Dynamic Macroeconomics

Chapter 4: The decentralized economy

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Overview

1. Private consumption: The Life-cycle/Permanent-income hypothesis
2. Labor supply
3. Firms
4. General equilibrium

Setup

- We consider a household with an infinite time horizon.
- The lifetime utility is given by:

$$V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{1}$$

where $U(\bullet)$ is the instantaneous (period-) utility function and c_t is consumption in period t.

- The period-utility function satisfies the following conditions:

$$U'(\bullet) > 0, \quad U''(\bullet) < 0 \tag{2}$$

- The household has initial wealth a_t.
- In each period, the household has (exogenous) income x_t.

Setup

- The household can save or borrow at an exogenous interest rate, r_t, which is assumed to be constant over time, i.e.

$$r_{t+s} = r, \, \forall s \geq 0, 1, 2, \ldots . \tag{3}$$

- The period t's budget constraint is given by:

$$c_t + a_{t+1} = (1 + r) \, a_t + x_t \tag{4}$$

The intertemporal/lifetime budget constraint

- The budget constraint for period $t+1$ is given by:

$$c_{t+1} + a_{t+2} = (1+r)\,a_{t+1} + x_{t+1} \Leftrightarrow a_{t+1} = \frac{1}{1+r}\left(c_{t+1} + a_{t+2} - x_{t+1}\right)$$

- Using the just derived expression for a_{t+1} to replace a_{t+1} in the period-t budget constraint yields:

$$c_t + \frac{1}{1+r}c_{t+1} + \frac{1}{1+r}a_{t+2} = (1+r)\,a_t + x_t + \frac{1}{1+r}x_{t+1} \tag{5}$$

- Repeating the last two steps for periods $t+2, t+3, \ldots$ we obtain:

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s c_{t+s} + \lim_{n\to\infty}\left(\frac{1}{1+r}\right)^n a_{t+n+1} = (1+r)\,a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s x_t \tag{6}$$

The intertemporal/lifetime budget constraint

- Assuming (no-Ponzi game condition) that

$$\lim_{n\to\infty}\left(\frac{1}{1+r}\right)^{n} a_{t+n+1}=0 \tag{7}$$

holds, the intertemporal budget constraint can be written as follows:

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} c_{t+s}=(1+r)\,a_t+\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} x_{t+s} \tag{8}$$

$\Longrightarrow$ Interpretation?

The intertemporal optimization problem

- The intertemporal optimization problem of the household is to maximize the lifetime utility function subject to the lifetime (intertemporal) budget constraint.
- Formally:

$$\max_{c_t, c_{t+1}, c_{t+2}, \dots} \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{9}$$

s.t.

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} = \left(1+r\right) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \tag{10}$$

Model solution

- The Lagrangian associated with the intertemporal optimization problem is given by:

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \quad + \quad \lambda \left[(1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} - \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} \right] \tag{11}$$

- The first-order condition with respect to c_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s U'\left(c_{t+s}\right) - \lambda \left(\frac{1}{1+r}\right)^s \overset{!}{=} 0 \Longleftrightarrow \lambda = \left[\beta\,(1+r)\right]^s U'\left(c_{t+s}\right)$$

- The first-order condition with respect to λ is given by:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c_{t+s} = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \tag{12}$$

Model solution

- If one computes the first-order condition for consumption in period $t+s+1$ (the period after $t+s$ for which we already computed the first-order condition) we obtain:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s+1}} = \beta^{s+1} U'\left(c_{t+s+1}\right) - \lambda \left(\frac{1}{1+r}\right)^{s+1} \overset{!}{=} 0 \iff$$
$$\lambda = \left[\beta\left(1+r\right)\right]^{s+1} U'\left(c_{t+s+1}\right)$$

