

**Volkswirtschaftliche  
Diskussionsbeiträge**



**Multidimensional Transitional Dynamics:  
A Simple Numerical Procedure**

**Timo Trimborn, University of Hamburg  
Karl-Josef Koch, University of Siegen  
Thomas M. Steger, ETH Zurich**

**Discussion Paper No. 121-05**

ISSN 1433-058x

**UNIVERSITÄT SIEGEN  
FACHBEREICH WIRTSCHAFTSWISSENSCHAFTEN**

# Multi-Dimensional Transitional Dynamics: A Simple Numerical Procedure

Timo Trimborn, University of Hamburg\*  
Karl-Josef Koch, University of Siegen  
Thomas M. Steger, ETH Zurich

December 2004

## Abstract

Growth models often give rise to saddle-point stable dynamic systems with multi-dimensional stable manifolds. It is argued that standard solution procedures used to numerically approximate the transition process are generally inadequate when the (stable) eigenvalues differ substantially in magnitude. Therefore, the relaxation procedure is proposed as a powerful method for simulating the transition process in dynamic macroeconomic models. We argue that this procedure is in general well-suited and highly efficient. The procedure can be easily applied to dynamic systems which exhibit the above mentioned structural characteristics. This is demonstrated by simulating the transition process of the well-known Jones (1995) model.

*JEL classification:* C61; C63; O40

*Keywords:* Saddlepoint problems; Transitional dynamics; Economic growth; Multidimensional stable manifolds

---

\*Timo Trimborn: Department of Economics, University of Hamburg, 20146 Hamburg, Germany, trimborn@econ.uni-hamburg.de, Phone: + 49.40.42838 - 2082, Fax: + 49.40.42838 - 6314.

## 1 Introduction

Dynamic macroeconomic theory nowadays relies heavily on infinite horizon optimization models which usually give rise to a system of nonlinear differential equations. This dynamic system is then interpreted to describe the evolution of the economy under consideration. Many studies in the field of growth theory have confined their analysis to the balanced growth path (BGP). A comprehensive understanding of the respective model under study requires, however, to additionally investigate the transition process. At least two important arguments support this view: First, the positive and normative implications might differ dramatically depending on whether an economy converges to the BGP or grows along the BGP (e.g. Jones, 1995). Second, dynamic macroeconomic models are often employed to conduct comparative welfare investigations of different policy regimes or instruments. In this context, the transition process needs to be taken into account. Linearizing the dynamic system might be appropriate in many cases but can be potentially misleading especially when the analysis aims at a Pareto-ranking of different policy instruments. This overall perspective is best summarized by the following statement due to Jonathan Temple (2003, p. 509): *Ultimately, all that a long-run equilibrium of a model denotes is its final resting point, perhaps very distant in the future. We know very little about this destination, and should be paying more attention to the journey.*

Especially in growth theory the models under study are very often multi-dimensional in the sense that there is more than one (predetermined) state variable. Usual stability properties then imply that the stable manifold is also multi-dimensional.<sup>1</sup> Examples for models which fit into this class comprise R&D-based growth models (e.g. Romer, 1990; Jones, 1995; Eicher and Turnovsky, 1999) as well as human-capital based growth models (e.g. Lucas,

---

<sup>1</sup>In the case of saddle-point stability, the dimension of the stable manifold equals the dimension of the state space, while indeterminacy implies that the dimension of the stable manifold exceeds the dimension of the state space.

1988; Mulligan and Sala-i-Martin, 1993; Benhabib and Perli, 1994). Moreover, if the dynamic system is characterized by (stable) eigenvalues which differ substantially in magnitude, then usual procedures are either not applicable or highly inefficient.<sup>2</sup> It is important to notice that this characteristic property is not at all a special (or even pathological) case but instead occurs quite frequently. Indeed, employing the well-known Jones (1995) model we will show that this property results from usual calibrations.

