Dynamic Macroeconomics

Chapter 2: The centralized economy

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Overview

Selected stylized facts of business cycles

- Stylized facts = empirical regularities.

  $\Longrightarrow$ Major objective of macroeconomics: Build models which can explain major stylized facts

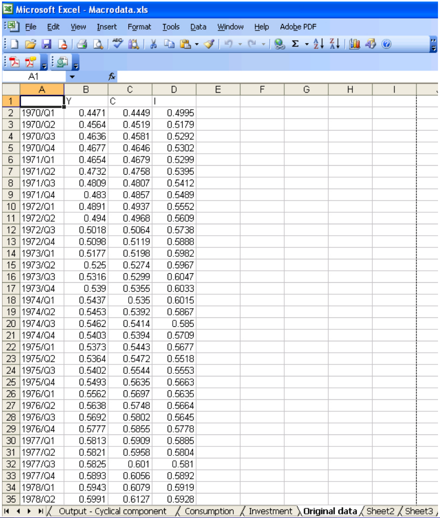
- In chapter 2: Analyze behavior of consumption and investment.

  $\Longrightarrow$ Necessary first step: Derive stylized facts concerning the behavior of consumption and investment.

- Procedure:
  - Obtain data (In our case: Euro area data)
  - Filter data (Decompose data into long-run and short-run component).
  - Compute statistics concerning the behavior of macroeconomic time series (Volatility and correlation of time series).

Selected stylized facts of business cycles

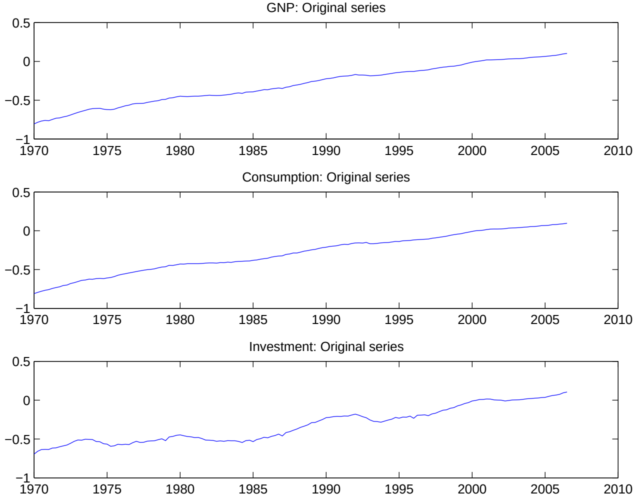
- Data for output, consumption and investment: Original data



	Y	C	I
1970/Q1	0.4471	0.4449	0.4995
1970/Q2	0.4564	0.4519	0.5179
1970/Q3	0.4636	0.4581	0.5292
1970/Q4	0.4677	0.4646	0.5302
1971/Q1	0.4654	0.4679	0.5299
1971/Q2	0.4732	0.4758	0.5395
1971/Q3	0.4809	0.4807	0.5412
1971/Q4	0.483	0.4857	0.5489
1972/Q1	0.4891	0.4937	0.5552
1972/Q2	0.494	0.4968	0.5609
1972/Q3	0.5018	0.5064	0.5738
1972/Q4	0.5098	0.5119	0.5888
1973/Q1	0.5177	0.5198	0.5982
1973/Q2	0.525	0.5274	0.5967
1973/Q3	0.5316	0.5299	0.6047
1973/Q4	0.539	0.5355	0.6033
1974/Q1	0.5437	0.535	0.6015
1974/Q2	0.5453	0.5392	0.5867
1974/Q3	0.5462	0.5414	0.585
1974/Q4	0.5403	0.5394	0.5709
1975/Q1	0.5373	0.5443	0.5677
1975/Q2	0.5364	0.5472	0.5518
1975/Q3	0.5402	0.5544	0.5553
1975/Q4	0.5493	0.5635	0.5663
1976/Q1	0.5562	0.5697	0.5635
1976/Q2	0.5638	0.5748	0.5664
1976/Q3	0.5692	0.5802	0.5645
1976/Q4	0.5777	0.5855	0.5778
1977/Q1	0.5813	0.5909	0.5885
1977/Q2	0.5821	0.5958	0.5804
1977/Q3	0.5825	0.601	0.581
1977/Q4	0.5893	0.6056	0.5892
1978/Q1	0.5943	0.6079	0.5919
1978/Q2	0.5991	0.6127	0.5928

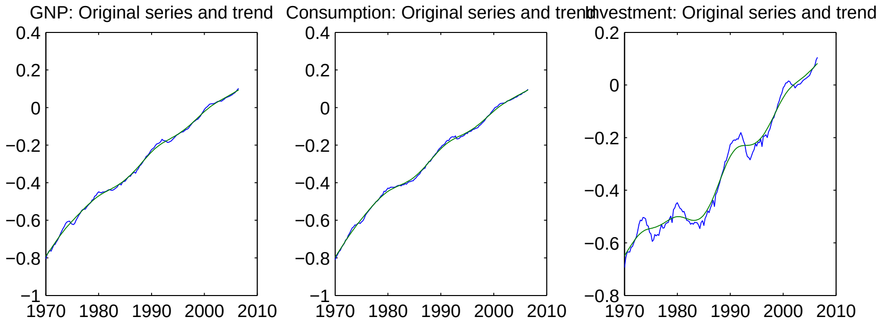
Selected stylized facts of business cycles