- Combining the two first-order conditions with respect to c_{t+s} and c_{t+s+1} one obtains:

$$\begin{aligned} \left[\beta\left(1+r\right)\right]^{s} U'\left(c_{t+s}\right) &= \left[\beta\left(1+r\right)\right]^{s+1} U'\left(c_{t+s+1}\right) \iff \qquad (13) \\ U'\left(c_{t+s}\right) &= \beta\left(1+r\right) U'\left(c_{t+s+1}\right) \end{aligned}$$

$\Longrightarrow$ Intertemporal Euler equation

Model solution

- If one assumes that the subjective discount factor is equal to the market discount factor, i.e. that $\beta = \frac{1}{1+r}$ holds the intertemporal Euler equation yields:

$$U'(c_{t+s}) = \beta (1+r) U'(c_{t+s+1}) \Longleftrightarrow U'(c_{t+s}) = U'(c_{t+s+1}) \tag{14}$$

$$\Longrightarrow c_t = c_{t+1} = c_{t+2} = \ldots = c$$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c = (1+r) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \tag{15}$$

$$\Longleftrightarrow c \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = (1+r) a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s}$$

Model solution

- Using the fact that:

$$\sum_{s=0}^{\infty} (a)^s = \frac{1}{1-a} \text{ for } 0 < a < 1 \tag{16}$$

the sum $\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s$ can be written as:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = \frac{1}{1-\frac{1}{1+r}} = \frac{1+r}{r} \tag{17}$$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s c = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \tag{18}$$

$$\Longleftrightarrow c \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s = (1+r)\, a_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s x_{t+s} \equiv W_t$$

Model solution

- Using the intertemporal budget constraint the solution for c is then given by:

$$c = \frac{r}{1+r} W_t \tag{19}$$

$\Longrightarrow$ Life-cycle/Permanent-income hypothesis.

- Interpretation:
  - Households divide lifetime income equally among each period of life.
  - Current consumption is determined not by current income, but by lifetime income.
  - Current saving is determined by the transitory income (the difference between current and permanent income), i.e.

$$s_t = x_t - \left(\frac{r}{1+r}\right) W_t \tag{20}$$

Labor supply: Setup

- In the previous section we assumed that the amount of labor supplied (n_t) is constant and given by 1.

  $\Longrightarrow$ Labor was neglected in the utility function.

- In this section, we want to model the labor supply decision.

  $\Longrightarrow$ Explicitly introduce labor into the utility function.

- We assume that the total time available to the household is 1.

- Assuming that leisure time is denoted by l_t the period-utility function is now given by:

$$U = U\left(c_t, l_t\right) = U\left(c_t, 1 - n_t\right) \tag{21}$$

with $U_c > 0$, $U_l > 0$, $U_{cc} < 0$, $U_{ll} < 0$ and $U_{n,t} = -U_{l,t}$

Labor supply: Setup

- Assuming that the wage rate is given by w_t, the household's budget constraint is given by:

$$\Delta a_{t+1} + c_t = w_t n_t + x_t + r_t a_t \tag{22}$$

where x_t now denotes exogenous income apart from labor income.

Labor supply: The intertemporal optimization problem

- The intertemporal optimization problem of the household is to maximize the lifetime utility function subject to the period budget constraints.
- Formally:

$$\max_{c_t, c_{t+1}, c_{t+2}, \ldots, n_t, n_{t+1}, n_{t+2}, \ldots} \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}, 1 - n_{t+s}\right) \tag{23}$$

s.t.

$$\Delta a_{s+1} + c_s = w_s n_s + x_s + r_s a_s, \ \forall s \geqslant t \tag{24}$$

Labor supply: Model solution

- The Lagrangian associated with the intertemporal optimization problem is given by:

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{ \beta^s U\left(c_{t+s}, 1 - n_{t+s}\right) + \right. \tag{25}$$

$$\left. + \lambda_{t+s}\left[w_{t+s} n_{t+s} + x_{t+s} + r_{t+s} a_{t+s} - \Delta a_{t+s+1} - c_{t+s}\right]\right\}$$