The paper at hand contributes to the literature on dynamic macroeconomic theory by proposing the relaxation procedure as a powerful method to simulate the transition process in growth models. We will argue that this procedure is in general well-suited and highly efficient. Moreover, this procedure can be easily applied even when the stable manifold is multi-dimensional and the stable eigenvalues differ substantially. This will be demonstrated by simulating the transition process of the well-known Jones (1995) model, which implies a two-dimensional stable manifold with potentially non-monotonic adjustments and shows (stable) eigenvalues which differ drastically in magnitude.<sup>3</sup>

Turning to the related literature, there are, of course, a number of procedures to simulate the transition process of dynamic macroeconomic models. In the context of growth theory, the most prominent approaches comprise shooting (e.g. Judd, 1998, Chapter 10), time elimination (Mulligan and Sala-i-Martin, 1991), backward integration (Brunner and Strulik, 2002), the projection method (Judd, 1992) as well as the discretization method of Merencier and Michel (1994). The similarities and differences of the relaxation procedure and the methods mentioned above will be discussed concisely in Section 2.4. We will argue that the relaxation procedure is largely superior to existing methods and can easily deal with problems such

---

<sup>2</sup>In the mathematical literature, differential equations exhibiting this structural characteristic are labelled "stiff differential equations".

<sup>3</sup>To the best of our knowledge, there is no study simulating the transition process of the Jones (1995) or related models such as Eicher and Turnovsky (1999).

as non-monotonic adjustments and stiff differential equations.

The paper is structured as follows: In Section 2, the relaxation procedure is first described concisely, then evaluated numerically employing the Ramsey-Cass-Koopmans model and eventually compared to other existing procedure. In Section 3, we apply this procedure to simulate the transition process of the well-known Jones (1995) model. Finally, Section 4 summarizes and concludes. The appendix (Section 5) provides a more formal description of the relaxation algorithm.

## **2 The relaxation procedure**

### **2.1 Description of the relaxation procedure**

The principle of relaxation can be applied to various numerical problems. Here we use it to solve a differential equation numerically. Relaxation type algorithms applied to differential equations have two very useful properties. First of all, they can easily scope with boundary conditions, such as initial conditions for state variables and transversality conditions of optimal growth. Second, additional equations, e.g. equilibrium conditions or feasibility constraints, can be incorporated straight away. Beyond, by transformation of the (independent) time variable one can solve infinite horizon problems with, as they arise from many dynamic optimization problems in economics.

Suppose we want to compute a numerical solution of a differential equation in terms of a large (finite) sequence of points representing the desired path. To start with, we take an arbitrary trial solution, typically not satisfying the slope conditions implied by the differential equation nor the boundary conditions. We measure the deviation from the true path by a multi-dimensional error function and use the derivative of the error function to improve the trial solution in a Newton type iteration. Hence, at each point of the path the correction is related to the particular inaccuracy in slope and

in solving the static equation. The crucial difference to the various shooting methods is the simultaneous adjustment along the path as a whole.

The outline of the algorithm proposed in this paper leans heavily on Press, Flannery, Teukolsky and Vetterling (Press et al. 1989, pp. 645-672). We have implemented the algorithm in matlab. The code is published for free download in the internet<sup>4</sup> and a print version is available on request.<sup>5</sup>

We apply the method to the following problem: Consider a system of  $\tilde{N}$  ordinary differential equations together with  $N - \tilde{N}$  (static) equations in  $N$  real variables. This system describes a vector field on a  $(N - \tilde{N})$ -dimensional surface in  $\mathbb{R}^N$ . We impose a list of  $n_1$  boundary conditions at the starting point and  $n_2$  at the end point of a path sufficient to determine a particular trajectory. To meet all dimensional requirements  $n_1$  and  $n_2$  must add up to  $\tilde{N}$ .

For the finite representation of the problem we fix a time mesh of  $M$  points in time. In case of an infinite time horizon we choose a transformation to map the interval  $[0, \infty]$  to  $[0, 1]$ . At each point of time an  $N$ -dimensional vector has to be determined. We approximate the differential equation by  $M-1$  systems of equations of dimension  $\tilde{N}$  for the slope between neighboring mesh points. Together with  $\tilde{N}$  boundary conditions we have an  $M \times \tilde{N}$  dimensional system of equations. After adding the  $n_2$  static equations which have to hold at each of the  $M$  mesh points we have incorporated all restrictions available. The final system of nonlinear equations is of dimension  $M \times N$  and involves the same number of unknowns.

We apply a Gauß-Newton procedure to compute a root of this system. Step by step we adjust the trial solution until the error is sufficiently small. This involves the solution of a linear equation with the Jacobian matrix of the system of nonlinear equations. At first glance there seems little chance to achieve good solutions because the complexity of the problem is proportional

---

<sup>4</sup><http://www.rrz.uni-hamburg.de/IWK/trimborn/relaxate.htm>

<sup>5</sup>In the appendix we give a detailed description of the algorithm.

to the size of the Jacobian matrix which is quadratic in  $M$ . However, the Jacobian is not an arbitrary matrix of dimension  $M \times N$ .

