Dynamic Macroeconomics

Problem Set 3

1. **Model of the centralized economy** Assume that we have an economy that can be described as follows:

   - **Households and preferences**
     - The economy is inhabited by one single representative consumer.
     - This consumer has preferences over an infinite stream of consumption $c_0, c_1, ... = \{c_t\}_{t=0}^{\infty}$.
     - The consumer's lifetime utility function is assumed to be **time-separable** and given by:
     $$U(\{c\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$
     - β is the individuals subjective time discount factor. We assume that $0 < \beta < 1$ holds.
     - $u\left(c_t\right)$ denotes the period utility function. We assume that it is strictly increasing and concave and is given by the following function:
     $$u(c) = \frac{c^{1-\sigma}}{1-\sigma}.$$
     We assume that $\sigma > 0$.
   - **Production**
     - The production technology of the economy is given by:
     $$y = f(k),$$
     where y denotes the output, k the used capital and $f(.)$ is a production function.
     - We further assume that $f(k)$ is Cobb-Douglas and thus given by:
     $$f(k) = Ak^{\alpha}.$$
     A denotes the level of 'knowledge'. Assume that $0 < \alpha < 1$.
     - We further assume that the consumer owns the production technology, i.e. there are no firms.

   a) Derive the budget constraint of the representative consumer.

   b) Setup the inter-temporal optimization problem of the individual.

c) Setup the Lagrange function associated with the inter-temporal optimization problem and show that the first-order conditions of the optimization problem can be written as follows:

$$u'(c_t) = \beta u'(c_{t+1}) \left[1 + f'(k_{t+1}) - \delta\right] \tag{1}$$

and

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{2}$$

d) Compute the models steady state values of capital k^* and consumption c^*. Discuss how the steady state values depend upon the parameters of the model.

From now one assume that $\sigma = 1$ (so that the utility function becomes $u(c_t) = \ln c_t$) and that $\delta = 1$. Also assume that $\alpha = 0.33, \beta = 0.97, A = 1$.

e) Show that the optimal solution for the evolution of the capital stock is given by

$$k_{t+1} = \alpha\beta k_t^{\alpha}.$$

f) Use the results from part e) to simulate the convergence process of the economy if it starts from a initial level of capital that is ten-percent below its steady state level.

g) Use the results from part e) to simulate the convergence process of the economy if the discount rate increases from $\beta = 0.97$ to $\beta = 0.98$.

2. **Hodrick-Prescott Filter** This problem introduces you to one of the main tools for decomposing a time-series in a trend and a cyclical component. Suppose you are given a sequence of data y_t for $t = 1, 2, ..., T$. (For example, a time-series on Gross Domestic Product (GDP) for T periods). The objective is to find a trend series $\{\tau_t\}_{t=1}^{T}$ that solves the following problem:

$$\min_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \tag{3}$$

where $\lambda \geq 0$ is a parameter.

a) Interpret the objective function in (3). (Assume that y_t and τ_t are expressed in (natural) logarithms).

b) Discuss what happens if $\lambda = 0$.

c) Discuss what happens if $\lambda \to \infty$.

d) Suppose $0 < \lambda < \infty$ and $T = 4$. Derive the first-order conditions for the optimal choice of $\{\tau_t\}_{t=1}^{4}$ and write them in matrix form.