- Data for output, consumption and investment: Plot of (ln) levels



Selected stylized facts of business cycles

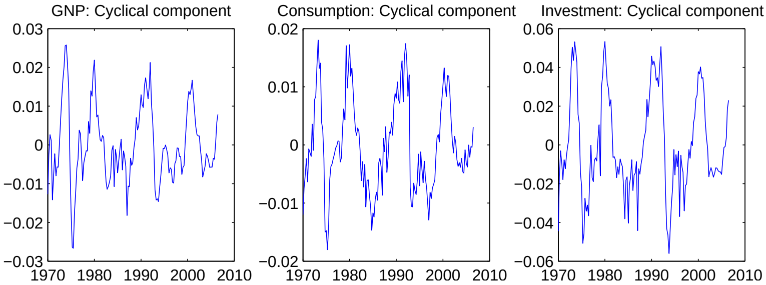
- Data for output, consumption and investment: Plot of level and trend component



$\Longrightarrow$ Observation: Variables exhibit long-run growth

Selected stylized facts of business cycles

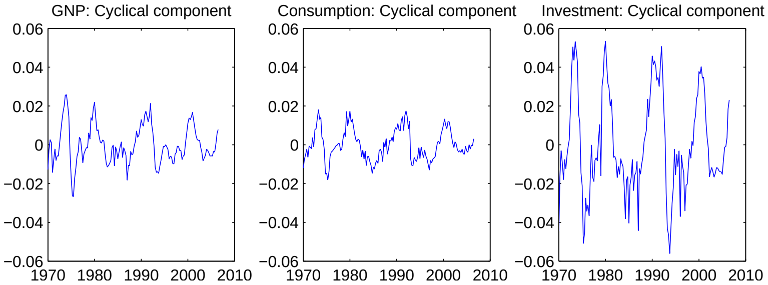
- Data for output, consumption and investment: Plot of cyclical component



$\Longrightarrow$ Observation: ?

Selected stylized facts of business cycles

- Data for output, consumption and investment: Plot of cyclical component (identical scale)



⟹ Observations:

⟹ Consumption is less volatile than output, investment is much more volatile than output.

⟹ Consumption and investment are strongly procyclical.

Selected stylized facts of business cycles

- To decompose the original time series: Filtering of the original data is necessary.
- Basic intuition:
  - Denote by $\{y_t\}_{t=1}^{T}$ the log of a time series (such as GDP, consumption, investment, ...) that you want to detrend.
  - y_t is considered to be composed of a long-run (y_t^{lr}) and a short-run (y_t^{sr}) component as follows:

$$y_t = y_t^{lr} + y_t^{sr} \tag{1}$$

  $\Longrightarrow$ To perform empirical growth or business cycle analysis: "Filtering" of the data is necessary to obtain either y_t^{lr} or y_t^{sr}.

- To filter data: Several possibilities exist.
- Most popular filter: Hodrick-Prescott filter.

Selected stylized facts of business cycles

- Hodrick-Prescott (HP) filter: Intuition
    - According to the Hodrick-Prescott filter, the long-run (growth or trend) component is obtained as the solution to the following minimization problem:

$$\min_{\{y_t^{lr}\}_{t=1}^{T}} \sum_{t=1}^{T} \left(y_t - y_t^{lr}\right)^2 + \lambda \sum_{t=2}^{T-1} \left[\left(y_{t+1}^{lr} - y_t^{lr}\right) - \left(y_t^{lr} - y_{t-1}^{lr}\right)\right]^2 \quad (2)$$

    where the parameter λ must be chosen by the researcher.
    - The higher the value of λ, the smoother the trend component becomes (Can you see why?).
    - For quarterly data, $\lambda = 1600$ is chosen.

Model setup: Motivation

- Build up a simple macroeconomic model which allows us to analyze the behavior of aggregate output, consumption and investment.
- Model is microfounded:

  $\Longrightarrow$ Model household and firm behavior explicitly.
- Behavior of macro variables is obtained by aggregating across households and firms.

  $\Longrightarrow$ Simplifying assumptions: All households are equal, all firms are owned by households.

  $\Longrightarrow$ It is sufficient to solve the decisions problems of the "representative" household/firm.

Model setup: Preferences

- Economy is inhabited by identical consumers.

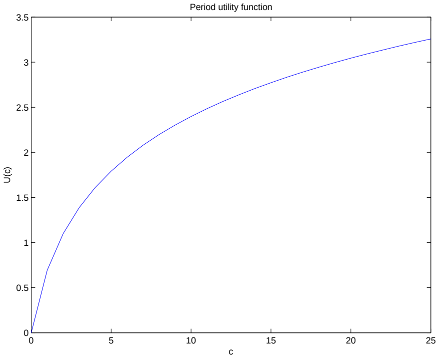
  $\Longrightarrow$ Individual variables are identical to aggregate variables.
- Consumers have preferences over an infinite stream of consumption $c_t, c_{t+1}, \ldots = \{c_{t+s}\}_{s=0}^{\infty}$.
- The consumer's lifetime utility function is assumed to be **time-separable** and given by:

$$V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{3}$$

- β is the individual's subjective time discount factor. We assume that $0 < \beta < 1$ holds.
- $U(.)$ denotes the period utility function. We assume that it is strictly increasing and concave.

Model setup: Preferences

- Period utility function: Graphical illustration:



$\Longrightarrow$ Positive marginal utility: $U'(.) > 0$.

$\Longrightarrow$ Diminishing positive marginal utility: $U''(.) < 0$.

