Dynamic Macroeconomics: Problem Set 2

Universität Siegen

1. Two period model - Problem 1
2. Two period model with borrowing constraint - Problem 2

1a) Maximization problem

- The maximization problem of the agent is

$$\max_{c_1,c_2,a_1,a_2} u(c_1) + \beta u(c_2)$$
$$\text{such that :}$$
$$c_1 + a_1 = y_1 + a_0 \tag{P1}$$
$$c_2 + a_2 = y_2 + (1+r)a_1$$
$$c_1 \geq 0, c_2 \geq 0$$

- To solve this problem, we will first
  - Get rid of the choice variable a_2 (Question 1b)).
  - Get rid of the non-negativity constraints on c_1 and c_2 (Question 1c)).

  and then take the first-order conditions for the simplified optimization problem.

1b) Why $a_2 > 0$ can't be optimal

- Suppose you have solved the problem in (P1) and denote the solution by
$$c_1^*, c_2^*, a_1^*, a_2^*$$
and suppose that
$$a_2^* > 0$$
- Now suppose you reduce a_2^* by ϵ to $a_2^* - \epsilon$ and increase c_2^* to $c_2^* + \epsilon$. This will increase life-time utility, since $u'(c_2) > 0$.
- Hence a_2^* can't be optimal.
- Intuitively: Saving for the time after your death does not yield utility.
- We will also assume that $a_2^* < 0$ is not allowed. (Banks won't lend to you, if they know you die tomorrow).

1c) Why $c_t = 0, t = 1, 2$ can't be optimal

- Suppose you have solved the problem in (P1) and denote the solution by

$$c_1^*, c_2^*, a_1^*, a_2^*$$

and suppose that

$$c_1^* = 0$$

- Now suppose you reduce c_2^* by ϵ to $c_2^* - \epsilon$ and increase c_1^* to $c_1^* + \epsilon$.
- This reduces utility in the second period by $u'(c_2^*)\epsilon$.
- It increases utility in the first period by $u'(c_1^*)\epsilon$.
- The overall change in life-time utility is

$$u'(c_1^*)\epsilon - \beta u'(c_2^*) = u'(0)\epsilon - \beta u'(c_2^*)\epsilon > 0$$

since we assumed that

$$\lim_{c_t \to 0} u'(c_t) = \infty$$

- Hence $c_1^* = 0$ can't be optimal.

1d) Simplified maximization problem

- We have simplified our optimization problem (P1) to

$$\begin{aligned} \max_{c_1,c_2,a_1} u(c_1) + \beta u(c_2) \\ \text{such that :} \\ c_1 + a_1 = y_1 + a_0 \\ c_2 = y_2 + (1+r)a_1 \end{aligned} \tag{P2}$$

- We will solve this problem on two ways:
  - Using the substitution method.
  - Using the Lagrange function.

1d) Substitution method

- Solving $c_1 + a_1 = y_1 + a_0$ for c_1 and using $c_2 = y_2 + (1+r)a_1$ we can write the optimization problem as

$$\max_{a_1} u(y_1 + a_0 - a_1) + \beta u(y_2 + (1+r)a_1)$$

- The first order condition is (using the chain rule)

$$\begin{aligned} -u'(y_1 + a_0 - a_1)) + \beta u'(y_2 + (1+r)a_1)(1+r) &= 0 \\ u'(y_1 + a_0 - a_1)) &= \beta u'(y_2 + (1+r)a_1)(1+r) \quad |\text{Def. of } c_i \\ u'(c_1) &= \beta(1+r)u'(c_2) \qquad \text{(Euler)} \end{aligned}$$

- Equation (Euler) is one of the fundamental building blocks of dynamic macro models.

1d) Lagrange function I

- The Lagrange function for the problem in (P2) is

$$L(c_1, c_2, a_1, \lambda_1, \lambda_2) = u(c_1) + \beta u(c_2) + \lambda_1[y_1 + a_0 - c_1 - a_1] \\ + \lambda_2[y_2 + (1+r)a_1 - c_2]$$

- The first order conditions are

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda_1 = 0 \qquad \text{(FOC 1)}$$

$$\frac{\partial L}{\partial c_2} = \beta u'(c_2) - \lambda_2 = 0 \qquad \text{(FOC 2)}$$

$$\frac{\partial L}{\partial a_1} = -\lambda_1 + (1+r)\lambda_2 = 0 \qquad \text{(FOC 3)}$$

$$\frac{\partial L}{\partial \lambda_1} = y_1 + a_0 - c_1 - a_1 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = y_2 + (1+r)a_1 - c_2 = 0$$

1d) Lagrange function II, Euler Equation

- From FOC 1, FOC 2 and FOC 3 we get

$$u'(c_1) = \beta(1 + r)u'(c_2) \tag{Euler}$$

which is again the Euler equation.
- The Euler equation describes the optimal consumption time path.
- Think of it in terms of costs and benefits of saving. LHS: Costs of saving. RHS: Benefits of saving.
- In the optimum the consumer cannot improve her utility by shifting consumption inter-temporally.
- We can rewrite the equation to

$$\frac{u'(c_1)}{\beta u'(c_2)} = 1 + r$$

- Marginal rate of substitution (LHS) has to be equal to relative price.

