Dynamic Macroeconomics
Problem Set 1

Universität Siegen

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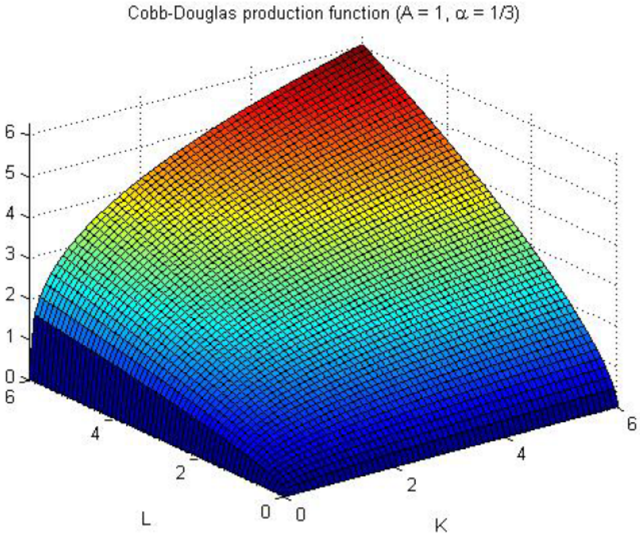
Cobb-Douglas Production Function



a) Properties i and ii

- Consider the Cobb-Douglas production function

$$Y = F(K, L) = \bar{A}K^{\alpha}L^{1-\alpha}$$

  where K is the capital stock and L the labor force.

- The Cobb-Douglas function is homogeneous of degree one, since

$$\begin{aligned} F(\lambda K, \lambda L) &= \bar{A}(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} \\ &= \bar{A}\lambda^{\alpha+1-\alpha}K^{\alpha}L^{1-\alpha} \\ &= \lambda F(K, L) \end{aligned}$$

- Both factors of production are necessary, since

$$F(0, L) = \bar{A} \times 0^{\alpha}L^{1-\alpha} = 0 = F(K, 0) = \bar{A} \times K^{\alpha}0^{1-\alpha}.$$

a) Properties iii and iv

- The marginal product of capital is

$$\frac{\partial F(K,L)}{\partial K} = \alpha \bar{A} K^{\alpha-1} L^{1-\alpha} = \underbrace{\alpha}_{>0} \underbrace{\bar{A}\left(\frac{L}{K}\right)^{1-\alpha}}_{>0} > 0$$

- For which

$$\lim_{K\to 0} F_K(K,L) = \lim_{K\to 0} \alpha \bar{A}\left(\frac{L}{K}\right)^{1-\alpha} = \alpha \bar{A} L^{1-\alpha} \lim_{K\to 0}\left(\frac{1}{K}\right)^{1-\alpha} = \infty$$

$$\lim_{K\to \infty} F_K(K,L) = \lim_{K\to \infty} \alpha \bar{A}\left(\frac{L}{K}\right)^{1-\alpha} = \alpha \bar{A} L^{1-\alpha} \lim_{K\to \infty}\left(\frac{1}{K}\right)^{1-\alpha} = 0$$

- For L the derivation is similar.

b) Elasticity of Y w.r.t. K

- The elasticity of Y w.r.t. K is given by

$$\begin{aligned}
\epsilon_{Y,K} &= \frac{\partial F(K,L)}{\partial K}\frac{K}{Y} \\
&= \alpha \bar{A} K^{\alpha-1} L^{1-\alpha} \frac{K}{Y} \\
&= \alpha \bar{A} K^{\alpha} L^{1-\alpha} \frac{1}{Y} \\
&= \alpha
\end{aligned}$$

- If K increase by one percentage point, Y will increase by α percentage points.
- The elasticity of Y with respect to L is $1-\alpha$.

c) and d) Wages and rental rates

- On competitive markets, factors of production are paid their marginal product.
- Therefore the wage is $w = F_L$ and the share of wage income in total income is