Production technology

- Output (GDP) is produced using the following production technology:

$$y_t = F\left(a_t, k_t, n_t\right) \tag{4}$$

with

  - y_t: Output
  - k_t: Capital input
  - n_t: Labor input
  - a_t: Level of technology, knowledge, efficiency of work

Production technology

- Assumptions concerning the production function (continued):
  - Constant returns to scale:

$$F(a, \phi k, \phi n) = \phi F(a, k, n) \quad \text{for all } \phi \geq 0 \tag{5}$$

  - Positive, but declining marginal products of capital and labor

$$\frac{\partial F(\bullet)}{\partial k} > 0, \frac{\partial^2 F(\bullet)}{\partial k^2} < 0, \frac{\partial F(\bullet)}{\partial n} > 0, \frac{\partial^2 F(\bullet)}{\partial n^2} < 0, \frac{\partial^2 F(\bullet)}{\partial n \partial k} \geq 0 \tag{6}$$

  - Both production factors are necessary

$$F(a, 0, n) = 0 \text{ and } F(a, k, 0) = 0 \tag{7}$$

  - Inada conditions are satisfied:

$$\lim_{k \to 0} \frac{\partial F(\bullet)}{\partial k} \to \infty, \ \lim_{k \to \infty} \frac{\partial F(\bullet)}{\partial k} = 0, \lim_{n \to 0} \frac{\partial F(\bullet)}{\partial n} \to \infty, \ \lim_{n \to \infty} \frac{\partial F(\bullet)}{\partial n} = 0 \tag{8}$$

Production technology

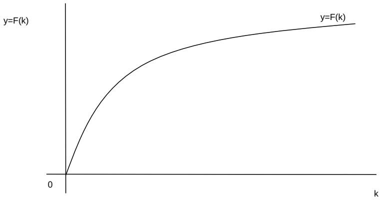
- For the moment, we assume that n_t is constant:

$$n_t = 1 \tag{9}$$

- Then:

$$y_t = F(a_t, k_t, 1) = F(a_t, k_t) \tag{10}$$

- Graphical illustration of the production function ($a = 1$):



Budget constraint

- Period t's budget constraint is given by:

$$y_t = c_t + i_t \tag{11}$$

$\Longrightarrow$ Budget constraint of a closed economy without government.

- Moreover, the household faces the following condition concerning the evolution of the capital stock:

$$k_{t+1} = k_t + i_t - \delta k_t \Longleftrightarrow i_t = k_{t+1} - (1-\delta)\, k_t \tag{12}$$

- Combining the two above equations, the household's budget constraint can be rewritten as (suppressing the a_t in the production function):

$$c_t + k_{t+1} = F(k_t) + (1-\delta) k_t \tag{13}$$

- In fact we assume $a_t = 1$ for the moment.

The maximization problem

- The household maximizes lifetime utility given the resource constraint:

  $\Longrightarrow$ Dynamic (constrained) intertemporal optimization problem.

- The intertemporal optimization problem is given by:

$$\max_{c_t, c_{t+1}, \ldots; k_{t+1}, k_{t+2}, \ldots} V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{14}$$

s.t.

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta) k_{t+s}, \; \forall s > 0 \tag{15}$$

- Solution approaches:
  - Transform constrained into unconstrained maximization problem.
  - Lagrange approach.
  - Dynamic programming.

Model solution: The two-period case

- To illustrate the basic intuition of the model we first solve it for the simple two-period case.
- In this case, the household's maximization problem is given by:

$$\max_{c_t, c_{t+1}, k_{t+1}, k_{t+2}} V_t = \sum_{s=0}^{1} \beta^s U\left(c_{t+s}\right) = U\left(c_t\right) + \beta U\left(c_{t+1}\right) \tag{16}$$

s.t.

$$c_t + k_{t+1} = F(k_t) + (1-\delta)k_t \tag{17}$$

$$c_{t+1} + k_{t+2} = F(k_{t+1}) + (1-\delta)k_{t+1} \tag{18}$$

- To solve the model we employ two different approaches:
  - Approach 1: Transform constrained into unconstrained maximization problem and solve the unconstrained problem.
  - Approach 2: Lagrange approach.

Model solution: The two-period case

- Solution approach 1: Transform constrained into unconstrained maximization problem and solve the unconstrained problem:
  - Solving the two budget constraint for consumption yields:

$$c_t = F(k_t) + (1-\delta)k_t - k_{t+1} \tag{19}$$

$$c_{t+1} = F(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2} \tag{20}$$

  - Since the household no longer lives in period $t+2$ it will disinvest its complete capital stock in period $t+1$ and consume it. That is, we have:

$$k_{t+2} = 0 \tag{21}$$

  - Period's $t+1$ budget constraint then becomes:

$$c_{t+1} = F(k_{t+1}) + (1-\delta)k_{t+1} \tag{22}$$

Model solution: The two-period case

- Solution approach 1 (continued):
  - Plugging the transformed budget constraints into the objective function yields:

$$\max_{k_{t+1}} V_t = U\left(c_t\right) + \beta U\left(c_{t+1}\right) =$$
$$= U\left(F(k_t) + (1-\delta)k_t - k_{t+1}\right) + \beta U\left(F(k_{t+1}) + (1-\delta)k_{t+1}\right)$$

  - The first-order condition is given by (Notation: $U'(.) = \frac{\partial U}{\partial c}$):

$$U'\left(c_t\right)(-1) + \beta U'\left(c_{t+1}\right)\left[F'\left(k_{t+1}\right) + 1 - \delta\right] \stackrel{!}{=} 0 \iff \quad (23)$$
$$U'\left(c_t\right) = \beta\left[F'\left(k_{t+1}\right) + 1 - \delta\right] U'\left(c_{t+1}\right)$$

$\Longrightarrow$ Intertemporal Euler equation

Model solution: The two-period case

- Solution approach 1 (continued):
  - Intuition for intertemporal Euler equation:
    - Assume that consumption is reduced by a small amount (denoted by Δc) in Period t.

      $\Longrightarrow$ Utility in period t is reduced by: $U'(c_t)\,\Delta c$.
    - The amount Δc is invested in capital. In period $t+1$ this investment leads to additional output of $F'(k_{t+1})\,\Delta c$.
    - Moreover, the household can transform the amount of consumption invested in period t back into consumption goods in period $t+1$. Since a proportion δ of Δc is lost through appreciation this leads to an increase in consumption by $(1-\delta)\,\Delta c$ in period $t+1$.
    - Overall, the household can increase consumption by $F'(k_{t+1}) + 1 - \delta$ in period $t+1$ which in turn leads to an increase in period's $t+1$ utility by $[F'[k_{t+1}] + 1 - \delta]\,U'(c_{t+1})$.