The Jacobian matrix inherits a specific structure from the approximation of the differential equation. The boundary conditions and the static equations each depend only on one respective vector, and the interior slope conditions only on neighboring vectors. Hence the Jacobian matrix shows nonzero entries only close to the diagonal. This can be used to solve the linear system by a special version of a Gauß algorithm carried out recursively on  $N$ -dimensional blocks along the diagonal. This recursive procedure allows to increase the number  $M$  of mesh points without increasing the dimension of the blocks. Only the number of blocks increases in proportion to  $M$ . The complexity of the problem is only linear in the number of mesh points and not quadratic. Hence, a fairly good approximation of the continuous path is possible without using too much computer time.

## 2.2 Implementation of the algorithm

In this section we describe how the relaxation algorithm is applied to solve the Ramsey-Cass-Koopmans model (see Ramsey (1928), Cass (1965), Koopmans (1965)), since it is a simple growth model where the problem of a saddle point arises. The model gives rise to a system of two differential equations for consumption and capital in effective labor (see Barro and Sala-i-Martin (2004)):

$$\dot{c} = \frac{c}{\theta} (\alpha k^{\alpha-1} - (\delta + \rho + x\theta)) \quad (1)$$

$$\dot{k} = k^\alpha - c - (n + x + \delta)k \quad (2)$$

The elasticity of capital in production is denoted by  $\alpha$ , population growth with  $n$ , depreciation with  $\delta$  and the exogenous growth of technology by  $x$ . The parameter for time preference is  $\rho$  and the inverse of the intertemporal elasticity of substitution is denoted by  $\theta$ . The steady state is at  $k^* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}$  and  $c^* = (k^*)^\alpha - (n + x + \delta)k^*$  and saddle point stable.

First, a time mesh has to be chosen where the solution is calculated. We choose the mesh to be uniform in  $\tau$ , the transformed time scale.

Second, the two differential equations have to be transformed into two difference equations, taking into account the transformation of time as described in the appendix. These difference equations has to be satisfied between every two mesh points. For  $M$  mesh points this leads to  $2 \cdot (M - 1)$  nonlinear equations.

Third, two boundary condition have to be chosen to complete the set of equations to  $2 \cdot M$ . In this example the relaxation algorithm needs one initial boundary condition and one final boundary condition. Capital is the state variable and therefore we choose capital to be at 10% of its steady state value at  $t = 0$ . For the final boundary condition there are several possibilities to formulate an equation. It would be possible to choose each of the two equations (1) or (2) and set the RHS equal to zero. However, here the steady state values for consumption and capital can be computed analytically so we choose consumption to equal its steady state value as the final boundary condition. Note, that only one final boundary condition is needed. Thus the algorithm does not make use of the knowledge of the steady state value of capital. It is reached automatically.

At last an initial guess for the solution has to be made. We choose  $c$  and  $k$  to be constant at their steady state values  $(c_t, k_t) \equiv (c^*, k^*)$ . The Newton procedure always converged quickly, indicating a high degree of robustness regarding to the initial guess.

### 2.3 Evaluation of the procedure

For the special parametrization  $\theta = \frac{\delta + \rho}{\alpha(\delta + n + x) - x}$  the solution can be expressed analytically, because the representative consumer chooses a constant saving rate  $s = \frac{1}{\theta}$  (see Barro and Sala-i-Martin (2004)).<sup>6</sup> This allows us to

---

<sup>6</sup>The analytical solution is  $k(t) = \left[ \frac{1}{(\delta + n + x)\theta} + \left( k_0^{1-\alpha} - \frac{1}{(\delta + n + x)\theta} \right) e^{-(1-\alpha)(\delta + n + x)t} \right]^{\frac{1}{1-\alpha}}$  and  $c(t) = \left( 1 - \frac{1}{\theta} \right) k(t)^\alpha$ .



compare the computed results with the analytical solution, which has a precision close to the machine epsilon. The relative error is computed for every mesh point. Table 1 shows the maximum relative error of consumption and capital per capita for different number of mesh points for a common set of parameter values. In addition, the quadratic mean error of combined  $c$  and  $k$  provides information about the distribution of the error.<sup>7</sup> It can be seen in table 1 that multiplying the number of mesh points by  $x$  reduces the maximum error of each solution vector by the factor  $\frac{1}{x^2}$ , which indicates the order 2 of the difference procedure. Even with a moderate number of mesh points and therefore a short computation time a sufficient high degree of accuracy can be achieved. Moreover the accuracy can be improved to a very high degree just with increasing the number of mesh points. It should be mentioned that the allocation of the mesh was chosen exogenously. The accuracy of the algorithm could be improved with a self allocating time mesh.

