Dynamic Macroeconomics

Chapter 11: Asset pricing and macroeconomics

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Overview

1. Expected utility and risk
2. Market efficiency
3. Asset pricing and contingent claims
4. General equilibrium asset pricing

Expected utility and risk

- Assume an investor is offered a "lottery" with two possible pay-offs x_1 or x_2 which occur with probability π and $1-\pi$, respectively.
- If we denote the (random) outcome of the lottery by W and the expected outcome by $E(W)$ then we have:

$$\pi x_1 + (1-\pi) x_2 = E(W) \tag{1}$$

- The investor is said to be

$$\left.\begin{array}{c}\text{risk-averse}\\ \text{risk-neutral}\\ \text{risk-loving}\end{array}\right\} \text{ if } \left\{\begin{array}{l} U[E(W)] > E[U(W)]\\ U[E(W)] = E[U(W)]\\ U[E(W)] < E[U(W)]\end{array}\right. \tag{2}$$

with $U[E(W)] = U[\pi x_1 + (1-\pi) x_2]$ and
$E[U(W)] = \pi_1 U(x_1) + (1-\pi) U(x_2)$.

$\Longrightarrow$ Interpretation?

Expected utility and risk

- Consider the expression for the expected utility from playing the lottery:

$$E\left[U\left(W\right)\right] \tag{3}$$

- Taking a second-order Taylor approximation of this expression around $W = E\left(W\right)$ yields:

$$\begin{aligned} E\left[U\left(W\right)\right] &\approx E\left[U\left(E\left(W\right)\right)\right] + E\left\{\frac{\partial\left[U\left(E\left(W\right)\right)\right]}{\partial W}\left(W - E\left(W\right)\right)\right\} + \\ &+E\left\{\frac{\partial^2\left[U\left(E\left(W\right)\right)\right]}{\partial W^2}\left(\frac{1}{2}\right)\left(W - E\left(W\right)\right)^2\right\} = \\ &= U\left(E\left(W\right)\right) + \frac{1}{2}E\left(W - E\left(W\right)\right)^2 U''\left(E\left(W\right)\right) \end{aligned}$$

Expected utility and risk

- The just derived equation shows that (Jensen's inequality):

$$\left.\begin{array}{l} U\left[E\left(W\right)\right] > E\left[U\left(W\right)\right] \\ U\left[E\left(W\right)\right] = E\left[U\left(W\right)\right] \\ U\left[E\left(W\right)\right] < E\left[U\left(W\right)\right] \end{array}\right\} \Longleftrightarrow \left\{ U''(.) \begin{array}{c} < \\ = \\ > \end{array} \right\} 0 \tag{4}$$

$\Longrightarrow$ Interpretation?

- We now consider a case in which the investor can buy one of two assets. These two assets have the following characteristics:
  - **Asset 1**: The return of the asset (denoted by r^f) is certain.

    $\Longrightarrow$ Risk-free asset
  - **Asset 2**: The return of the asset (denoted by r) is subject to uncertainty. The expected rate of return is denoted by $E\left(r\right)$, the degree of uncertainty is captured by the variance of r, denoted by $V\left(r\right)$.

    $\Longrightarrow$ Risky asset

Expected utility and risk

- Assuming that an investor has initial wealth W_0, the expected return on the safe asset (denoted by $E\left(W^f\right)$) is given by:

$$E\left(W^f\right) = E\left[\left(1 + r^f\right) W_0\right] = \left(1 + r^f\right) W_0 = W^f \tag{5}$$

- The expected return on the risky asset (denoted by $E\left(W^r\right)$) is given by:

$$E\left(W^r\right) = E\left[\left(1 + r\right) W_0\right] = \left(1 + E\left(r\right)\right) W_0 \tag{6}$$

$\Longrightarrow$ Assuming $E\left(r\right) = r^f$, which asset should the investor buy?

- Answer: The investor should buy the asset which yields the higher expected utility.