Model solution: The two-period case

- Solution approach 1 (continued):
  - Intuition for intertemporal Euler equation (continued):
    - From today's perspective the utility gain tomorrow is "worth": $\beta\left[F'\left[k_{t+1}\right]+1-\delta\right]U'\left(c_{t+1}\right)$.
    - In the optimum, the utility loss from saving more today must be equal to the discounted utility gain tomorrow (why?). Thus, we must have:

$$U'\left(c_t\right)=\beta\left[F'\left(k_{t+1}\right)+1-\delta\right]U'\left(c_{t+1}\right) \tag{24}$$

  - Interpretation of the term $F'\left(k_{t+1}\right)+1-\delta$:
    - Assume you invest one unit of consumption in period 0. Then, your consumption in period 1 increases by:

$$F'\left(k_{t+1}\right)+1-\delta \tag{25}$$

$\Longrightarrow F'\left(k_{t+1}\right)+1-\delta$ represents the gross real interest rate.

Model solution: The two-period case

- Solution approach 1 (continued):
  - Implications of the Euler equation (1):
    - Assume that the subjective discount factor (β) is equal to the market discount factor ($\frac{1}{F'(k_{t+1})+1-\delta}$).
    - Then, the Euler equation becomes:

$$U'\left(c_t\right) = \beta\left[F'\left(k_{t+1}\right) + 1 - \delta\right] U'\left(c_{t+1}\right) \Longleftrightarrow U'\left(c_t\right) = U'\left(c_{t+1}\right) \tag{26}$$

$\Longrightarrow$ Consumption in the two periods would be equal:

$$c_t = c_{t+1} \tag{27}$$

$\Longrightarrow$ Perfect consumption smoothing

Model solution: The two-period case

- Solution approach 1 (continued):
  - Why do households want to smooth consumption?
  - Illustrative example:
    - Household has log-utility function ($U(c_t) = \ln c_t$).
    - Household lives for two periods.
    - There is no discounting: $\beta = 1$.
    - Household can choose between two consumption patterns:

    $\Longrightarrow$ Pattern 1: $c_t = 9$, $c_{t+1} = 1$.

    $\Longrightarrow$ Pattern 2 (smooth pattern): $c_t = 5$, $c_{t+1} = 5$.

  $\Longrightarrow$ Which consumption pattern do households prefer?

  - Lifetime utility from pattern 1:

$$V_t^1 = \ln(9) + \ln(1) \approx 2.2 \tag{28}$$

  - Lifetime utility from pattern 2:

$$V_t^2 = \ln(5) + \ln(5) \approx 3.2 > 2.2 = V_t^1 \tag{29}$$

$\Longrightarrow$ Households prefer (lifetime-maximizing) smooth pattern 2.

Model solution: The two-period case

- Solution approach 1 (continued):
  - Implications of the Euler equation (2):
    - How does β (= subjective discount factor) influence the consumption pattern over time?

      $\Longrightarrow$ For illustrative purposes, we assume that $U(c_t) = \ln c_t$ $\left(U'(c_t) = \frac{1}{c_t}\right)$.
    - From the Euler equation:

$$U'(c_t) = \beta \left[F'(k_{t+1}) + 1 - \delta\right] U'(c_{t+1})$$

we get:

$$\frac{1}{c_t} = \beta \left[F'(k_{t+1}) + 1 - \delta\right] \frac{1}{c_{t+1}} \Longleftrightarrow c_{t+1} = \beta \left[F'(k_{t+1}) + 1 - \delta\right] c_t$$

$\Longrightarrow$ A higher value of β (everything else held constant) implies that c_{t+1} is relatively higher compared to c_t.

Model solution: The two-period case

- Solution approach 1: (continued):
  - Implications of the Euler equation (2):
    - How does $F'(k_{t+1})$ (= marginal product of next period's capital stock) influence the consumption pattern over time?

      $\Longrightarrow$ For illustration purposes, we again assume that $U(c_t) = \ln c_t$ ($U'(c_t) = \frac{1}{c_t}$).
    - From above we know that the dynamics of c is then given by:

$$c_{t+1} = \beta \left[F'(k_{t+1}) + 1 - \delta\right] c_t \tag{30}$$

$\Longrightarrow$ A higher value of $F'(k_{t+1})$ implies (everything else held constant) that c_{t+1} is relatively higher compared to c_t (**= intertemporal substitution effect**).

Model solution: The two-period case

- Solution approach 2: Lagrange approach:
  - The household's maximization problem is given by:

$$\max_{c_t, c_{t+1}, k_{t+1}, k_{t+2}} V_t = \sum_{s=0}^{1} \beta^s U(c_{t+s}) = U(c_t) + \beta U(c_{t+1}) \tag{31}$$

s.t.

$$c_t + k_{t+1} = F(k_t) + (1-\delta)k_t \tag{32}$$

$$c_{t+1} = F(k_{t+1}) + (1-\delta)k_{t+1} \tag{33}$$

where we have used that

$$k_{t+2} = 0 \tag{34}$$

Model solution: The two-period case

- Solution approach 2 (continued):
    - The associated Lagrange function is given by:

$$\begin{aligned} \mathcal{L} &= U(c_t) + \beta U(c_{t+1}) + \\ &\quad + \lambda_t \left[F(k_t) + (1-\delta)k_t - c_t - k_{t+1} \right] + \\ &\quad + \lambda_{t+1} \left[F(k_{t+1}) + (1-\delta)k_{t+1} - c_{t+1} \right] \\ &= \sum_{s=0}^{1} \left\{ \beta^s U(c_{t+s}) + \lambda_{t+s} \left[F(k_{t+s}) + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1} \right] \right\} \end{aligned} \quad (35)$$

with $k_{t+2} = 0$

Model solution: The two-period case

- Solution approach 2 (continued):
  - The first-order conditions of the maximization problem are given by:
    - With respect to c_t:

$$\frac{\partial \mathcal{L}}{\partial c_t} \overset{!}{=} 0 \Longleftrightarrow U'\left(c_t\right) - \lambda_t = 0 \Longleftrightarrow \beta^0 U'\left(c_t\right) = \lambda_t \tag{36}$$