number of mesh points	max error $c$	max error $k$	mean error
10	$< 1.3 \cdot 10^{-2}$	$< 3.4 \cdot 10^{-2}$	$< 3.0 \cdot 10^{-3}$
100	$< 1.1 \cdot 10^{-4}$	$< 8.6 \cdot 10^{-5}$	$< 2.7 \cdot 10^{-6}$
1,000	$< 1.1 \cdot 10^{-6}$	$< 8.5 \cdot 10^{-7}$	$< 8.2 \cdot 10^{-9}$
10,000	$< 1.1 \cdot 10^{-8}$	$< 8.5 \cdot 10^{-9}$	$< 2.6 \cdot 10^{-11}$
100,000	$< 1.1 \cdot 10^{-10}$	$< 8.5 \cdot 10^{-11}$	$< 8.2 \cdot 10^{-14}$

Table 1: Accuracy of the algorithm

The treatment of higher dimensional systems even with multi-dimensional stable manifolds is analogous, therefore the performance of the algorithm should be similar at more complicated models.

<sup>7</sup>It is defined as  $\epsilon = \frac{1}{NM} \cdot \sqrt{\sum_{i=1}^N \epsilon_{c_i}^2 + \sum_{i=1}^N \epsilon_{k_i}^2}$  with  $\epsilon_{c_i}$  and  $\epsilon_{k_i}$  defining the relative error of  $k$  and  $c$  respectively at mesh point  $i$ .

## 2.4 Comparison to other procedures

In this section we compare very briefly the relaxation procedure to common alternative solution methods. Other solution techniques for this problem are the backward integration (Brunner and Strulik 2002), multiple shooting (e.g. Judd (1998), Chapter 10), time elimination (Mulligan and Sala-i-Martin 1991), projection methods (e.g. Judd (1992), Judd (1998)) and the method of Mercenier and Michel (1994). We keep this section brief and restrict our comparison to related methods, because most of the procedures and their advantages are described in Judd (1998) and Brunner and Strulik (2002).

The method of backward integration as described in Brunner and Strulik (2002) exploits the numerical stability of the backward looking system by inverting time. By starting near the steady state of the transformed system the resulting initial value problem is stable and the solution converges towards the stable manifold quickly. Therefore the method can solve systems with single dimensional stable manifolds very conveniently. For multi-dimensional manifolds Brunner and Strulik (2002) suggest to generate starting values on an orbit around the steady state. To pass through a pre-specified point, which is determined by a specific shock under study, it is necessary to iterate until the trajectory hits this point. However, if the real parts of the eigenvalues associated with the stable manifold differ substantially, the problem of a stiff system of differential equations occurs. It is well-known that these problems are very hard to handle numerically. If the difference of the stable eigenvalues is sufficiently high, it is impossible to meet the pre-specified point, because the backward shooting trajectories will be attracted by the submanifold associated with the eigenvalue with the smallest real part. Therefore the resulting trajectories can not represent a specified shock and potentially have no economic meaning.

Mercenier and Michel (1994) propose to transform the continuous time, infinite horizon problem into a finite horizon maximization problem in discrete time with the same steady state. Maximizing the transformed problem

leads to a system of non-linear equations, which can be solved with a Newton algorithm similar as in the relaxation algorithm. Our approach is to solve the system of differential equations. Here the discretisation is done at a later stage. To apply our algorithm the researcher simply has to paste the system of differential equation into the algorithm instead of converting the whole maximization problem. Apart from the simplicity our approach has some further advantages.