Expected utility and risk

- The expected utility from investing in the safe asset (denoted by $E\left[U\left(W^f\right)\right]$) is given by:

$$E\left[U\left(W^f\right)\right] = U\left(W^f\right) = U\left[E\left(W^f\right)\right] \tag{7}$$

- To evaluate the expected utility from investing in the risky asset (denoted by $E\left[U\left(W^r\right)\right]$) we compute a second-order Taylor approximation of this expression around $r = r^f$. This yields (Remember: $W^f = \left(1 + r^f\right) W_0$):

$$\begin{aligned} E\left[U\left(W^r\right)\right] \;\approx\; & E\left[U\left(W^f\right)\right] + E\left\{\frac{\partial\left[U\left(W^f\right)\right]}{\partial W}\left(\frac{\partial W^f}{\partial r}\right)_{r=r^f}\left(r - r^f\right)\right\} \\ & + E\left\{\frac{\partial^2\left[U\left(W^f\right)\right]}{\partial W^2}\left(W_0^2\right)\left(\frac{1}{2}\right)\left(r - r^f\right)^2\right\} \end{aligned}$$

Expected utility and risk

- The latter expression can be simplified to:

$$
\begin{aligned}
E\left[U\left(W^{r}\right)\right] &\approx E\left[U\left(W^{f}\right)\right]+\left(\frac{1}{2}\right) W_{0}^{2} U''\left(W^{f}\right) E\left\{\left(r-r^{f}\right)^{2}\right\} \\
&= E\left[U\left(W^{f}\right)\right]+\left(\frac{1}{2}\right)\left(W_{0}^{2}\right) U''\left(W^{f}\right) V\left(r\right)
\end{aligned}
$$

- If the investor is risk-averse, we thus have:

$$
E\left[U\left(W^{r}\right)\right] < E\left[U\left(W^{f}\right)\right] \tag{8}
$$

$\Longrightarrow$ Interpretation?

Expected utility and risk

- The equation

$$E\left[U\left(W^{r}\right)\right] < E\left[U\left(W^{f}\right)\right] \tag{9}$$

  shows that a risk-averse investor will prefer to hold a risk-free asset relative to a risky asset given that the expected returns of the two assets are equal.
- We now want to examine how much compensation (in terms of additional return, ρ) the investor must be offered to be willing to hold the risky asset.
- In other words, we want to examine how large ρ in the following equation must be:

$$E\left[U\left(W^{r}\right)\right] = E\left[U\left(\left(1 + r + \rho\right) W_0\right)\right] = E\left[U\left(W^{f}\right)\right] \tag{10}$$

  (Please note that we now assume that the expected return of the risky asset is: $E\left(r\right) = r^{f} + \rho$.)

Expected utility and risk

- A second-order Taylor approximation of the term $E\left[U\left(W^r\right)\right] = E\left[U\left(\left(1+r\right)W_0\right)\right]$ around $r = r^f + \rho$ yields:

$$E\left[U\left(\left(1+r\right)W_0\right)\right] \approx E\left[U\left(\left(1+r^f+\rho\right)W_0\right)\right] + \\ + \left(\frac{1}{2}\right)\left(W_0^2\right)U''\left(.\right)E\left\{\left(r-r^f-\rho\right)^2\right\}$$

- A first-order Taylor approximation of $E\left[U\left(\left(1+r^f+\rho\right)W_0\right)\right]$ around $\rho = 0$ yields:

$$E\left[U\left(\left(1+r^r+\rho\right)W_0\right)\right] \approx E\left[U\left(\left(1+r^f\right)W_0\right)\right] + \\ +U'\left(.\right)W_0\left(\rho-0\right) \tag{11}$$

Expected utility and risk

- For $E\left[U\left(W^r\right)\right]$ we then obtain:

$$\begin{aligned} E\left[U\left(W^r\right)\right] &\approx E\left[U\left(W^f\right)\right] + U'(.)\, W_0 \rho + \\ &+ \left(\frac{1}{2}\right)\left(W_0^2\right) U''(.)\, E\left\{\left(r - r^f - \rho\right)^2\right\} \end{aligned}$$

- Then $E\left[U\left(W^r\right)\right] = E\left[U\left(W^f\right)\right]$ if:

$$\rho = -\frac{W_0 U''}{U'} \frac{V(r)}{2} \tag{12}$$

$\Longrightarrow$ Interpretation?