    - With respect to c_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} \overset{!}{=} 0 \Longleftrightarrow \beta U'\left(c_{t+1}\right) - \lambda_{t+1} = 0 \Longleftrightarrow \beta^1 U'\left(c_{t+1}\right) = \lambda_{t+1} \tag{37}$$

    - With respect to k_{t+1}:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial k_{t+1}} \overset{!}{=} 0 \quad &\Longleftrightarrow \quad -\lambda_t + \lambda_{t+1}\left[F'(k_{t+1}) + (1-\delta)\right] = 0 \qquad (38) \\ &\Longleftrightarrow \quad \lambda_t = \lambda_{t+1}\left[F'(k_{t+1}) + (1-\delta)\right] \end{aligned}$$

Model solution: The two-period case

- Solution approach 2 (continued):
  - First-order conditions of the maximization problem (continued):
    - With respect to λ_t:

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} \overset{!}{=} 0 \iff F(k_t) + (1-\delta)k_t - c_t - k_{t+1} = 0 \tag{39}$$
$$\iff c_t + k_{t+1} = F(k_t) + (1-\delta)k_t$$

    - With respect to λ_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+1}} \overset{!}{=} 0 \iff F(k_{t+1}) + (1-\delta)k_{t+1} - c_{t+1} = 0 \tag{40}$$
$$\iff c_{t+1} = F(k_{t+1}) + (1-\delta)k_{t+1}$$

  - Using equations (36) and (37) to replace λ_t and λ_{t+1} in equation (38) we obtain **the intertemporal Euler equation:**

$$U'(c_t) = \beta\left[F'(k_{t+1}) + 1 - \delta\right] U'(c_{t+1}) \tag{41}$$

Model solution: The infinite-horizon case

- In the infinite-horizon case, the household's maximization problem is given by:

$$\max_{c_t, c_{t+1}, \ldots; k_{t+1}, k_{t+2}, \ldots} V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s}\right) \tag{42}$$

s.t.

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta)k_{t+s}, \; \forall s \geq 0 \tag{43}$$

- To solve the model we employ the Lagrange approach.
- The Lagrange function is given by:

$$\mathcal{L} = \sum_{s=0}^{\infty} \left\{\beta^s U\left(c_{t+s}\right) + \lambda_{t+s}\left[F(k_{t+s}) + (1-\delta)k_{t+s} - c_{t+s} - k_{t+s+1}\right]\right\}$$

$\Longrightarrow$ Maximize with respect to $\left\{c_{t+s}, k_{t+s+1}, \lambda_{t+s}; \; s \geq 0\right\}$

Model solution: The infinite-horizon case

- The first-order condition with respect to c_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = 0 \Leftrightarrow \beta^s U'(c_{t+s}) = \lambda_{t+s} \tag{44}$$

- The first-order condition with respect to k_{t+s+1} is given by:

$$\frac{\partial \mathcal{L}}{\partial k_{t+s+1}} = 0 \Leftrightarrow \lambda_{t+s} = \lambda_{t+s+1}\left[F'(k_{t+s+1}) + 1 - \delta\right] \tag{45}$$

- The first-order condition with respect to λ_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+s}} = 0 \Leftrightarrow c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta)k_{t+s} \tag{46}$$

- Additionally, the following transversality condition must be satisfied:

$$\lim_{s\to\infty} \lambda_{t+s} k_{t+s+1} = \lim_{s\to\infty} \beta^s U'(c_{t+s}) k_{t+s+1} = 0 \tag{47}$$

Model solution: The infinite-horizon case

- Putting together the two first-order conditions yields:

$$U'(c_t) = \beta\left[F'(k_{t+1}) + 1 - \delta\right] U'(c_{t+1}) \iff \frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{1}{1 + F'(k_{t+1}) - \delta} \tag{48}$$

$\Longrightarrow$ Intertemporal Euler equation.

- Alternative interpretation: In the optimum, the marginal rate of substitution between consumption today and tomorrow must be equal to the physical rate of transformation.

Model solution: The infinite-horizon case

- An equilibrium/The optimum of the model is characterized by the following:
  - Consumption levels c_{t+s} and capital stock choices k_{t+s+1} must solve the following coupled system of non-linear difference equations

$$U'(c_{t+s}) = \beta U'(c_{t+s+1}) \left[1 + F'(k_{t+s+1}) - \delta\right] \tag{49}$$

    and

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta)k_{t+s} \tag{50}$$

    $\Longrightarrow$ The two equation constitute a system of two nonlinear difference equations in c and k.

  - The boundary (nonnegativity) conditions, the given initial conditions k_0 and the transversality condition must be satisfied.