- The first-order condition with respect to c_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = \beta^s U_{c,t+s} - \lambda_{t+s} \overset{!}{=} 0 \Longleftrightarrow \lambda_{t+s} = \beta^s U_{c,t+s}$$

- The first-order condition with respect to n_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial n_{t+s}} = -\beta^s U_{l,t+s} + \lambda_{t+s} w_{t+s} \overset{!}{=} 0 \Longleftrightarrow \beta^s U_{l,t+s} = \lambda_{t+s} w_{t+s}$$

Labor supply: Model solution

- The first-order condition with respect to a_{t+s+1} is given by:

$$\frac{\partial \mathcal{L}}{\partial a_{t+s+1}} = -\lambda_{t+s} + \lambda_{t+s+1}\left(1 + r_{t+s+1}\right) \overset{!}{=} 0 \iff$$
$$\lambda_{t+s} = \lambda_{t+s+1}\left(1 + r_{t+s+1}\right)$$

- The first-order condition with respect to λ_{t+s} is given by:

$$w_{t+s}n_{t+s} + x_{t+s} + r_{t+s}a_{t+s} = \Delta a_{t+s+1} + c_{t+s} \tag{26}$$

- Dividing the second first-order condition by the first, yields:

$$\frac{U_{l,t+s}}{U_{c,t+s}} = w_{t+s} \tag{27}$$

$\Longrightarrow$ Interpretation?

Labor supply: Model solution

- Combining the first and third first-order conditions one obtains:

$$U_c\left(c_{t+s}\right) = \beta\left(1+r_{t+s+1}\right)U_c\left(c_{t+s+1}\right) \tag{28}$$

$\Longrightarrow$ Intertemporal Euler equation

Labor supply: Model solution

- We again assume that the subjective discount factor is equal to the (constant) market discount factor, i.e. that $\beta = \frac{1}{1+r}$ holds.
- Then, the intertemporal Euler equation becomes:

$$U_c\left(c_{t+s}\right) = \beta\left(1+r\right)U_c\left(c_{t+s+1}\right) \Longleftrightarrow U_c\left(c_{t+s}\right) = U_c\left(c_{t+s+1}\right) \tag{29}$$

$\Longrightarrow c_t = c_{t+1} = c_{t+2} = \ldots = c$

- Plugging this result into the lifetime budget constraint yields:

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s c = \left(1+r\right)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\left[x_{t+s} + w_{t+s}n_{t+s}\right]$$

$$\Longleftrightarrow c\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s = \left(1+r\right)a_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s\left[x_{t+s} + w_{t+s}n_{t+s}\right]$$

Labor supply: Model solution

- Solving for c yields:

$$
\begin{aligned}
c\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s} &= (1+r)\,a_t+\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s}\left[x_{t+s}+w_{t+s}n_{t+s}\right] \\
\iff c &= \frac{r}{1+r}W_t
\end{aligned}
$$

with

$$
W_t \equiv (1+r)\,a_t+\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^{s}\left[x_{t+s}+w_{t+s}n_{t+s}\right] \tag{30}
$$

- Interpretation:
  - Households divide lifetime income equally among each period of life.
  - Current consumption is determined not by current income, but by lifetime income.
  - Current saving is determined by the transitory income (the difference between current and permanent income).

Labor supply: Model solution

- Assuming that the period utility function is given by

$$U\left(c_t, l_t\right) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \ln l_t \quad (\text{with } \sigma > 0) \tag{31}$$

the optimality condition for labor supply (in period t) can be written as:

$$\frac{U_{l,t}}{U_{c,t}} = w_t \iff \frac{1}{l_t c_t^{-\sigma}} = w_t \tag{32}$$

$$l_t = \frac{c_t^{\sigma}}{w_t} \iff 1 - n_t = \frac{c_t^{\sigma}}{w_t} \iff n_t = 1 - \frac{c_t^{\sigma}}{w_t}$$

$\Longrightarrow$ What are the effects of an increase in wages on n_t?

$$\frac{\partial n_t}{\partial w_t} = -\frac{\frac{\partial c_t^{\sigma}}{\partial w_t} - c_t^{\sigma}}{w_t^2} \gtreqless 0? \tag{33}$$

Firm behavior: Assumptions

- Firms use labor, n_t, and capital, k_t, to produce output, y_t, according to the following production function:

$$y_t = F\left(k_t, n_t\right) \tag{34}$$

  where $F(.)$ represents a classical production function.