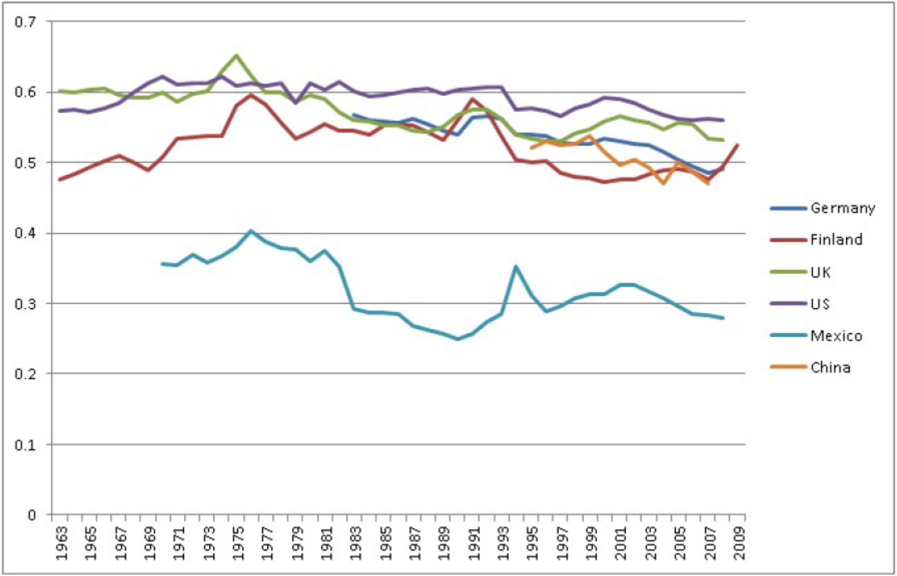
$$\frac{wL}{Y} = \frac{F_L L}{Y} = \frac{(1-\alpha)\bar{A}K^{\alpha}L^{-\alpha}L}{\bar{A}K^{\alpha}L^{1-\alpha}} = \frac{(1-\alpha)F(K,L)}{F(K,L)} = 1-\alpha$$

- Therefore the rental rate of capital is $r = F_K$ and the share of capital income in total income is

$$\frac{rK}{Y} = \frac{F_K K}{Y} = \frac{\alpha\bar{A}K^{\alpha-1}L^{1-\alpha}K}{\bar{A}K^{\alpha}L^{1-\alpha}} = \frac{\alpha F(K,L)}{F(K,L)} = \alpha$$

- For Cobb-Douglas production function, the income share going to the different factors of production is constant.

Wage shares in some countries

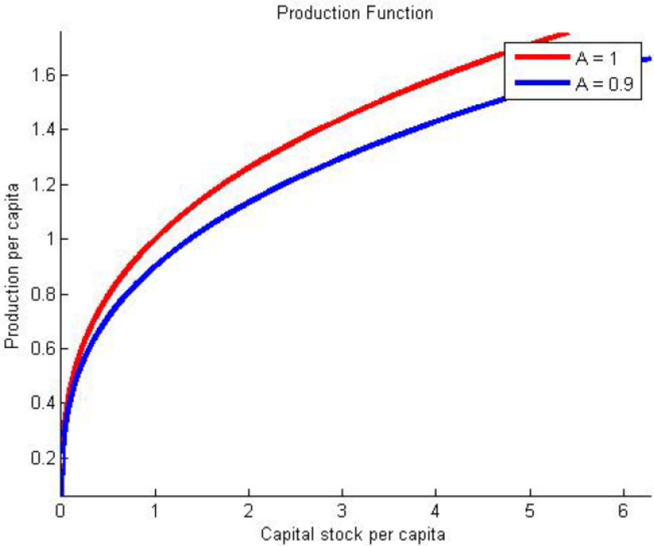


e) Per-capita production function

- To derive the production function per capita, start with the production function for total output and divide by the labor force:

$$
\begin{aligned}
Y &= \bar{A}K^{\alpha}L^{1-\alpha} && |:L \\
\frac{Y}{L} &= \bar{A}K^{\alpha}\frac{L^{1-\alpha}}{L} && |y \equiv \frac{Y}{L} \\
y &= \bar{A}K^{\alpha}L^{-\alpha} && \\
y &= \bar{A}\left(\frac{K}{L}\right)^{\alpha} && |k \equiv \frac{K}{L} \\
y &= \bar{A}k^{\alpha} &&
\end{aligned}
$$

f) Interpreting $\bar{A}$



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a) Firms optimization problem

- The firms optimization problem is

$$\max_{K,L} \bar{A}K^{\alpha}L^{1-\alpha} - wL - rK \tag{1}$$

taking the factor prices w and r as given.