Model solution: Long-run equilibrium

- In the long-run equilibrium/steady state we have:

$$c_t = c_{t+1} = c^* \tag{51}$$

and

$$k_t = k_{t+1} = k^* \tag{52}$$

- For the first-order conditions (equations (49) and (50)) we then obtain:

$$U'(c^*) = \beta U'(c^*) \left[1 + F'(k^*) - \delta\right] \tag{53}$$

and

$$c^* + k^* = F(k^*) + (1 - \delta)k^* \tag{54}$$

Model solution: Long-run equilibrium

- This can be simplified to:

$$1 = \beta \left[1 + F'(k^*) - \delta\right] \tag{55}$$

and

$$c^* = F(k^*) - \delta k^* \tag{56}$$

- The only unknown variable in the first equation is k^*.
- To obtain the steady-state value of k we thus can simply solve the first equation for k.
- The solution is given by:

$$F'\left(k^*\right) = \frac{1}{\beta} - 1 + \delta \Longleftrightarrow k^* = F'^{-1}\left(\frac{1}{\beta} - 1 + \delta\right) \tag{57}$$

Model solution: Long-run equilibrium

- Thus,
  - a higher degree of patience (a higher value of β) corresponds to a higher value of k and
  - a higher depreciation rate corresponds to a lower steady-state level of k.
- Please note that the steady-state capital stock is independent of consumption.
- The steady-state level of c^* is then given by:

$$c^* = F(k^*) - \delta k^* \tag{58}$$

Model solution: Model dynamics (graphical solution)

- As shown above the dynamics of the model is determined by the two difference equations:

$$U'(c_{t+s}) = \beta U'(c_{t+s+1}) \left[1 + F'(k_{t+s+1}) - \delta\right] \tag{59}$$

and

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta)k_{t+s} \tag{60}$$

- To obtain a concrete solution we make specific assumptions concerning the utility and the production function.
- We assume that the consumer's period utility function is given by:

$$U(c_t) = \ln(c_t) \tag{61}$$

- The production technology of the economy is Cobb-Douglas and thus given by:

$$y_t = F(k_t) = k_t^{\alpha} \text{ with } 0 < \alpha < 1 \tag{62}$$

Model solution: Model dynamics (graphical solution)

- The two first-order conditions then become:

$$\begin{aligned} U'(c_{t+s}) &= \beta U'(c_{t+s+1}) \left[1 + F'(k_{t+s+1}) - \delta\right] \quad (63) \\ \frac{1}{c_{t+s}} &= \beta \frac{1}{c_{t+s+1}} \left[1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right] \Longleftrightarrow \\ c_{t+s+1} &= \beta \left[1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right] c_{t+s} \Longleftrightarrow \\ c_{t+s+1} - c_{t+s} = \Delta c_{t+s+1} &= \beta \left[1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right] c_{t+s} - c_{t+s} \Longleftrightarrow \\ c_{t+s+1} - c_{t+s} = \Delta c_{t+s+1} &= \left\{\beta \left[1 + \alpha k_{t+s+1}^{\alpha-1} - \delta\right] - 1\right\} c_{t+s} \end{aligned}$$

and

$$\begin{aligned} c_{t+s} + k_{t+s+1} &= F(k_{t+s}) + (1-\delta)k_{t+s} \Longleftrightarrow \quad (64) \\ k_{t+s+1} - k_{t+s} &= \Delta k_{t+s+1} = F(k_{t+s}) - \delta k_{t+s} - c_{t+s} \end{aligned}$$

Model solution: Model dynamics (graphical solution)

- To illustrate the dynamics of the model we can use a phase diagram.
- To construct such a diagram we proceed as follows:
  - First, set the left-hand side of the Euler equation equal to zero and solve for the right-hand side for c_{t+s}. This yields:

$$\left\{\beta\left[1+\alpha k_{t+s+1}^{\alpha-1}-\delta\right]-1\right\} c_{t+s}=0 \Longleftrightarrow \tag{65}$$

$$k_{t+s+1}=k^{*}=\left(\frac{\alpha}{\frac{1}{\beta}-1+\delta}\right)^{\frac{1}{1-\alpha}}$$

$\Longrightarrow$ Plot this "function" in a c-k diagram.

  - Secondly, set the left-hand side of the budget constraint equal to zero and solve for the right-hand side for c_{t+s}. This yields:

$$F(k_{t+s})-\delta k_{t+s}-c_{t+s}=0 \Longleftrightarrow \tag{66}$$

$$c_{t+s}=F(k_{t+s})-\delta k_{t+s}$$

$\Longrightarrow$ Plot this "function" in a c-k diagram.

Model solution: Model dynamics (graphical solution)

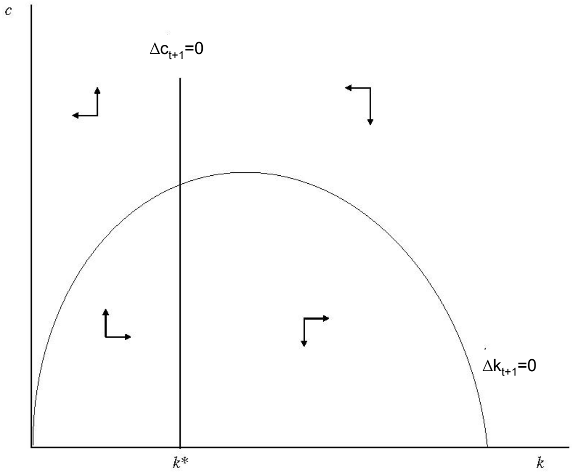
- Construction of a phase diagram (continued):
  - The intersection of both steady-state relations defines the steady state of the system. At this steady state, all first-order conditions of households and firms as well as the budget and resource constraints are satisfied.
  - To characterize the dynamics around steady state, consider the dynamics of capital if consumption is below/above the level that would stabilize k, i.e., below/above the steady-state budget constraint:

    $\Longrightarrow$ A low/high level of c_t implies that k_t is increasing/falling.
  - Next, consider the dynamics of c_t if k_t is below/above the level that would stabilize consumption, i.e., "below/above the steady-state Euler equation:"

    $\Longrightarrow$ A low/high level of k_t implies that c_t is increasing/falling.
  - Indicate the just derived dynamics of c_t and k_t apart from the zero-movement lines with corresponding arrows.