- A firm's profits in period t, denoted by Π_t, are given by:

$$\Pi_t = y_t - w_t n_t - i_t + \Delta b_{t+1} - r b_t \tag{35}$$

  where i_t denotes investment and b_t represents the stock of outstanding debt at the beginning of period t.

Firm behavior: Decision problem

- Firms maximize the present value of their profits given the production function and given the capital accumulation equation.
- Formally:

$$\max_{n_{t+s}, i_{t+s}, b_{t+s+1}; s \geq 0} \mathcal{P}_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+r} \right)^s \left[y_{t+s} - w_{t+s} n_{t+s} - i_{t+s} + \right.$$
$$\left. + \Delta b_{t+s+1} - r b_{t+s} \right]$$

s.t.

$$y_{t+s} = F\left(k_{t+s}, n_{t+s}\right) \tag{36}$$

$$\Delta k_{t+s+1} = i_{t+s} - \delta k_{t+s} \Longrightarrow i_{t+s} = k_{t+s+1} - (1-\delta)\, k_{t+s} \tag{37}$$

Firm behavior: Decision problem

- Plugging the constraints into the objective function we can write the firm's decision problem as follows:

$$\max_{n_{t+s},k_{t+s+1},b_{t+s+1};\ s\geq 0} \mathcal{P}_t = \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s \left[F\left(k_{t+s}, n_{t+s}\right) - w_{t+s}n_{t+s} \right.$$
$$\left. -k_{t+s+1} + \left(1-\delta\right)k_{t+s} + b_{t+s+1} - \left(1+r\right)b_{t+s}\right]$$

Firm behavior: Optimality conditions

- The first-order conditions with respect to n_{t+s} are given by:

$$\frac{\partial \mathcal{P}_t}{\partial n_{t+s}} = \left(\frac{1}{1+r}\right)^s \left[F_{n,t+s} - w_{t+s}\right] \overset{!}{=} 0 \iff F_{n,t+s} = w_{t+s} \quad (38)$$

- The first-order conditions with respect to k_{t+s+1} are given by:

$$\frac{\partial \mathcal{P}_t}{\partial k_{t+s+1}} = -\left(\frac{1}{1+r}\right)^s + \left(\frac{1}{1+r}\right)^{s+1} \left[F_{k,t+s+1} + 1 - \delta\right] \overset{!}{=} 0 \iff$$

$$F_{k,t+s+1} = r + \delta \implies k_{t+s+1} = F_{k,t+s+1}^{-1}\left(r+\delta\right)$$

- The first-order conditions with respect to b_{t+s+1} are given by:

$$\frac{\partial \mathcal{P}_t}{\partial b_{t+s+1}} = -\left(\frac{1}{1+r}\right)^s + \left(\frac{1}{1+r}\right)^{s+1} \left[1+r\right] \overset{!}{=} 0 \quad (39)$$

Consolidating the household and firm budget constraints

- The national income identity is given by:

$$y_t = F\left(k_t, n_t\right) = c_t + i_t \tag{40}$$

- Solving the household budget constraint (holding r constant) for c_t yields:

$$c_t = w_t n_t + x_t + r a_t - \Delta a_{t+1} \tag{41}$$

- Solving the capital accumulation equation for i_t yields:

$$i_t = \Delta k_{t+1} + \delta k_t \tag{42}$$

- Using the so obtained expressions to replace c_t and i_t in the national income identity yields:

$$F\left(k_t, n_t\right) = w_t n_t + x_t + r a_t - \Delta a_{t+1} + \Delta k_{t+1} + \delta k_t \tag{43}$$

