t^{-\alpha} - w_t = 0$$

$$\alpha A K_t^{\alpha-1} N_t^{1-\alpha} = R_t$$
$$(1-\alpha) A K_t^{\alpha} N_t^{-\alpha} = w_t$$

which state that the marginal product of each factor of production has to be equalized to its marginal cost.

2.b Use the results from 2.a to show that $w_t N_t / Y_t = 1 - \alpha$, where $Y_t = AK_t^{\alpha} N_t^{1-\alpha}$

From question a), we know what w_t is. Using this in the expression $w_t N_t / Y_t$ gives

$$\frac{w_t N_t}{Y_t} = \frac{(1-\alpha) A K_t^{\alpha} N_t^{-\alpha} N_t}{Y_t}$$
$$= \frac{(1-\alpha) A K_t^{\alpha} N_t^{1-\alpha}}{Y_t}$$
$$= \frac{(1-\alpha) Y_t}{Y_t}$$
$$= 1 - \alpha$$

Similar, using results from a) for R_t we can find that

$$\frac{R_t K_t}{Y_t} = \alpha$$

2.c Show that the right-hand side of (1) is equal to Y_t.

$$C_t + I_t = w_t N_t + R_t K_t + \Pi_t \tag{1}$$

Using the results form question 2.b we can represent equation (1) as follows

$$w_t N_t + R_t K_t + \Pi_t = (1-\alpha) Y_t + \alpha Y_t + \Pi_t = Y_t + \Pi_t,$$

Using the results form question 2.a we get

$$\begin{aligned} \Pi_t &= A K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t \\ &= Y_t - (1-\alpha) Y_t - \alpha Y_t \\ &= 0. \end{aligned}$$

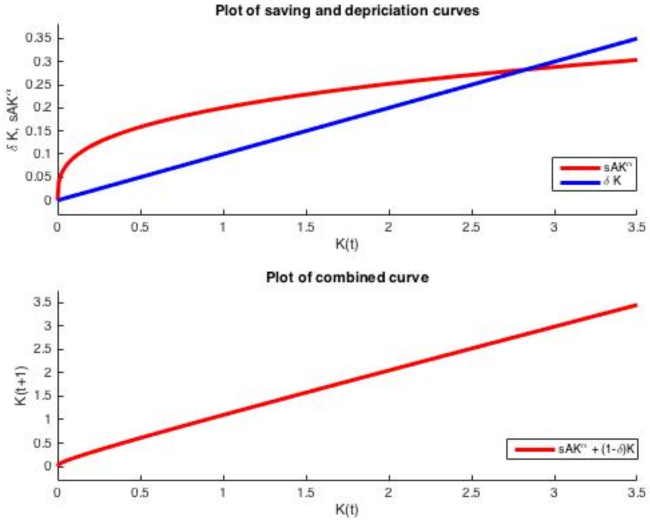
Taking into account that $\Pi_t = 0$:

$$w_t N_t + R_t K_t + \Pi_t = Y_t + \Pi_t = Y_t + 0 = Y_t$$

2.d Write down a difference equation relating K_{t+1} to K_t and parameters of the model.

$$
\begin{aligned}
K_{t+1} &= sY_t + (1-\delta)K_t \\
&= sAK_t^{\alpha} + (1-\delta)K_t
\end{aligned}
$$

2.e Plot K_{t+1} against K_t. Use the graph to argue that there is a steady-state with $K_{t+1} = K_t = \bar{K}$



2.f Solve for steady-state levels of capital, production and consumption. Discuss how they depend upon parameter values.

The steady-state capital stock (for which $K_{t+1} = K_t = \bar{K}$) can be computed as follows

$$\begin{aligned}
\bar{K} &= sA\bar{K}^{\alpha} - (1-\delta)\bar{K} \\
1 &= sA\bar{K}^{\alpha-1} - (1-\delta) \\
\frac{\delta}{sA} &= \bar{K}^{\alpha-1} \\
\left(\frac{\delta}{sA}\right)^{\frac{1}{\alpha-1}} &= \bar{K} \\
\bar{K} &= \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} .
\end{aligned}$$

2.f Solve for steady-state levels of capital, production and consumption. Discuss how they depend upon parameter values.