Market efficiency

- **Definition of market efficiency (Wickens, 2011):**

  *A market is said to be **efficient** if there are no unexploited arbitrage opportunities.*

- An arbitrage portfolio is a self-financing portfolio with a zero or negative cost that has a positive payoff.

  $\Longrightarrow$ If unexploited arbitrage opportunities exist the investor *gets something for nothing*.

- Implication: For any risky asset i with return $r_{i,t+1}$ the absence of arbitrage opportunities implies:

$$E_t r_{i,t+1} = r_t^f + \rho_{i,t} \tag{13}$$

  where $\rho_{i,t}$ denotes the risk premium of asset i.

Asset pricing and contingent claims

- Basic principle in asset pricing: The value of any investment is found by computing the value today (present value) of all cash flows the investment will generate over its lifetime.
- Problem: Future payoffs depend on unknown future economic conditions.

  $\Longrightarrow$ There is a high degree of uncertainty about future payoffs.

  $\Longrightarrow$ To price assets: Modeling of uncertainty necessary.
- Approach to model uncertainty:
  - We assume that in each period one out of $s = 1, 2, \ldots, S$ possible states of nature can occur.
  - The probability that state s occurs is denoted by $\pi(s)$. Since exactly one state of nature s occurs in each period we have:

$$\sum_{s=1}^{S} \pi(s) = 1 \tag{14}$$

Asset pricing and contingent claims

- **Specification of the asset market:**
  - There are $s = 1, 2, \ldots, S$ different types of assets.
  - An asset of type s pays off one euro if state s occurs and zero otherwise.

    $\Longrightarrow$ State-contingent claim.

    $\Longrightarrow$ If one state-contingent claim exists for each possible state of nature we have **complete markets**.
  - The price of state-contingent claim s is denoted by $q(s)$.
  - The vector $q = \left[q(1) \;\; q(2) \;\; \ldots \;\; q(S)\right]'$ is denoted as state-price vector.

Asset pricing and contingent claims

- **Pricing of non-state contingent assets:**
  - Assume we have an asset that pays off $x(s)$ euros in state s (for $s = 1, 2, \ldots, S$).
  - Note that a portfolio which contains $x(s)$ units of the state-contingent claim s has the same pay-off as the asset that provides a pay-off $x(s)$ in each state of nature.

    $\Longrightarrow$ The prices of the asset and the portfolio must be equal.
  - Assuming that the prices of the state-contingent claims are given we therefore have that the price of the asset, denoted by p, must satisfy:

$$p = \sum_{s=1}^{S} q(s)\, x(s) \tag{15}$$

Asset pricing and contingent claims

- **Pricing of non-state contingent assets (continued):**
  - The "pricing formula" of the last page can be reformulated as follows:

$$\begin{aligned} p &= \sum_{s=1}^{S} q(s)\, x(s) = \sum_{s=1}^{S} \pi(s) \frac{q(s)}{\pi(s)} x(s) \qquad (16) \\ &= \sum_{s=1}^{S} \pi(s)\, m(s)\, x(s) = E(mx) \end{aligned}$$

$\Longrightarrow$ $m(s)$: Stochastic discount factor of 1 euro in state s.

$\Longrightarrow$ Interpretation?

