Dynamic Macroeconomics

Chapter 3: Economic growth

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Overview

1 Introduction

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Introduction

- Development of GDP in the United States: 1947 - 2002.

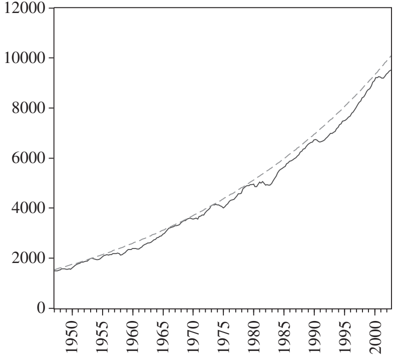

Figure 3.1. U.S. GDP (solid line) and capacity GDP (dashed line) 1947–2002.

$\Longrightarrow$ Positive growth trend over 55 years.

Stylized facts of economic growth

- In the short run, important fluctuations: Output, employment, investment, and consumption vary a lot across booms and recessions.
- In the long run, balanced growth: Output per worker and capital per worker (Y/N and K/N) grow at roughly constant, and certainly not vanishing, rates.
- The capital-to-output ratio (K/Y) is nearly constant.
- The return to capital (r) is roughly constant, whereas the wage rate (w) grows at the same rates as output.
- The income shares of labor and capital (wN/Y and rK/Y) stay roughly constant.
- Substantial cross-country differences in both income levels and growth rates.

The Solow-Swan model of growth

- Firms:
  - Firms produce output (GDP) using the following production function:

$$Y_t = F(K_t, N_t, t) = (1+\mu)^t K_t^{\alpha} N_t^{1-\alpha} \tag{1}$$

    where the notation for Y, K and N is as in chapter 2.
  - μ denotes the (exogenously determined) rate of technological progress.
  - Dividing by N_t yields:

$$y_t = \frac{Y_t}{N_t} = (1+\mu)^t \frac{K_t^{\alpha} N_t^{1-\alpha}}{N_t} = (1+\mu)^t k_t^{\alpha} \tag{2}$$

    where small letters denote per-capita terms ($k = K/N$, $y = Y/N$).

The Solow-Swan model of growth

- Households:
  - Households generate income from production and use this income to consume and save/invest.
  - We assume that in each period they decide to save a fixed proportion of their income.
  - The (exogeneously given) saving rate is denoted by s.
  - Savings in a given period, denoted by S_t, are then given by:

$$S_t = sY_t \tag{3}$$

  - We assume that the working population, N_t, grows by the rate n, i.e. we have

$$N_{t+1} = (1 + n) N_t \tag{4}$$

The Solow-Swan model of growth

- The economy's "budget constraint":
  - We assume that the economy is closed, i.e., we have:

$$S_t = I_t \tag{5}$$

    where I_t denote period's t investment.

  - The capital stock then evolves according to:

$$K_{t+1} = (1-\delta)\, K_t + I_t \Longleftrightarrow K_{t+1} - K_t = I_t - \delta K_t = sY_t - \delta K_t \tag{6}$$

    where δ denotes the depreciation rate.

  - Diving both sides of this term by K_t yields:

$$\frac{K_{t+1} - K_t}{K_t} = s\frac{Y_t}{K_t} - \delta = s\frac{\frac{Y_t}{N_t}}{\frac{K_t}{N_t}} - \delta = s\frac{y_t}{k_t} - \delta \tag{7}$$

The Solow-Swan model of growth

- The economy's "budget constraint" (... continued):
  - To derive an expression for the growth rate of per-capita capital ($\frac{k_{t+1}-k_t}{k_t}$) we consider the difference in the growth rate of the capital stock and the population and transform this difference appropriately.
  - This yields (Remember: $N_t = \frac{N_{t+1}}{1+n}$):

$$\frac{K_{t+1}-K_t}{K_t} - \frac{N_{t+1}-N_t}{N_t} = \frac{K_{t+1}}{K_t} - \frac{K_t}{K_t} - \frac{N_{t+1}}{N_t} + \frac{N_t}{N_t} =$$

$$= \frac{K_{t+1}}{K_t} - \frac{N_{t+1}}{N_t} = \frac{\frac{K_{t+1}}{N_t}}{\frac{K_t}{N_t}} - \frac{(1+n)\,N_t}{N_t} =$$

$$= \frac{\frac{K_{t+1}}{\frac{N_{t+1}}{1+n}}}{\frac{K_t}{N_t}} - \frac{(1+n)\,\frac{K_t}{N_t}}{\frac{K_t}{N_t}} = \frac{\frac{K_{t+1}}{\frac{N_{t+1}}{1+n}} - (1+n)\,\frac{K_t}{N_t}}{\frac{K_t}{N_t}}$$

The Solow-Swan model of growth

- The economy's "budget constraint" (... continued):
  - The latter expression can be rearranged as follows:

$$\frac{\frac{K_{t+1}}{\frac{N_{t+1}}{1+n}} - \frac{K_t}{N_t}(1+n)}{\frac{K_t}{N_t}} = \frac{(1+n)(k_{t+1}-k_t)}{k_t} = (1+n)\frac{\Delta k_{t+1}}{k_t} \tag{8}$$

  - Given that both the growth rate of population, n, and the per-capita capital growth rate, $\frac{\Delta k_{t+1}}{k_t}$ are small numbers their product is an even smaller number and can be treated as being (almost) equal to zero, i.e.,

$$n\frac{\Delta k_{t+1}}{k_t} \approx 0 \tag{9}$$

  - Then, the per-capita capital growth rate is given by:

$$\begin{aligned} \frac{\Delta k_{t+1}}{k_t} &\approx \frac{K_{t+1}-K_t}{K_t} - \frac{N_{t+1}-N_t}{N_t} = \\ &= s\frac{y_t}{k_t} - \delta - n = s\frac{y_t}{k_t} - (n+\delta) \end{aligned} \tag{10}$$

