Dynamic Macroeconomics

Problem Set 1

1. **Cobb-Douglas production function** Consider the Cobb-Douglas production function

$$Y = F(K, L) = \bar{A}K^{\alpha}L^{1-\alpha},$$

where $\bar{A}$ and α are parameters, K denotes capital and L denotes labor.

   a) Show that the production function satisfies the following properties
      - i F is homogeneous of degree 1. (A function is homogeneous of degree 1, if $F(\lambda K, \lambda L) = \lambda F(K, L)$ holds for all $\lambda > 0$).
      - ii Both factors of production are necessary to produce positive output.
      - iii Both factors of production have a positive but diminishing marginal product.
      - iv The Inada conditions are satisfied:

$$\lim_{K\to 0} \frac{\partial F(K,L)}{\partial K} = \infty, \quad \lim_{L\to 0} \frac{\partial F(K,L)}{\partial L} = \infty$$
$$\lim_{K\to \infty} \frac{\partial F(K,L)}{\partial K} = 0, \quad \lim_{L\to \infty} \frac{\partial F(K,L)}{\partial L} = 0$$

   b) Show that α is the elasticity of output Y with respect to K. (Hint: For a function $y = f(x)$ the elasticity is computed as $f'(x)\frac{x}{y}$).
   c) Find an expression for the wage w and the rental rate on capital r. (Remember: On competitive markets, factors of production are rewarded according to their marginal product).
   d) Calculate the share of wage income (and capital income) of output.
   e) Define $y = Y/L$. Show that you can transform the production function into $y \equiv f(k) = \bar{A}k^{\alpha}$, where $k = K/L$.
   f) Provide an interpretation of $\bar{A}$.

2. **A Simple Macroeconomic Model** Consider the following simple macroeconomic model to explain differences in GDP per captia across countries. Households supply $\bar{L}$ units of labor and $\bar{K}$ units of capital on competitive factor markets. Firms produce final output using a Cobb-Douglas production function of the type

$$Y = \bar{A}K^{1/3}L^{2/3} \tag{1}$$

and try to maximize their profits, given by

$$\Pi = Y - rK - wL \tag{2}$$

by choosing their demand for capital K and labor L.

   a) Derive the first-order conditions for K and L for the firms optimization problem. Provide an economic interpretation of them.

In analyzing a model it is useful to distinguish between endogenous variables (i.e. variables we like to explain within the model) and exogenous variables (i.e. variables that are explained outside the model).

   b) Classify the variables into the endogenous and exogenous.

The goal of analyzing a model is to find the solution for the endogenous variables. Most of the time this is done by looking at the General Equilibrium of the economy. Mathematically this amounts to listing all the equations describing the behavior of agents in the economy and all market clearing conditions and then manipulating the equations to obtain expressions for the endogenous variables.

   c) List the equations describing the General Equilibrium of the economy and solve them for the endogenous variables. Also show that the models predicts that output per capita is given by $y^* = \bar{A}\bar{k}^{1/3}$.

To evaluate the models performance, we can compare the models predictions with empirical observations. Our model makes predictions of the relationship between output per capita and the captial stock per capita.

d) Fill in the missing values in the following table, assuming $\bar{A} = 1$ for all countries. Discuss how good/bad the our model is at explaining the observed data on capital and output.

Table 1: Model prediction for GDP per capita relative to USA

Country	Observed k	Observed y	Predicted y
United States	1	1	1
Japan	1.0949	0.7276	
Italy	1.0385	0.6949	
Brazil	0.2420	0.2129	
China	0.2299	0.1816	
India	0.0057	0.0812	
South Africa	0.1410	0.1907	

Now we drop the assumption that all countries share the same level of $\bar{A}$. Note that we can express $\bar{A}$ as

$$\bar{A} = \frac{y^*}{\bar{k}^{1/3}}. \tag{3}$$

e) Fill in the missing values in the following table and interpret your results.

Table 2: Model prediction for $\bar{A}$ relative to USA

Country	Observed k	Observed y	Implied $\bar{A}$
United States	1	1	1
Japan	1.0949	0.7276	
Italy	1.0385	0.6949	
Brazil	0.2420	0.2129	
China	0.2299	0.1816	
India	0.0057	0.0812	
South Africa	0.1410	0.1907	

On the course web-site you can find a MatLab data-set and program that applies the ideas of this exercise to a greater sample of countries.

f) Read the program code and discuss what it does. Then run it and interpret the output.

3. **Reading Exercise** Read pages 83-97 of the article "Why do some countries produce so much more output per worker than others?" by Robert Hall and Charles Jones, which you can find at http://web.stanford.edu/~chadj/HallJonesQJE.pdf