- Now: Express this equation in terms of returns instead of the asset price ($\rightarrow$ stochastic discount factor representation of returns)

Asset pricing and contingent claims

- **Deriving an expression for the risk premium:**
  - Dividing equation (15) by p and defining $1+r(s)=x(s)/p$ yields:

$$p=\sum_{s=1}^{S} q(s)\,x(s) \Longleftrightarrow 1 = \sum_{s=1}^{S} q(s)\,\frac{x(s)}{p}=\sum_{s=1}^{S} q(s)\,(1+r(s))$$
$$= \sum_{s=1}^{S} \pi(s)\,\frac{q(s)}{\pi(s)}\,(1+r(s)) =$$
$$= E\left[m(1+r)\right] \quad (17)$$

  - For the **risk** $-$ **free** asset (with a rate of return of r^f) we obtain (since $1+r^f=x/p^f$):

$$1=\sum_{s=1}^{S} \pi(s)\,m(s)\left(1+r^f\right)=E(m)\left(1+r^f\right) \Longleftrightarrow \quad (18)$$
$$\Longleftrightarrow 1=\frac{x}{p^f}E(m) \Longleftrightarrow p^f=xE(m)$$

Asset pricing and contingent claims

- **Deriving an expression for the risk premium (continued):**
  - The expectational term $E\left[m\left(1+r\right)\right]$ (equation (17) can be written as follows:

$$E\left[m\left(1+r\right)\right]=E\left(m\right)E\left(1+r\right)+\text{Cov}\left(m,1+r\right) \tag{19}$$

(Remember: $\text{Cov}\left(X,Y\right)=E\left(XY\right)-E\left(X\right)E\left(Y\right)$)

  - Then equation (17) can be written as:

$$1=E\left[m\left(1+r\right)\right]=E\left(m\right)E\left(1+r\right)+\text{Cov}\left(m,1+r\right)\Longleftrightarrow \tag{20}$$

$$E\left(1+r\right)=\frac{1}{E\left(m\right)}-\frac{\text{Cov}\left(m,1+r\right)}{E\left(m\right)}$$

  - Using the expression for $E\left(m\right)$ derived for the risk-free asset ($E\left(m\right)=\frac{1}{\left(1+r^f\right)}$) we can write:

$$E\left(1+r\right)=\frac{1}{\frac{1}{\left(1+r^f\right)}}-\frac{\text{Cov}\left(m,1+r\right)}{\frac{1}{\left(1+r^f\right)}} \tag{21}$$

Asset pricing and contingent claims

- **Deriving an expression for the risk premium (continued):**
  - The just derived equation can be rearranged as follows:

$$E\left(1+r\right)=\left(1+r^{f}\right)-\left(1+r^{f}\right)\text{Cov}\left(m,1+r\right)\Longleftrightarrow \qquad (22)$$

$$\Longleftrightarrow E\left(r\right)=r^{f}-\left(1+r^{f}\right)\text{Cov}\left(m,1+r\right)$$

$\Longrightarrow$ Interpretation?

  - From the equation

$$E_t r_{i,t+1}=r_t^{f}+\rho_{i,t} \qquad (23)$$

we see that the risk premium for the risky asset is given by:

$$\rho=-\left(1+r^{f}\right)\text{Cov}\left(m,1+r\right) \qquad (24)$$

- $\rho>0$ when $\text{Cov}(m,1+r)=\text{Cov}(m,r)<0 \Longrightarrow$ Interpretation?

General equilibrium asset pricing

- Basic idea: Prices of financial assets and real variables are determined jointly.
- Model setup:
  - Two-period model.
  - Economy is inhabited by one representative household.
  - Period t's income (denoted by y) is certain, period $t+1$'s income is uncertain.
  - Period $t+1$'s income depends on the state of nature in that period (denoted by s) and is given by: $y(s)$.
  - Future income/consumption is discounted at rate $q(s)$.
  - The household's lifetime utility function is given by:

$$V = U(c_t) + \beta E_t U(c_{t+1}) \tag{25}$$

General equilibrium asset pricing

- Optimization problem of the household:
  - The household maximizes expected lifetime utility, i.e., the objective function of the household is given by:

$$\max_{c, c(s)_{s=1}^{S}} V = U(c_t) + \beta E_t U(c_{t+1}) \equiv U(c) + \beta \sum_{s=1}^{S} \pi(s) U(c(s)) \tag{26}$$

    where S denotes the number of possible states of nature in period $t+1$, $\pi(s)$ denotes the probability that state s will occur and the period utility function $U(.)$ is strictly concave in c.
  - The household's intertemporal budget constraint is given by:

$$c + \sum_{s=1}^{S} q(s) c(s) = y + \sum_{s=1}^{S} q(s) y(s), \tag{27}$$

    where $q(s)$ denotes the state price for contingent claims that are used to value future consumption and income in state s.