The Solow-Swan model of growth

- The economy's "budget constraint" (... continued):
  - The change in the per-capita capital is then given by:

$$\Delta k_{t+1} = sy_t - (n + \delta) k_t \tag{11}$$

The Solow-Swan model of growth

- Relationship between the growth rate of capital and the capital stock:
  - Denoting the growth rate of capital as $\gamma(k_t)$ we can write:

$$\gamma(k_t) = \frac{\Delta k_{t+1}}{k_t} = s\frac{y_t}{k_t} - (n+\delta) \tag{12}$$

  - The relationship between the growth rate of capital and the capital stock can then be derived as follows:

$$\begin{aligned} \frac{\partial \gamma(k_t)}{\partial k_t} &= s\left(\frac{\frac{\partial y_t}{\partial k_t}k_t - y_t}{k_t^2}\right) = s\left(\frac{\frac{\partial y_t}{\partial k_t}k_t}{k_t^2}\frac{y_t}{y_t} - \frac{y_t}{k_t^2}\right) = \\ &= \frac{sy_t}{k_t^2}\left[\frac{k_t}{y_t}\frac{\partial y_t}{\partial k_t} - 1\right] < 0 \end{aligned} \tag{13}$$

$\Longrightarrow$ Interpretation?

The Solow-Swan model of growth

- Balanced growth:
  - Using the same approximation as above (for the growth rate of per-capita capital) we can write:

$$\frac{\Delta y_{t+1}}{y_t} = \frac{\Delta Y_{t+1}}{Y_t} - \frac{\Delta N_{t+1}}{N_t} \tag{14}$$

  - The term $\frac{\Delta Y_{t+1}}{Y_t}$ can be written as:

$$\begin{aligned} \frac{\Delta Y_{t+1}}{Y_t} &= \frac{Y_{t+1}}{Y_t} - 1 = \frac{(1+\mu)^{t+1} K_{t+1}^{\alpha} N_{t+1}^{1-\alpha}}{(1+\mu)^{t} K_t^{\alpha} N_t^{1-\alpha}} - 1 = \\ &= (1+\mu)\left(\frac{K_{t+1}}{K_t}\right)^{\alpha}\left(\frac{N_{t+1}}{N_t}\right)^{1-\alpha} - 1 = \\ &= (1+\mu)\left(1+\frac{\Delta K_{t+1}}{K_t}\right)^{\alpha}\left(1+\frac{\Delta N_{t+1}}{N_t}\right)^{1-\alpha} - 1 \end{aligned} \tag{15}$$

The Solow-Swan model of growth

- Balanced growth (... continued):
  - The growth of output is thus a function f of the growth rate of technological progress, capital growth and population growth, i.e.,

$$\frac{\Delta Y_{t+1}}{Y_t} = f\left(\mu, \frac{\Delta K_{t+1}}{K_t}, \frac{\Delta N_{t+1}}{N_t}\right) \tag{16}$$

  - A first-order Taylor approximation around $\mu = \frac{\Delta K_{t+1}}{K_t} = \frac{\Delta N_{t+1}}{N_t} = 0$ yields:

$$\begin{aligned} \frac{\Delta Y_{t+1}}{Y_t} \approx\ & (1+0)(1+0)^{\alpha}(1+0)^{1-\alpha} - 1 \\ & + (1+0)^{\alpha}(1+0)^{1-\alpha}\mu \\ & + \alpha(1+0)(1+0)^{\alpha-1} 1 (1+0)^{1-\alpha}\frac{\Delta K_{t+1}}{K_t} \\ & + (1-\alpha)(1+0)(1+0)^{\alpha}(1+0)^{-\alpha} 1 \frac{\Delta N_{t+1}}{N_t} \end{aligned} \tag{17}$$

The Solow-Swan model of growth

- Balanced growth (... continued):
  - For the growth rate in per capita income we then obtain:

$$\frac{\Delta y_{t+1}}{y_t} = \frac{\Delta Y_{t+1}}{Y_t} - \frac{\Delta N_{t+1}}{N_t} = \mu + \alpha \frac{\Delta K_{t+1}}{K_t} + (1-\alpha)\frac{\Delta N_{t+1}}{N_t} - \frac{\Delta N_{t+1}}{N_t}$$

$$= \mu + \alpha \left[\frac{\Delta K_{t+1}}{K_t} - \frac{\Delta N_{t+1}}{N_t} \right] = \mu + \alpha \frac{\Delta k_{t+1}}{k_t} = \mu + \alpha \gamma (k_t)$$

$\Longrightarrow$ Interpretation?

  - For the growth rate in per capita consumption we obtain:

$$c_t = (1-s)\, y_t \Longrightarrow \frac{\Delta c_{t+1}}{c_t} = \frac{\Delta y_{t+1}}{y_t} \tag{18}$$

The Solow-Swan model of growth

- Balanced growth (... continued):
  - On a balanced growth path (in the steady state):

$$\gamma\left(k_{t+1}\right)=\gamma\left(k_t\right)=\gamma\left(k^*\right) \tag{19}$$

  - Since $\gamma\left(k_t\right)=s\frac{y_t}{k_t}-(n+\delta)$ this requires that the output-capital ratio must remain constant in the steady-state:

    $\Longrightarrow$ In the steady-state the growth of per-capita income and per-capita capital must be equal.

  - Thus:

$$\mu+\alpha\gamma\left(k^*\right)=\gamma\left(k^*\right) \Leftrightarrow \gamma\left(k^*\right)=\frac{\mu}{1-\alpha} \tag{20}$$

    $\Longrightarrow$ Interpretation?