Using this result in the production function gives us the expression for steady-state production:

$$\bar{Y} = A\bar{K}^{\alpha} = A\left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

To find steady-state consumption we use the fact that $\bar{C} = (1-s)\bar{Y}$ and obtain

$$\bar{C} = (1-s)A\left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

2.g What saving rate s would maximize steady-state production? Discuss.
2.h What saving rate s would maximize current period consumption? Discuss.

2.g $s = 1$, then $\bar{C} = 0$, because the economy converges to a steady state with a lot of capital, but none of it is consumed.
2.h $s = 0$, then $\bar{C} = 0$, because everything is consumed in current period and the economy converges to a steady state with no capital.

2.i What saving rate s would maximize steady-state consumption? Discuss.

We start with the expression for steady-state consumption:

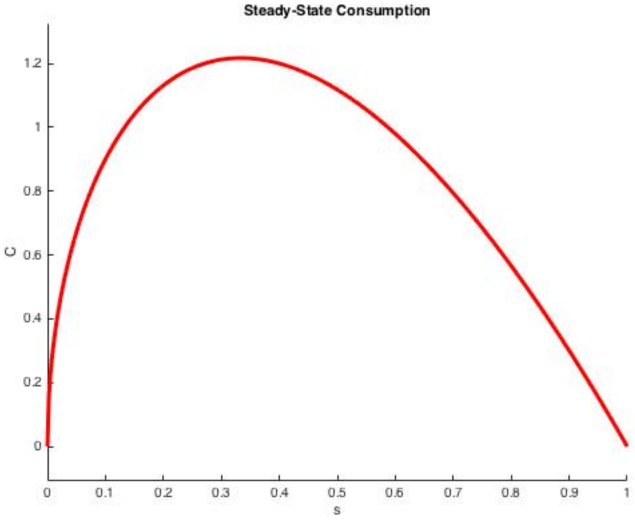
$$\bar{C} = (1-s)A\left(\frac{sA}{\delta}\right)^{\frac{\alpha}{1-\alpha}} = (1-s)s^{\frac{\alpha}{1-\alpha}}A\left(\frac{A}{\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

The first-order necessary condition for a maximum is then given by (neglecting the irrelevant multiplicative constants

$$\begin{aligned} s^{\frac{\alpha}{\alpha-1}} \cdot (-1) + (1-s)s^{\frac{\alpha}{1-\alpha}-1}\frac{\alpha}{1-\alpha} &= 0 \\ s^{\frac{\alpha}{\alpha-1}}\left[-1 + (1-s)s^{-1}\frac{\alpha}{1-\alpha}\right] &= 0 \\ \frac{1-s}{s} &= \frac{1-\alpha}{\alpha} \\ s &= \alpha \end{aligned}$$

$s = \alpha$ – "Golden Rule" saving rate, that maximizes steady state consumption

2.i What saving rate s would maximize steady-state consumption? Discuss.



2.j Suppose $\alpha = 0.33$. Currently the saving rate in the United States is somewhere between 10 and 15 percent. Would you recommend to increase the saving-rate to the one found in question i)?

Not necessarily. It depends on the preferences of households. "Colden Rule" saving rate aims at optimal "long run" consumption, however in the discussed case it demands immediate decline in current consumption. Without knowing how much a household values both – current and "long run" consumptions – we can not definitely encourage an increase in saving rate.

3.a Derive the following equation $(1+g_a)(1+g_n)\hat{k}_{t+1} = s\hat{k}_t^\alpha + (1-\delta)\hat{k}_t$.

$\hat{k}_t$ is capital per efficiency units of labor, defined as $\hat{k}_t = K_t/A_tN_t$. Take capital accumulation equation $K_{t+1} = I_t + (1-\delta)K_t$, combine it with the production function $Y_t = K_t^\alpha(A_tN_t)^{1-\alpha}$ and the investment equation $I_t = sY_t$:

$$K_{t+1} = sK_t^\alpha(A_tN_t)^{1-\alpha} + (1-\alpha)K_t \qquad |: A_tN_t$$

$$\frac{K_{t+1}}{A_tN_t} = sK_t^\alpha\frac{(A_tN_t)^{1-\alpha}}{A_tN_t} + (1-\delta)\frac{K_t}{A_tN_t}$$

$$\frac{A_{t+1}N_{t+1}}{A_tN_t}\frac{K_{t+1}}{A_{t+1}N_{t+1}} = s\hat{k}_t^\alpha + (1-\delta)\hat{k}_t \qquad | * \frac{A_{t+1}N_{t+1}}{A_{t+1}N_{t+1}}$$

$$(1+g_a)(1+g_n)\hat{k}_{t+1} = s\hat{k}_t^\alpha + (1-\delta)\hat{k}_t$$

3.b Derive the steady-state value of $\hat{k}$ and discuss how it depends upon the parameters of the model.

