Dynamic Macroeconomics

Chapter 6: Overlapping generations model

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Overview

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Setup: Population

- We assume that the economy is inhabited by individuals who live for two periods.
- In the first period of their life individuals are young and work, in the second they are old and are retired and do not work.
- In each time period, therefore two generations live: A young generation and an old generation.
- If $N_{1,t}$ denotes the number of young people in period t and $N_{2,t}$ denotes the number of old people in period t, then the total population N_t in period t is given by:

$$N_t = N_{1,t} + N_{2,t} \tag{1}$$

- Furthermore, since $N_{2,t} = N_{1,t-1}$ we have:

$$N_t = N_{1,t} + N_{1,t-1} \tag{2}$$

Setup: Population

- We assume that the population grows at the fixed rate $n > 0$.
- Thus:

$$N_{1,t} = (1+n)\, N_{1,t-1} \tag{3}$$

- For the total population, we then obtain:

$$N_t = N_{1,t} + N_{1,t-1} = N_{1,t} + \frac{1}{1+n} N_{1,t} = \left[1 + \frac{1}{1+n}\right] N_{1,t} \tag{4}$$

Setup: Preferences

- We assume that the preferences of the two generations are identical.
- Lifetime utility of a young person is given by:

$$\mathcal{U} = U\left(c_{1,t}\right) + \beta U\left(c_{2,t+1}\right) \tag{5}$$

where β denotes the subjective discount factor (with $0 < \beta < 1$) and $U(.)$ represents the (strictly concave) period utility function. $c_{1,t}$ denotes consumption per capita of the young whereas $c_{2,t+1}$ is consumption per capita when the young is old.

Setup: Production technolgoy

- Aggregate output, Y_t, is produced using the aggregate capital stock, K_t, and overall labor supply, $N_{1,t}$.
- We assume that the production function is given by

$$Y_t = F\left(K_t, N_{1,t}\right) \tag{6}$$

- The production function satisfies all assumptions made for the production function in the Solow model.
- Per capita output is given by:

$$\frac{Y_t}{N_t} = \frac{F\left(K_t, N_{1,t}\right)}{N_t} = \frac{N_{1,t}}{N_t} F\left(\frac{K_t}{N_{1,t}}, 1\right) \tag{7}$$

$$= \frac{N_{1,t}}{\left[1 + \frac{1}{1+n}\right] N_{1,t}} F\left(\frac{K_t}{N_{1,t}}, 1\right) = \frac{1}{\left[1 + \frac{1}{1+n}\right]} f\left(k_t\right)$$

where $k_t = \frac{K_t}{N_{1,t}}$

Resource constraints

- The national income identity is given by:

$$Y_t = C_t + I_t \tag{8}$$

where C_t denotes aggregate consumption and I_t denotes aggregate investment.

- Aggregate consumption is given by:

$$C_t = C_{1,t} + C_{2,t} = N_{1,t} c_{1,t} + N_{1,t-1} c_{2,t} = \left(c_{1,t} + \frac{1}{1+n} c_{2,t} \right) N_{1,t} \tag{9}$$

where $C_{1,t}$ denotes aggregate consumption of young individuals in period t and $C_{2,t}$ denotes aggregate consumption of old individuals in period t.

Resource constraints

- The capital accumulation equation is given by:

$$\Delta K_{t+1} = K_{t+1} - K_t = I_t - \delta K_t \Longleftrightarrow I_t = K_{t+1} - (1-\delta) K_t \quad (10)$$

- The national income account identity can then be expressed in per capita terms as follows:

$$Y_t = C_t + I_t \Leftrightarrow \frac{Y_t}{N_t} = \frac{C_t}{N_t} + \frac{I_t}{N_t}$$

$$y_t = \frac{N_{1,t}}{N_t}\left(c_{1,t} + \frac{1}{1+n} c_{2,t}\right) + \frac{N_{1,t}}{N_t}\frac{N_{1,t+1}}{N_{1,t}}\frac{K_{t+1}}{N_{1,t+1}} - (1-\delta)\frac{N_{1t}}{N_t}\frac{K_t}{N_{1t}}$$

$$y_t = \frac{1}{1+\frac{1}{1+n}}\left[\left(c_{1,t} + \frac{1}{1+n} c_{2,t}\right) + (1+n) k_{t+1} - (1-\delta) k_t\right]$$

with $k_t = \frac{K_t}{N_{1,t}}$ and $k_{t+1} = \frac{K_{t+1}}{N_{1,t+1}}$ but $y_t = \frac{Y_t}{N_t}$

Resource constraints

- Using the expression for per capita income derived from the production function (equation (7)) we obtain:

$$y_t = \frac{1}{1+\frac{1}{1+n}}\left[\left(c_{1,t} + \frac{1}{1+n}c_{2,t}\right) + (1+n)\,k_{t+1} - (1-\delta)\,k_t\right] \Longleftrightarrow$$

$$f\,(k_t) = c_{1,t} + \frac{1}{1+n}c_{2,t} + (1+n)\,k_{t+1} - (1-\delta)\,k_t$$

This is the resource constraint per capita.