General equilibrium asset pricing

- Optimization problem of the household (continued):
  - The Lagrangian of the household is given by:

$$\begin{aligned}\mathcal{L} &= U(c) + \beta \sum_{s=1}^{S} \pi(s) U[c(s)] + \\ &+ \lambda \left[y + \sum_{s=1}^{S} q(s) y(s) - c - \sum_{s=1}^{S} q(s) c(s) \right]\end{aligned} \tag{28}$$

- Model solution:
  - The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial c} = U'(c) - \lambda \overset{!}{=} 0 \Longleftrightarrow U'(c) = \lambda \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial c(s)} = \beta \pi(s) U'(c(s)) - \lambda q(s) \overset{!}{=} 0 \Longleftrightarrow \beta \frac{\pi(s)}{q(s)} U'(c(s)) = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y + \sum_{s=1}^{S} q(s) y(s) - c - \sum_{s=1}^{S} q(s) c(s) \overset{!}{=} 0$$

General equilibrium asset pricing

- Model solution (continued):
  - Combining the first two first-order conditions yields:

$$U'(c) = \beta \frac{\pi(s)}{q(s)} U'(c(s)) \Longleftrightarrow q(s) = \beta \pi(s) \frac{U'(c(s))}{U'(c)} \tag{30}$$

  - Above, we defined the stochastic discount factor $m(s)$ as:

$$m(s) = \frac{q(s)}{\pi(s)} \Longleftrightarrow q(s) = m(s)\,\pi(s) \tag{31}$$

  - Using this expression and the result for the combined two first-order conditions we obtain:

$$q(s) = \beta \pi(s) \frac{U'(c(s))}{U'(c)} \Longleftrightarrow m(s) = \beta \frac{U'(c(s))}{U'(c)} \tag{32}$$

$\Longrightarrow$ Interpretation?

General equilibrium asset pricing

- Implications:
  - Above, we showed that the price of an asset which has the payoff $x(s)$ in state s (with $s = 1, 2, \ldots, S$) is given by:
$$p = \sum_{s=1}^{S} \pi(s)\, m(s)\, x(s) \tag{33}$$
  - The price of tomorrow's output $y(s)$ is thus given by (using the result for the stochastic discount factor derived above):
$$p = \sum_{s=1}^{S} \pi(s)\, m(s)\, y(s) = \sum_{s=1}^{S} \pi(s) \frac{\beta U'(c(s))}{U'(c)} y(s) \tag{34}$$
  - Dividing by p (and remembering that $y(s)/p = 1 + r(s)$ we obtain:

$$1 = \sum_{s=1}^{S} \pi(s) \frac{\beta U'(c(s))}{U'(c)} [1 + r(s)] \iff$$

$$1 = E_t \left\{ \frac{\beta U'(c_{t+1})}{U'(c_t)} [1 + r_{t+1}] \right\} \iff U'(c_t) = \beta E_t \left\{ U'(c_{t+1}) [1 + r_{t+1}] \right\}$$

$\Longrightarrow$ Interpretation?