Steady-state capital per efficiency units of labor is given by $\hat{k}_t = \hat{k}_{t+1}$
Use equation from part a), and drop the time-subscripts to solve for steady-state capital stock. It is given by

$$\bar{\hat{k}} = \left(\frac{s}{(1+g_a)(1+g_n) - (1-\delta)} \right)^{\frac{1}{1-\alpha}} .$$

3.c Calculate the steady-state growth rates of Y and Y/N.

Capital growth: Steady-state capital per efficiency units of labor is given by $\hat{k}_t = \hat{k}_{t+1}$, so we can write $\frac{K_t}{A_t N_t} = \frac{K_{t+1}}{A_t N_{t+1}}$.

$$\frac{K_t}{A_t N_t} = \frac{K_{t+1}}{A_{t+1} N_{t+1}}$$

$$\frac{A_{t+1} N_{t+1}}{A_t N_t} = \frac{K_{t+1}}{K_t}$$

$$\frac{A_t(1+g_a) N_t(1+g_n)}{A_t N_t} = \frac{K_{t+1}}{K_t}$$

$$K_{t+1} = (1+g_a)(1+g_n)K_t$$

$$ln(K_{t+1}) = ln(1+g_a) + ln(1+g_n) + ln(K_t)$$

$$ln(K_{t+1}) - ln(K_t) \approx g_a + g_n$$

3.c Calculate the steady-state growth rates of Y and Y/N.

Output growth:

$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}^{\alpha}(A_{t+1}N_{t+1})^{1-\alpha}}{K_t^{\alpha}(A_tN_t)^{1-\alpha}}$$

$$\frac{Y_{t+1}}{Y_t} = \frac{((1+g_a)(1+g_n)K_t)^{\alpha}((1+g_a)A_t(1+g_n)N_t)^{1-\alpha}}{K_t^{\alpha}(A_tN_t)^{1-\alpha}}$$

$$\frac{Y_{t+1}}{Y_t} = (1+g_a)^{\alpha}(1+g_n)^{\alpha}(1+g_a)^{1-\alpha}(1+g_n)^{1-\alpha}$$