Model solution: Model dynamics (graphical solution)

- Phase diagram for model solution:



Model simulation and discussion

- To draw quantitative implications the model is simulated.
- Unfortunately, the system of the two nonlinear difference equations in c and k which characterize the dynamics of the economy in the optimum does not have an analytical solution.
  $\Longrightarrow$ To simulate the model the nonlinear difference equations are linearly approximated around the long-run equilibrium.
- Basic procedure:
  - First, compute the long-run steady state.
  - Secondly, log-linearize the system around the steady-state (All variables are expressed in terms of percentage deviations from the steady state).
  - Thirdly, calibrate the model (i.e. determine values for the model parameters.)
  - Forthly, simulate the model and compare its dynamic properties with those found in the data.

Model simulation and discussion

- Model setup:
  - The consumer's period utility function is given by:
$$U(c_t) = \ln(c_t) \tag{67}$$
  - The production technology of the economy is Cobb-Douglas and thus given by:
$$y_t = a_t F(k_t) = a_t k_t^\alpha \tag{68}$$
  - We assume that $0 < \alpha < 1$.
  - (Log) Total factor productivity is random and follows an AR(1) process
$$\ln(a_{t+1}) = \rho \ln(a_t) + \varepsilon_{t+1} \tag{69}$$
where $0 < \rho < 1$ and ε_{t+1} is Gaussian white noise with initial realization a_0 given.

Model simulation and discussion

- Calibration:
  - We assume that the parameters take the following values:

$$\alpha = 0.33 \tag{70}$$

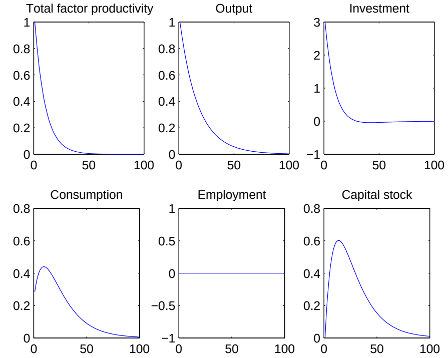
$$\delta = 0.04 \tag{71}$$

$$\beta = 0.99 \tag{72}$$

$$\rho = 0.95 \tag{73}$$

Model simulation and discussion

- Effects of a one-time increase in total factor productivity:

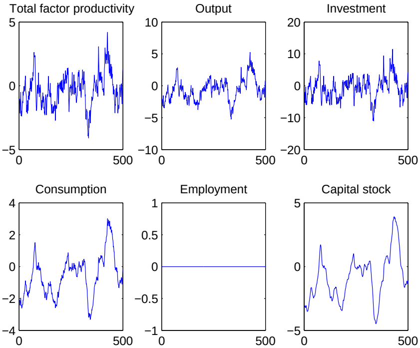


$\Longrightarrow$ Positive effect on output, consumption and investment.

$\Longrightarrow$ Investment reacts stronger than consumption.

Model simulation and discussion

- Model simulation over 500 periods:



$\Longrightarrow$ Positive comovements: $corr(y, c) \approx 0.73$, $corr(y, i) \approx 0.71$

$\Longrightarrow$ Relative volatilities: $\frac{\sigma_c}{\sigma_y} \approx 0.77$, $\frac{\sigma_i}{\sigma_y} \approx 2.01$

Labor in the basic model

- Thus far, we assumed that the household supplies a fixed amount of labor, n_t, in every period.
- More specifically, we assumed that the overall amount of time in a given period is 1 and that

$$n_t = 1 \tag{74}$$

- In this subsection, we model the labor supply decision explicitly.
- To this end, we include labor (leisure) both into the period-utility function and the production function.
- The period-utility function is now given by:

$$U(.) = U\left(c_t, l_t\right) \tag{75}$$

where l_t denotes leisure time.

Labor in the basic model

- We continue to assume that the overall amount of time is normalized to 1.
- Then we have:

$$n_t + l_t = 1 \tag{76}$$

- We assume that the period-utilitiy function satisfies the following conditions:

$$\frac{\partial U(c,l)}{\partial c} = U_c(c,l) > 0, \quad U_{cc}(c,l) < 0 \tag{77}$$

and

$$U_l(c,l) > 0, \quad U_{ll}(c,l) < 0, \quad U_{cl}(c,l) = 0 \tag{78}$$

Labor in the basic model

- Output (GDP) is produced using the following production technology:

$$y_t = F\left(a_t, k_t, n_t\right) \tag{79}$$

- The production function is assumed to satisfy all the conditions stated above.
- For simplicity of notation we assume $a_t = 1$.

Labor in the basic model

- The household's maximization problem is given by:

$$\max_{c_t,c_{t+1},\dots;k_{t+1},k_{t+2},\dots;l_t,l_{t+1},\dots} V_t = \sum_{s=0}^{\infty} \beta^s U\left(c_{t+s,l_{t+s}}\right) \tag{80}$$

s.t.

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}, n_{t+s}) + (1-\delta)k_{t+s}, \; \forall s \geq 0 \tag{81}$$

and

$$n_t + l_t = 1 \tag{82}$$

- To solve the model we employ the Lagrange approach.

Labor in the basic model

- The Lagrange function is given by:

$$
\begin{aligned}
\mathcal{L} \;=\; & \sum_{s=0}^{\infty} \left\{ \beta^s U\left(c_{t+s}, l_{t+s}\right) + \lambda_{t+s}\left[F(k_{t+s}, n_{t+s}) + (1-\delta)k_{t+s} - c_{t+s} \right.\right. \\
& \left.\left. -k_{t+s+1}\right] + \mu_{t+s}\left[1 - n_{t+s} - l_{t+s}\right]\right\}
\end{aligned}
$$

$\Longrightarrow$ Maximize with respect to
$\{c_{t+s}, l_{t+s}, n_{t+s}, k_{t+s+1}, \lambda_{t+s}, \mu_{t+s};\ s \geq 0\}$

Labor in the basic model

- The first-order condition with respect to c_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = 0 \Leftrightarrow \beta^s U_c\left(c_{t+s}, l_{t+s}\right) = \lambda_{t+s} \tag{83}$$