General equilibrium asset pricing

- Implications (continued):
  - Since

$$E_t\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right][1+r_{t+1}]\right\} = E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\}E_t\{[1+r_{t+1}]\} + \\ +\text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right],[1+r_{t+1}]\right\}$$

we can write

$$1 = E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}[1+r_{t+1}]\right\}$$

$$= E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\}E_t\{[1+r_{t+1}]\} + \text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right],[1+r_{t+1}]\right\}$$

General equilibrium asset pricing

- Implications (continued):
  - If future return was certain ($r_{t+1} = r_t^f$) we would have that

$$\text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right]\left[1+r_t^f\right]\right\} = 0 \tag{35}$$

and therefore

$$\begin{aligned} 1 &= E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\} E_t\left\{\left[1+r_t^f\right]\right\} \\ \iff \frac{1}{\left[1+r_t^f\right]} &= E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\} \end{aligned} \tag{36}$$

General equilibrium asset pricing

- Implications (continued):
  - Combining the expression for the risk-free asset and the risky asset we obtain:

$$1 = E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\} E_t\{[1+r_{t+1}]\} + \text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right], [1+r_{t+1}]\right\}$$

$$E_t\left\{\frac{\beta U'(c_{t+1})}{U'(c_t)}\right\} E_t\{[1+r_{t+1}]\} = 1 - \text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right], [1+r_{t+1}]\right\}$$

$$\frac{1}{\left[1+r_t^f\right]} E_t\{[1+r_{t+1}]\} = 1 - \text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right], [1+r_{t+1}]\right\}$$

$$E_t\{[1+r_{t+1}]\} = \left[1+r_t^f\right] - \left[1+r_t^f\right]\text{Cov}\left\{\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}\right], [1+r_{t+1}]\right\}$$

$\Longrightarrow$ Consumption-based capital asset-pricing model (C-CAPM)

$\Longrightarrow$ Interpretation?

General equilibrium asset pricing

- Implications (continued):
  - Taking a Taylor approximation of the marginal utility in period $t+1$ around $c_{t+1} = c_t$ we obtain:

$$U'(c_{t+1}) \approx U'(c_t) + U''(c_t)(c_{t+1} - c_t) = U'(c_t) + U''(c_t)\,\Delta c_{t+1}$$

  - For the term $\frac{\beta U'(c_{t+1})}{U'(c_t)}$ we then obtain:

$$\begin{aligned}\frac{\beta U'(c_{t+1})}{U'(c_t)} = \beta \frac{U'(c_t) + U''(c_t)\,\Delta c_{t+1}}{U'(c_t)} &= \beta\left[1 + \frac{U''(c_t)\,\Delta c_{t+1}}{U'(c_t)}\right] \\ &= \beta\left[1 + \frac{c_t U''(c_t)}{U'(c_t)}\frac{\Delta c_{t+1}}{c_t}\right] = \beta\left[1 - \sigma_t \frac{\Delta c_{t+1}}{c_t}\right]\end{aligned}$$

$\Longrightarrow$ Interpretation of σ_t?

General equilibrium asset pricing

- Implications (continued):
  - Then the expression for the expected rate of return from the previous slide can be written as:

$$
\begin{aligned}
E_t\left[1+r_{t+1}\right] &= \left[1+r_t^f\right] - \left[1+r_t^f\right]\mathrm{Cov}\left\{\left[\frac{\beta U'\left(c_{t+1}\right)}{U'\left(c_t\right)}\right],\left[1+r_{t+1}\right]\right\} \\
E_t\left[r_{t+1}\right] &= r_t^f - \left[1+r_t^f\right]\mathrm{Cov}\left\{\beta\left[1-\sigma_t\frac{\Delta c_{t+1}}{c_t}\right],\left[1+r_{t+1}\right]\right\} \\
&= r_t^f + \left[1+r_t^f\right]\beta\sigma_t\mathrm{Cov}\left\{\left[\frac{\Delta c_{t+1}}{c_t}\right],r_{t+1}\right\}
\end{aligned}
$$

$\Longrightarrow$ Interpretation?

(Note: If a is a scalar and X and Y are random variables we have

$$\mathrm{Cov}\left(a+X,Y\right)=E\left(\left(a+X\right)Y\right)-E\left(a+X\right)E\left(Y\right)= \quad (37)$$

$$=E\left(aY\right)+E\left(XY\right)-\left(E\left(a\right)E\left(Y\right)+E\left(X\right)E\left(Y\right)\right)=\mathrm{Cov}\left(X,Y\right))$$