The household optimization problem: Young household

- A young household maximizes lifetime utility given the budget constraint it faces when being young and old.
- Assuming that the household is young in period t, then the budget constraint for this period is given by:

$$c_{1,t} = w_t - s_{1,t} \tag{11}$$

where w_t denotes the wage in period t (we assume that a young household works one unit of time) and $s_{1,t}$ denotes the savings in period t.

- Assuming that the real interest rate between periods t and $t+1$ is given by r_{t+1}, the period's $t+1$ budget constraint is given by:

$$c_{2,t+1} = (1 + r_{t+1})\, s_{1,t} = (1 + r_{t+1})\, (w_t - c_{1,t}) \tag{12}$$

The household optimization problem: Young household

- Rearranging, we obtain:

$$c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} = w_t \tag{13}$$

$\Longrightarrow$ Intertemporal budget constraint

- The decision problem of a young household in period t is thus given by:

$$\max_{c_{1,t}, c_{2,t+1}} \mathcal{U} = U\left(c_{1,t}\right) + \beta U\left(c_{2,t+1}\right) \tag{14}$$

s.t.

$$c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} = w_t \tag{15}$$

The household optimization problem: Young household

- The associated Lagrange function is given by:

$$\mathcal{L} = U\left(c_{1,t}\right) + \beta U\left(c_{2,t+1}\right) + \lambda\left[w_t - c_{1,t} - \frac{c_{2,t+1}}{1+r_{t+1}}\right] \tag{16}$$

- The first-order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial c_{1,t}} = U_{c_{1,t}} - \lambda \overset{!}{=} 0 \Leftrightarrow U_{c_{1,t}} = \lambda \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} = \beta U_{c_{2,t+1}} - \lambda \frac{1}{1+r_{t+1}} \overset{!}{=} 0 \Leftrightarrow \beta U_{c_{2,t+1}} = \frac{\lambda}{1+r_{t+1}} \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} \overset{!}{=} 0 \Leftrightarrow c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = w_t \tag{19}$$

The household optimization problem: Young household

- Combining the first two first-order conditions yields:

$$U_{c_1,t} = \beta\left(1 + r_{t+1}\right) U_{c_2,t+1} \tag{20}$$

$\Longrightarrow$ Interpretation?

The household optimization problem: Old household

- When old, the household's decision problem is straightforward: It consumes all accumulated savings. Why?
- We thus have:

$$c^*_{2,t+1} \quad = \quad (1 + r_{t+1})\, s^*_{1,t} = (1 + r_{t+1}) \left(w_t - c^*_{1,t}\right) \tag{21}$$

where an asterisk (*) denotes here the optimal value of the respective variable.

The firm's optimization problem

- Firms hire labor and capital to maximize life-time profits.
- The optimal capital stock is determined by the following first-order condition (recall Chapter 4):

$$f'(k_t) - \delta = r_t \Leftrightarrow f'(k_t) = r_t + \delta \quad (22)$$

where δ denotes the depreciation rate.

$\Longrightarrow$ Interpretation?

- Given that we have a constant-returns-to-scale production function we have (proof?):

$$Y_t = w_t N_{1,t} + f'(k_t) K_t \quad (23)$$

- Using the fact that $Y_t = N_{1,t} f(k_t)$, dividing by $N_{1,t}$ and solving for w_t, we obtain:

$$w_t = f(k_t) - f'(k_t) k_t \quad (24)$$

Aggregate savings

- The economy's capital accumulation is given by:

$$\begin{aligned} K_{t+1} &= K_t + \left(Y_t - c_{1,t}N_{1,t} - c_{2,t}N_{1,t-1}\right) - \delta K_t \\ &= K_t + w_t N_{1,t} + \left(r_t + \delta\right) K_t - c_{1,t}N_{1t} - c_{2,t,}N_{1,t-1} - \delta K_t \\ &= K_t + w_t N_{1,t} + r_t K_t - \left(w_t - s_{1,t}\right) N_{1,t} - \left(1 + r_t\right) s_{1,t-1}N_{1,t-1} \\ &= \left(1 + r_t\right)\left(K_t - s_{1,t-1}N_{1,t-1}\right) + s_{1,t}N_{1,t} \end{aligned}$$

$\Longrightarrow$ unstable difference equation. Why?

$\Longrightarrow$ Satisfied only for the degenerate solution:

$$K_{t+1} = s_{1,t}N_{1,t} \tag{25}$$

- In per capita terms:

$$\frac{K_{t+1}}{N_{1,t}} = s_{1,t} \Leftrightarrow s_{1,t} = \frac{N_{1,t+1}K_{t+1}}{N_{1,t}N_{1,t+1}} = \left(1 + n\right) k_{t+1} \tag{26}$$

Short-run dynamics and steady state

- We assume that the period utility function is given by:

$$U(c_t) = \ln c_t \tag{27}$$

- We further assume that:

$$\beta = \frac{1}{1+\theta} \tag{28}$$

- For the Euler equation we then obtain:

$$U_{c_1,t} = \beta (1 + r_{t+1}) U_{c_2,t+1} \Leftrightarrow c_{2,t+1} = \frac{1 + r_{t+1}}{1+\theta} c_{1,t} \tag{29}$$