1e) $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$

- For the period utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

- We have

$$u'(c) = c^{-\sigma} = \frac{1}{c^{\sigma}} > 0$$

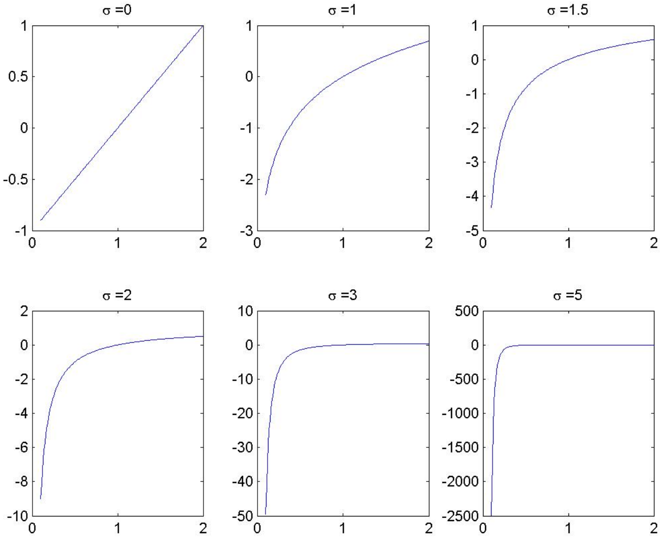
and

$$u''(c) = -\sigma c^{-\sigma-1} < 0$$

- The Euler equation reads then as

$$c_1^{-\sigma} = \beta(1 + r)c_2^{-\sigma} \tag{Euler}$$

1e) Plot of $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$



1f) Consumption growth

- Rewrite the Euler equation as

$$
\begin{aligned}
\left(\frac{c_2}{c_1}\right)^{\sigma} &= \beta(1+r) && \left|\beta = \frac{1}{1+\rho}\right. \\
\left(\frac{c_2}{c_1}\right)^{\sigma} &= \frac{1+r}{1+\rho} && |\ \ln \\
\sigma \ln\left(\frac{c_2}{c_1}\right) &= \ln(1+r) - \ln(1+\rho) && |\ln(1+x) \approx x \\
\ln\left(\frac{c_2}{c_1}\right) &\approx \frac{1}{\sigma}(r-\rho)
\end{aligned}
$$

- ρ can be interpreted as a personal discount rate (a personal interest rate).
- Consumption grows, if the market interest rate is higher than the private interest rate.
- Movements in r have a higher effect if σ is small.

1g) Solving for c_1

- Optimal allocation is implicitly given by

$$c_1 + a_1 = y_1 + a_0 \tag{1}$$

$$c_2 = y_2 + (1+r)a_1 \tag{2}$$

$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \beta(1+r) \tag{Euler}$$

- Combining (1) and (2) gives

$$\underbrace{c_1 + \frac{c_2}{1+r}}_{i)} = \underbrace{a_0 + y_1 + \frac{y_2}{1+r}}_{ii)} \tag{LB}$$

- i): Present value of lifetime expenditure on consumption goods.
- ii): Present value of lifetime income.

1g) Solving for c_1

- From Euler equation have

$$c_2 = (\beta(1+r))^{\frac{1}{\sigma}} c_1$$

- Combine it with (LB) to get

$$c_1 = \frac{1}{1 + (\beta)^{1/\sigma} [(1+r)]^{1/\sigma - 1}} \left[a_0 + y_1 + \frac{y_2}{1+r} \right]$$

  - Consumption in period 1 depends upon lifetime income.
  - If $\beta(1+r) = 1$, then split consumption equally between periods 1 and 2. Perfect consumption smoothing.
  - σ determines how much differences in subjective and market interest rate affect the allocation of consumption.

1g) Income, substitution and wealth effects

See additional file

1h) Optimal consumption in three cases

- Note that for all three case, lifetime income is given by

$$100 = 100 + \frac{0}{1+0} = 50 + \frac{50}{1+0} = 0 + \frac{100}{1+0}$$

- Additionally, since $\beta(1+r) = 1(1+0) = 1$, it is optimal to have $c_1 = c_2 = 50$.
- This implies the following saving choices
  i) $a_1 = y_1 - c_1 = 100 - 50 = 50$
  ii) $a_1 = y_1 - c_1 = 50 - 50 = 0$
  iii) $a_1 = y_1 - c_1 = 0 - 50 = -50$
- Does this make sense? (Who lends to somebody with 0 income?).