Consolidating the household and firm budget constraints

- Solving the just derived equation for x_t yields:

$$x_t = F(k_t, n_t) - w_t n_t - ra_t + \Delta a_{t+1} - \Delta k_{t+1} - \delta k_t \tag{44}$$

- Subtracting firms' profits from x_t yields:

$$\begin{aligned} x_t - \Pi_t &= F(k_t, n_t) - w_t n_t - ra_t + \Delta a_{t+1} - \Delta k_{t+1} - \delta k_t - \\ &\quad - \left[F(k_t, n_t) - w_t n_t - \Delta k_{t+1} - \delta k_t + b_{t+1} - (1+r) b_t\right] \\ &= \Delta(a_{t+1} - b_{t+1}) - r(a_t - b_t) \end{aligned} \tag{45}$$

- Since $a_t = b_t$ (Why?) we obtain:

$$x_t = \Pi_t \tag{46}$$

Consolidating the household and firm budget constraints

- Since the production function is assumed to have constant returns to scale (i.e. is assumed to be homogenous of degree one), we have (Euler's theorem):

$$F(k_t, n_t) = F_{n,t} n_t + F_{k,t} k_t = w_t n_t + (r+\delta) k_t$$
$$\Longleftrightarrow \delta k_t = F(k_t, n_t) - w_t n_t - r k_t$$

- Using this result, firms' profits can be written as:

$$\begin{aligned} \Pi_t &= F(k_t, n_t) - w_t n_t - \Delta k_{t+1} - \delta k_t + b_{t+1} - (1+r) b_t \qquad (47) \\ &= F(k_t, n_t) - w_t n_t - k_{t+1} + k_t - F(k_t, n_t) + w_t n_t + r k_t + b_{t+1} - \\ &\quad - (1+r) b_t = \\ &= -(k_{t+1} - b_{t+1}) + (1+r)(k_t - b_t) \end{aligned}$$

Consolidating the household and firm budget constraints

- Solving for $k_t - b_t$ yields (assuming $\lim_{s\to\infty} \frac{k_{t+s}-b_{t+s}}{(1+r)^s} = 0$):

$$k_t - b_t = \frac{\Pi_t + (k_{t+1} - b_{+1})}{1+r} = \sum_{s=0}^{\infty} \frac{\Pi_{t+s}}{(1+r)^{s+1}} \tag{48}$$

- If profits are constant and equal to Π then

$$k_t - b_t = \frac{\Pi}{r} \tag{49}$$

- With $x = \Pi$ and $a = b$ it follows that total household asset income is given by

$$x + ra = rk \tag{50}$$

General equilibrium: The labor market

- Labor demand is determined by the following equation:

$$F_{n,t} = w_t \tag{51}$$

- Labor supply is determined by the following equation:

$$-\frac{U_{n,t}}{U_{c,t}} = w_t \tag{52}$$

- Setting these two equations equal yields:

$$-\frac{U_{n,t}}{U_{c,t}} = F_{n,t} \tag{53}$$

$\Longrightarrow$ Interpretation?

General equilibrium: The goods market

- Aggregate demand, y_t^d is given by:

$$\begin{aligned} y_t^d &= c_t + i_t = \frac{r}{1+r} W_t + k_{t+1} - (1-\delta) k_t = \\ &= \frac{r}{1+r} W_t + F_{k,t+1}^{-1} (r+\delta) - (1-\delta) k_t \end{aligned} \tag{54}$$

- Aggregate supply is given by the following equation:

$$y_t^s = F(k_t, n_t) . \tag{55}$$

- In equilibrium:

$$y_t^d = y_t^s \Longleftrightarrow \frac{r}{1+r} W_t + F_{k,t+1}^{-1} (r+\delta) - (1-\delta) k_t = F(k_t, n_t) \tag{56}$$