- Replacing $c_{1,t}$ from the period's t budget constraint ($s_{1,t} = w_t - c_{1,t}$), we obtain:

$$s_{1,t} = w_t - \left[\frac{1 + r_{t+1}}{1+\theta}\right]^{-1} c_{2,t+1} \tag{30}$$

Short-run dynamics and steady state

- Plugging the just derived optimal $c_{2,t+1}$ into the intertemporal budget constraint and rearranging terms yields:

$$(2+\theta)\, c_{2,t+1} = (1+r_{t+1})\, w_t. \tag{31}$$

- Using $s_{1,t}$ from (26) the solution for the "old" household's optimization (21) can be written as (without asterisks):

$$c_{2,t+1} = (1+r_{t+1})\, s_{1,t} = (1+r_{t+1})\,(1+n)\, k_{t+1} \tag{32}$$

- Using this above yields

$$\begin{aligned} (2+\theta)\,(1+r_{t+1})\,(1+n)\, k_{t+1} &= (1+r_{t+1})\, w_t \Leftrightarrow \\ k_{t+1} &= \frac{w_t}{(2+\theta)\,(1+n)} \Leftrightarrow \\ k_{t+1} &= \frac{1}{(2+\theta)\,(1+n)} \left(f\,(k_t) - f'\,(k_t)\, k_t\right) \end{aligned} \tag{33}$$

Short-run dynamics and steady state

- For a Cobb-Douglas production function $\left(f\left(k_t\right) = k_t^{\alpha}\right)$ we obtain:

$$k_{t+1} = \frac{1}{(2+\theta)(1+n)}\left(f\left(k_t\right) - f'\left(k_t\right)k_t\right) \Leftrightarrow \tag{34}$$

$$k_{t+1} = \frac{1-\alpha}{(2+\theta)(1+n)}k_t^{\alpha} = \phi k_t^{\alpha}$$

with $\phi = \frac{1-\alpha}{(2+\theta)(1+n)}$

- Taking logs of both sides of this equation yields:

$$\ln k_{t+1} = \ln \phi + \alpha \ln k_t \tag{35}$$

$\Longrightarrow$ Linear difference equation in $\ln k_t$

Short-run dynamics and steady state

- The just derived linear difference equation in $\ln k_t$ can be solved as follows:

$$\begin{aligned} \ln k_t &= \ln \phi + \alpha \ln k_{t-1} = \ln \phi + \alpha \left(\ln \phi + \alpha \ln k_{t-2} \right) = \\ &= \left(1 + \alpha\right) \ln \phi + \alpha^2 \ln k_{t-2} = \ldots \\ &= \frac{1 - \alpha^t}{1 - \alpha} \ln \phi + \alpha^t \ln k_0 \end{aligned} \tag{36}$$

- For $t \to \infty$ we obtain (since $0 < \alpha < 1$):

$$\lim_{t \to \infty} \ln k_t = \frac{1}{1 - \alpha} \ln \phi = \ln k^* \tag{37}$$

$\Longrightarrow$ Stationary linear difference equation in $\ln k_t$

- The steady-state level k^* of k_t is then given by:

$$k^* = e^{\ln k^*} = e^{\frac{1}{1-\alpha} \ln \phi} = \phi^{\frac{1}{1-\alpha}} = \left[\frac{1 - \alpha}{\left(2 + \theta\right)\left(1 + n\right)} \right]^{\frac{1}{1-\alpha}} \tag{38}$$

Short-run dynamics and steady state

- The steady-state level of the wage is given by (asterisks denote steady state values here):

$$w^* = (1-\alpha)(k^*)^\alpha \tag{39}$$

- The steady-state consumption of young individuals is given by:

$$c_1^* = \frac{1+\theta}{2+\theta} w^* \tag{40}$$

$$s^* = \frac{1}{2+\theta} w^* \tag{41}$$

$$c_2^* = (1+r^*) s^* = \frac{1+r^*}{2+\theta} w^* \tag{42}$$

- The difference in consumption when being young and old, we obtain:

$$c_2^* - c_1^* = \frac{r^* - \theta}{2+\theta} w^* \tag{43}$$

$\Longrightarrow$ Interpretation?

Short-run dynamics and steady state

- The dynamics of the capital stock can be illustrated graphically as follows:

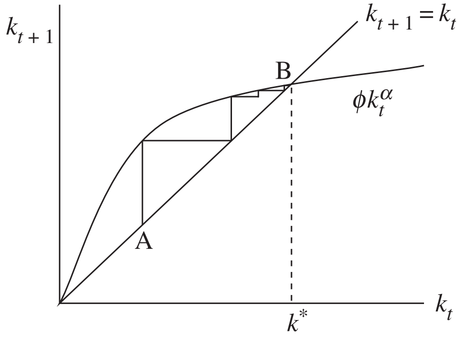

Figure 6.1. The adjustment path of capital.