- The first-order condition with respect to l_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial l_{t+s}} = 0 \Leftrightarrow \beta^s U_l\left(c_{t+s}, l_{t+s}\right) = \mu_{t+s} \tag{84}$$

- The first-order condition with respect to n_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial n_{t+s}} = 0 \Leftrightarrow \lambda_{t+s} F_n\left(k_{t+s}, n_{t+s}\right) = \mu_{t+s} \tag{85}$$

Labor in the basic model

- The first-order condition with respect to k_{t+s+1} is given by:

$$\frac{\partial \mathcal{L}}{\partial k_{t+s+1}} = 0 \Leftrightarrow \lambda_{t+s} = \lambda_{t+s+1} \left[F_k(k_{t+s+1}, n_{t+s+1}) + 1 - \delta \right] \quad (86)$$

- The first-order condition with respect to λ_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial \lambda_{t+s}} = 0 \Leftrightarrow c_{t+s} + k_{t+s+1} = F(k_{t+s}, n_{t+s}) + (1-\delta) k_{t+s} \quad (87)$$

- The first-order condition with respect to μ_{t+s} is given by:

$$\frac{\partial \mathcal{L}}{\partial \mu_{t+s}} = 0 \Leftrightarrow n_{t+s} + l_{t+s} = 1 \quad (88)$$

Labor in the basic model

- Combining the first-order condition with respect to consumption (equation (83)) with the first-order condition with respect to capital (equation (86)) yields:

$$U_c\left(c_{t+s}, l_{t+s}\right) = \beta\left[F_k\left(k_{t+s+1}, n_{t+s+1}\right) + 1 - \delta\right] U_c\left(c_{t+s+1}, l_{t+s+1}\right)$$
$$\frac{\beta U_c(c_{t+s+1}, l_{t+s+1})}{U_c(c_{t+s}, l_{t+s})} = \frac{1}{1 + F_k(k_{t+s+1}, n_{t+s+1}) - \delta}$$

$\Longrightarrow$ Intertemporal Euler equation.

- Combining equations (83), (84) and (85) yields:

$$\beta^s U_l\left(c_{t+s}, l_{t+s}\right) = \beta^s U_c\left(c_{t+s}, l_{t+s}\right) F_n\left(k_{t+s}, n_{t+s}\right) \Leftrightarrow \tag{89}$$
$$U_l\left(c_{t+s}, l_{t+s}\right) = F_n\left(k_{t+s}, n_{t+s}\right) U_c\left(c_{t+s}, l_{t+s}\right)$$

$\Rightarrow$ Interpretation?

Labor in the basic model

- Example:
  - The consumer's period utility function is given by:

$$U\left(c_t, l_t\right) = \ln(c_t) + b \ln l_t \tag{90}$$

  - The production technology of the economy is Cobb-Douglas and given by:

$$y_t = F\left(a_t, k_t, n_t\right) = a_t k_t^{\alpha} n^{1-\alpha} \tag{91}$$

  - We assume that $0 < \alpha < 1$.
  - The first derivatives of the utility function are given by:

$$U_c\left(c_t, l_t\right) = \frac{1}{c_t} \text{ and } U_l\left(c_t, l_t\right) = \frac{b}{l_t} \tag{92}$$

  - The first derivatives of the production function are given by:

$$F_k\left(a_t, k_t, n_t\right) = \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} \text{ and } F_n\left(a_t, k_t, n_t\right) = \left(1-\alpha\right) a_t k_t^{\alpha} n_t^{-\alpha}$$

Labor in the basic model

- Example (... continued):
  - For the optimality condition

$$\frac{U_l\left(c_{t+s}, l_{t+s}\right)}{U_c\left(c_{t+s}, l_{t+s}\right)} = F_n\left(a_{t+s}, k_{t+s}, n_{t+s}\right). \tag{94}$$

  we then get:

$$\frac{bc_t}{l_t} = (1-\alpha)\, a_t k_t^{\alpha} n_t^{-\alpha} \Leftrightarrow l_t = \frac{bc_t}{(1-\alpha)\, a_t k_t^{\alpha} n_t^{-\alpha}} \tag{95}$$

  - Using $n_t = 1 - l_t$ we obtain:

$$1 - n_t = \frac{bc_t}{(1-\alpha)\, a_t k_t^{\alpha} n_t^{-\alpha}} \Leftrightarrow n_t = 1 - \frac{bc_t}{(1-\alpha)\, a_t k_t^{\alpha} n_t^{-\alpha}} \tag{96}$$

  $\Rightarrow$ Interpretation? Exercise: Calculate the result for the other optimality condition.

Labor in the basic model

- An equilibrium/The optimum of the model (assuming general functional forms) is characterized by the following:
  - Consumption levels c_{t+s}, leisure (l_{t+s}) and labor (n_{t+s}) decisions and capital stock choices k_{t+s+1} must satisfy the following system of equations

$$U_c(c_{t+s}, l_{t+s}) = \beta U_c(c_{t+s+1}, l_{t+s+1}) \left[1 + F_k(k_{t+s+1}, n_{t+s+1}) - \delta\right]$$

$$\frac{U_l\left(c_{t+s}, l_{t+s}\right)}{U_c\left(c_{t+s}, l_{t+s}\right)} = F_n\left(k_{t+s}, n_{t+s}\right)$$

$$c_{t+s} + k_{t+s+1} = F(k_{t+s}) + (1-\delta)k_{t+s}$$

and

$$n_t + l_t = 1 \tag{97}$$

$\Longrightarrow$ The first and third equation represent two nonlinear difference equations in c and k, the second and forth equations are "intra-temporal" equations.

Labor in the basic model

- An equilibrium/The optimum of the model is characterized by the following (... continued):
  - The boundary (nonnegativity) conditions, the given initial conditions k_0 and the transversality condition must be satisfied.