- The first order conditions for K and L are

$$\alpha\bar{A}K^{\alpha-1}L^{1-\alpha} = r$$
$$(1-\alpha)\bar{A}K^{\alpha}L^{-\alpha} = w$$

- Interpretation?

a) Firms optimization problem ii

- How does demand for L change, if w changes?
- Start with F.O.C. for general function $F(K, L)$ and use the implicit function theorem

$$F_L(K, L) = w$$
$$F_L(K, h(L)) = w$$

- Take derivative w.r.t. wage (using the chain rule) gives

$$F_{L,L}(K, h(L)) * h'(L) = 1$$
$$h'(L) = \frac{1}{F_{L,L}}$$

- Sign?

a) Firms optimization problem iii

- What determines the trade-off between K and L?
- From F.O.C get

$$\frac{\alpha}{1-\alpha}\frac{L}{K} = \frac{r}{w}$$

- First assume $\alpha = 0.5$ and $r = w$. Then $\frac{L}{K} = 1$. Interpretation?
- Now assume $\alpha = 0.5$ but $r > w$. Then $\frac{r}{w} > 1$ and thus $\frac{L}{K} > 1$. Interpretation?
- Now assume $r = w$ but $\alpha < 0.5$. Then

$$\frac{\alpha}{1-\alpha}\frac{L}{K} = 1$$

$$\frac{L}{K} = \underbrace{\frac{1-\alpha}{\alpha}}_{>1} 1 > 1$$

  Interpretation?
- (General tip: Look at the extreme case to make sense of the economic mechanisms at hand)

b) Endogenous and exogenous variables

Table: Endogenous and Exogenous Variables

Symbol	Name	Type
$L^s = \bar{L}$	Supply of Labor by Households	Exogenous
$K^s = \bar{K}$	Supply of Capital by Households	Exogenous
$\bar{A}$	Productivity parameter	Exogenous
α	Production function parameter	Exogenous
L^d	Demand of Labor by firms	Endogenous
K^d	Demand of Capital by firms	Endogenous
L^*	Equilibrium Level of Labor	Endogenous
K^*	Equilibrium Level of Capital	Endogenous
Y^*	Equilibrium Level of Production	Endogenous

c) General Equilibrium I

- Market clearing condition for Labor

$$L^s = L^d$$

- Market clearing condition for Capital

$$K^s = K^d$$

- Firm demand for capital and labor from first order conditions.
- Solution for labor