1. Two period model - Problem 1

2. Two period model with borrowing constraint - Problem 2

2a) Budget constraints

- Start with the budget constraints for period 1 and 2 given by

$$\begin{aligned} c_1 + a_1 &= y_1 \\ c_2 &= y_2 + (1+r)a_1 \end{aligned}$$

and combine them to yield

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}.$$

- This is the life-time budget constraint. It states that the present value of life-time consumption expenditure (the stuff on the left-hand side) cannot exceed the present value of life-time income (the stuff on the right hand side.
- For the given income levels, the life-time budget constraint becomes

$$c_1 + \frac{c_2}{1+r} = 4 + \frac{10}{1+r}.$$

2b) Optimal consumption

- The optimal consumption in the first period can be obtained by combining the Euler equation,

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2},$$

and the life-time budget constraint. One obtains

$$c_1 = \frac{1}{1+\beta}\left[4 + \frac{10}{1+r}\right]$$

- Savings in the first period are given by

$$\begin{aligned} s &= y_1 - c_1 \\ &= y_1 - \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \\ &= \frac{1}{(1+\beta)(1+r)}\left(y_1\beta(1+r) - y_2\right). \end{aligned}$$

2b) Optimal consumption

- The consumer will be a borrower if the following condition holds

$$\begin{aligned} s &< 0 \\ \frac{1}{(1+\beta)(1+r)}\left(y_1\beta(1+r) - y_2\right) &< 0 \\ \beta(1+r) &< \frac{y_2}{y_1} \\ \beta(1+r) &< \frac{10}{4}. \end{aligned}$$

2c) Comparison with Keynesian consumption function

- Above we found

$$c_1 = \frac{1}{1+\beta}\left[4 + \frac{10}{1+r}\right]$$

- A typical Keynesian consumption function from undergraduate course looks like

$$c_1 = 0.9y_1$$

- Differences:
  - Where does the 0.9 come form?
  - Different response to changes in first period income.

2c) Comparison with Keynesian consumption function

- At low incomes, people will spend a high proportion of their income. When you have low income, you do not have the luxury of being able to save. You need to spend everything you have on essentials. Consumption depends only on y_1. However, it depends also on life-cycle circles, behavioural factors.
- When you get more income, it is possible to save.
- In Keynesian function: a consumer always spends the income's fraction of 0.9 on consumption.

2d) Marginal propensity to consume

- Suppose that the income in the first period increases by ϵ (while the income in the second period remains constant).
- The marginal propensity to consume in the first-period is given by $\Delta c_0 / \Delta y_0$
- If first-period income increases by one unit, then first-period consumption increases by

$$\frac{1}{1+\beta}$$

  units in the optimization case. Most likely $\beta \in [0.9, 1]$ so that $\frac{1}{1+\beta} \approx 0.5$.
- For the Keynesian consumption function a one unit increase in income leads to a 0.9 unit increase in consumption.

2d) Marginal propensity to consume

- At low incomes, people will spend a high proportion of their income. When you have low income, you do not have the luxury of being able to save. You need to spend everything you have on essentials. Consumption depends only on y_1. However, it depends also on life-cycle circles, behavioural factors. When you get more income, it is possible to save.
- In Keynesian function: a consumer always spends the income's fraction of 0.9 on consumption.

2d) Borrowing constraints

- Suppose that the consumer faces a liquidity constraint, which means that he can save but is unable to borrow in the first-period.
- For $r = 0$ and $\beta = 1$ we see from question b) that the consumer would like to consume

$$c_1 = \frac{1}{1+1}\left[4 + \frac{10}{1}\right] = 7$$

which would entail a borrowing of

$$s_1 = y_1 - c_1 = 4 - 7 = -3.$$

- In the presence of borrowing constraints this is not attainable. The consumer tries to come as close to the optimal level of consumption without borrowing constraints and hence consumes his entire first period income, i.e

$$c_1 = 4, s_1 = 4 - 4 = 0$$

2f) Marginal propensity to consume with borrowing constraints

- Continue to assume that the consumer is liquidity constrained in the first-period.
- Suppose that the income in the first period increases by ϵ (while the income in the second period remains constant).
- The marginal prospensity to consume for consumer faced by borrowing constraint is 1. Why?
  - Without constraint he would like to consume at least 7 units in period 1. But actually he can only consume 4 units.
  - If he now has one more unit available, he wants to get closer to the 7 units and consumes all additional income in the first period.
- Behaves more like a Keynesian consumer.
- Importance for policy?