$$\frac{Y_{t+1}}{Y_t} = (1+g_a)(1+g_n)$$

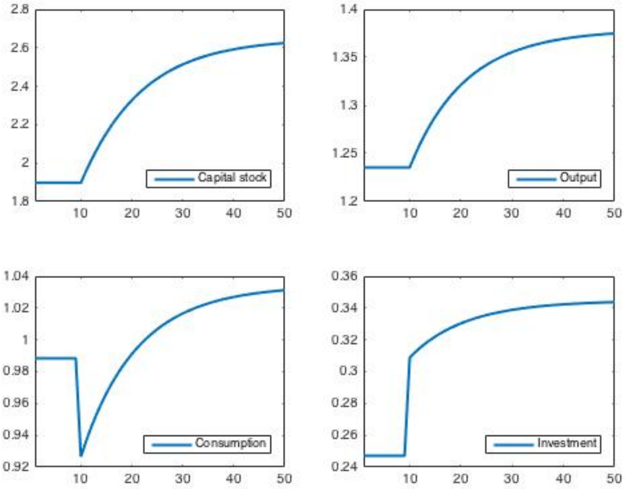
$$Y_{t+1} = (1+g_a)(1+g_n)Y_t$$

$$ln(Y_{t+1}) = ln(1+g_a) + ln(1+g_n) + ln(Y_t)$$

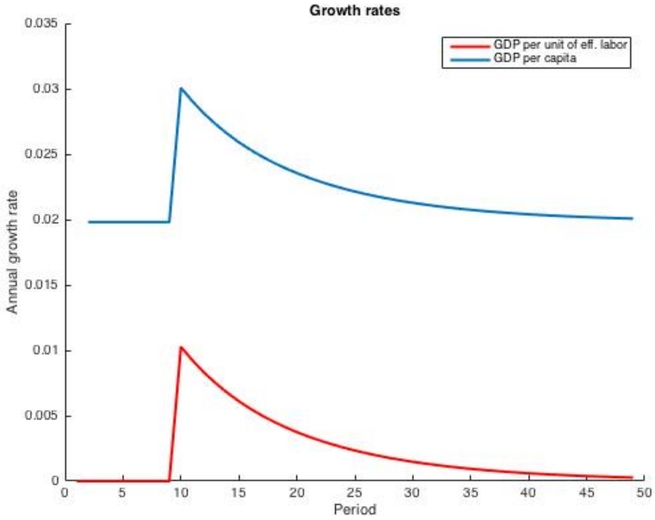
$$ln(Y_{t+1}) - ln(Y_t) \approx g_a + g_n$$

Output per capita growth: $ln(\frac{Y_{t+1}}{N_{t+1}}) - ln(\frac{Y_t}{N_t}) \approx g_a$

3.d Suppose that for periods 1-9 $\hat{k}$ is equal to its steady-state value. Then, in period $t = 10$, the saving rate increases to $s = 0.25$. Discuss what happens.



3.f Calculate the growth rate of production in the periods after the change in the saving rate.



4.a As a first step, construct a time series for the capital stock using the equation $K_{t+1} = I_t + (1-\delta)K_t$. To set an initial value of K_0 use the formula $K_0 = \frac{1}{\delta}\frac{\bar{I}}{\bar{Y}}Y_0$.

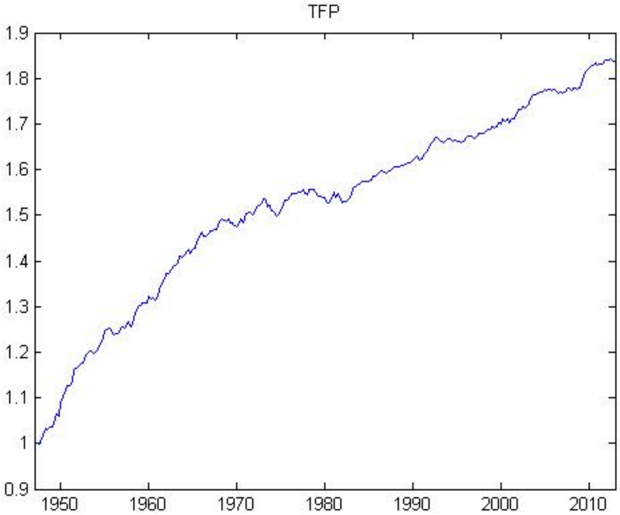
4.b Compute time series on $\Delta \ln Y_t, \Delta \ln K_t$ and $\Delta \ln N_t$. Then compute the time series of $\Delta \ln A_t$.

4.c Compute the accumulated logarithm of A_t by computing the cumulative sum of $\Delta \ln A_t$. Plot the resulting series. Discuss.

4.d Calculate averages for each decade (i.e. $1950-1959$, $1960-1969$, ...) of annual GDP growth rates and annual technology growth rates and display them in a table.

See Matlab code for the solutions. Following variable are avaliable in the data: HOANBS - Nonfarm Business Sector: Hours of All Persons, GDPC1 - Real Gross Domestic Product, GPDIC96 - Real Gross Private Domestic Investment

4.c Compute the accumulated logarithm of A_t by computing the cumulative sum of $\Delta \ln A_t$. Plot the resulting series. Discuss.



4.d Calculate averages for each decade (i.e. 1950 – 1959, 1960 – 1969, ...) of annual GDP growth rates and annual technology growth rates and display them in a table.

Variable	1950s	1960s	1970s	1980s	1990s	2000s
GDP	0.0107	0.0116	0.0084	0.0079	0.0080	0.0041
Labor	0.0038	0.0046	0.0044	0.0044	0.0039	-0.0017
Capital	0.0049	0.0109	0.0106	0.0089	0.0097	0.0097
TFP	0.0065	0.0049	0.0020	0.0019	0.0021	0.0021