$$L^* = \bar{L}$$

- Solution for capital

$$K^* = \bar{K}$$

- Solution for production

$$Y^* = \bar{A}\bar{K}^{\alpha}\bar{L}^{1-\alpha}$$

d) Observed and predicted y

Table: Model prediction for GDP per capita relative to USA

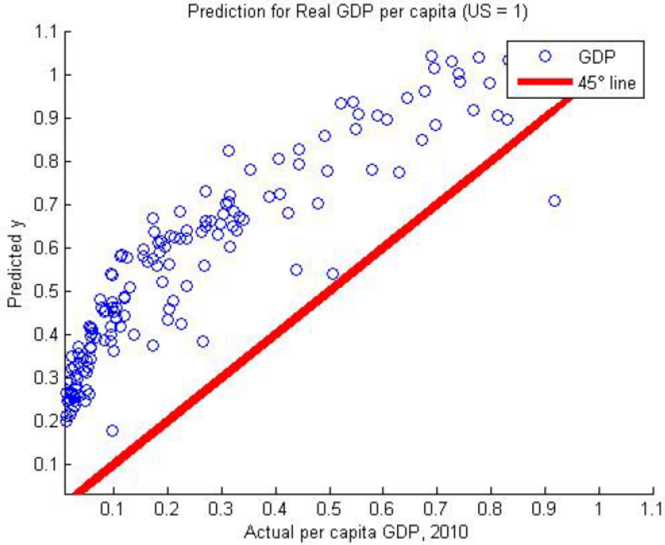
Country	Observed k	Observed y	Predicted y^*
United States	1	1	1
Japan	1.0949	0.7276	1.0307
Italy	1.0385	0.6949	1.0127
Brazil	0.2420	0.2129	0.6232
China	0.2299	0.1816	0.6126
India	0.0057	0.0812	0.1786
South Africa	0.1410	0.1907	0.5205

e) Implied $\bar{A}$

Table: Model prediction for $\bar{A}$ relative to USA

Country	Observed k	Observed y	Implied A
United States	1	1	1
Japan	1.0949	0.7276	0.7059
Italy	1.0385	0.6949	0.6171
Brazil	0.2420	0.2129	0.3402
China	0.2299	0.1816	0.2964
India	0.0057	0.0812	0.4546
South Africa	0.1410	0.1907	0.3664

f) Results from MatLab i)



f) Results from MatLab ii)

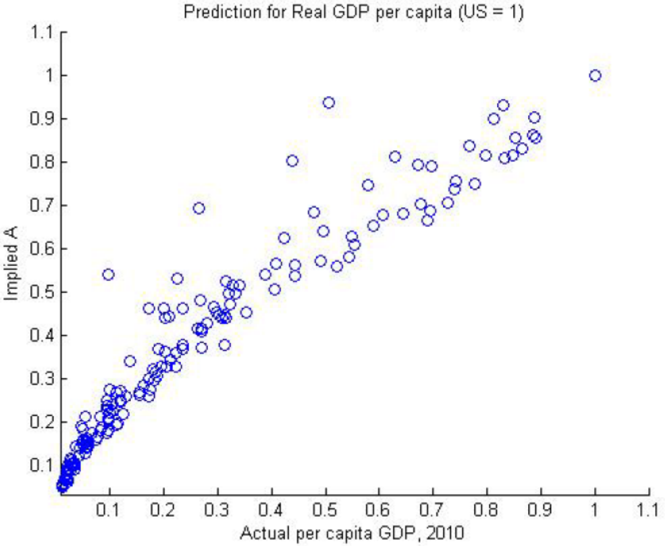


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Table 1 of Hall/Jones (1999)

TABLE I

PRODUCTIVITY CALCULATIONS: RATIOS TO U. S. VALUES

Country	Y/L	Contribution from $(K/Y)^{\alpha/(1-\alpha)}$	H/L	A
United States	1.000	1.000	1.000	1.000
Canada	0.941	1.002	0.908	1.034
Italy	0.834	1.063	0.650	1.207
West Germany	0.818	1.118	0.802	0.912
France	0.818	1.091	0.666	1.126
United Kingdom	0.727	0.891	0.808	1.011
Hong Kong	0.608	0.741	0.735	1.115
Singapore	0.606	1.031	0.545	1.078
Japan	0.587	1.119	0.797	0.658
Mexico	0.433	0.868	0.538	0.926
Argentina	0.418	0.953	0.676	0.648
U.S.S.R.	0.417	1.231	0.724	0.468
India	0.086	0.709	0.454	0.267
China	0.060	0.891	0.632	0.106
Kenya	0.056	0.747	0.457	0.165
Zaire	0.033	0.499	0.408	0.160
Average, 127 countries:	0.296	0.853	0.565	0.516
Standard deviation:	0.268	0.234	0.168	0.325
Correlation with Y/L (logs)	1.000	0.624	0.798	0.889
Correlation with A (logs)	0.889	0.248	0.522	1.000

The elements of this table are the empirical counterparts to the components of equation (3), all measured