First, our approach is more general, since the system of differential equations can be attained in different ways, not only by a single maximization problem. In particular the approach of Mercenier and Michel for steady state invariance requires the discount factor to be constant. However, if the firm also faces an intertemporal optimization problem the discount factor is the real interest rate which is not constant during transition. Therefore steady state invariance can not be guaranteed and the performance of the method deteriorates. Second, the relaxation algorithm we propose can deal with a compactification of the time interval. Therefore it is not necessary to choose an adequate terminal time where the optimization is truncated. Also the treatment of a post terminal stationary phase does not apply. Third, in the approach of Mercenier and Michel the way of discretisation is fixed. The relaxation algorithm leaves room for selecting different discretisation rules, also of higher order. This leads to a higher level of accuracy with the same number of mesh points. The discretisation rule of the method of Mercenier and Michel is a first order rule, where our approach uses a second order rule.<sup>8</sup>

Projection methods as they are introduced in Judd (1992) and Judd (1998) cover a very wide range of algorithms. They are considered to be fast and accurate, but they need a high programming effort. However, very often they are applied to solve for the policy function, not for the system of

---

<sup>8</sup>When multiplying the number of mesh point with  $x$  a first order rule leads to a reduction of the global error by  $\frac{1}{x}$  where a second order rule reduces the error by  $\frac{1}{x^2}$ .

differential equations. This does not work for cyclical adjustment or problems with multi-dimensional stable manifolds, because the policy function can not be computed at certain points. In addition, the polynomial bases and therefore the computation costs grow exponentially when the dimension of the problem increases. To avoid this “curse of dimensionality” a special complete polynomial basis is chosen. But then also the dimension of the basis grows polynomially compared to the relaxation algorithm where the cost of computation grows quadratically.<sup>9</sup>

For the method of time elimination part of the same critique holds. With multi-dimensional stable manifolds the policy functions can not be computed at certain points and therefore the method does not work.

### 3 An illustrative application

The relaxation procedure is employed to investigate the transition process of the well-known R&D-based semi-endogenous growth model of Jones (1995). This model is chosen since it implies a two-dimensional stable manifold. Moreover, for standard calibrations the two stable eigenvalues differ drastically (by about a factor of ten) and, hence, usual procedures are inappropriate to solve the underlying dynamic system.

#### 3.1 The Jones model

As in Jones (1995), the focus here is on the market solution. The final-output technology is given by  $Y = \alpha_F (\phi L)^{\sigma_L} \int_0^A x(i)^{1-\sigma_L} di$ , where  $Y$  denotes final output,  $\phi$  the share of labor allocated to final-output production,  $x(i)$  the amount of differentiated capital goods of type  $i$ ,  $A$  the number of differentiated capital goods,  $\alpha_F$  a constant overall productivity parameter and  $\sigma_L$

<sup>9</sup>For the example of the Jones (1995) model presented below this means a basis of 28 elements if the solution is approximated with linear functions, a basis of 55 elements if the solution is approximated with quadratic polynomials and smaller and a basis of 91 elements for an approximation with polynomials of degree three and smaller.

the elasticity of labor in final-output production. Noting the general symmetry among  $x(i)$  and using the definition of aggregate capital  $K := Ax$ , the final-output technology can be written as  $Y = \alpha_F (A\phi L)^{\sigma_L} K^{1-\sigma_L}$ . The R&D technology is  $\dot{A} = J = \alpha_J A^{\eta_A} [(1-\phi)L]^{\eta_L}$  with  $\eta_L := \eta_L^p + \eta_L^e$ ,  $\eta_L^p = 1, -1 < \eta_L^e < 0$ , where  $\dot{A} := dA/dt$ ,  $\alpha_J$  denotes a constant overall productivity parameter,  $\eta_A$  the elasticity of technology in R&D and  $\eta_L$  the elasticity of labor in R&D.

To simulate the transition process, one needs the complete dynamic system governing the evolution the economy under study. Moreover, we express all endogenous variables as normalized or scale-adjusted variables such that their long-run levels are constant. This dynamic system can be summarized as follows:<sup>10</sup>

$$\dot{k} = y - c - \delta k - \beta_K n k \quad (3)$$

$$\dot{a} = j - \beta_A n k \quad (4)$$

$$\dot{c} = \frac{c}{\gamma} [r - \delta - \rho - (1-\gamma)n] - \beta_K n c \quad (5)$$

$$v_a = v_a [r - (\beta_K - \beta_A)n] - \pi \quad (6)$$

$$\frac{\sigma_L y}{\phi} = v_a \frac{\eta_L^p j}{1-\phi} \quad (7)$$

with  $y = \alpha_F (a\phi)^{\sigma_L} k^{1-\sigma_L}$ ,  $j = \alpha_J a^{\eta_A} (1-\phi)^{\eta_L}$ ,  $r = \frac{(1-\sigma_L)^2 y}{k}$ ,  $\pi = \frac{\sigma_L(1-\sigma_L)y}{a}$ ,  $\beta_K = \frac{1-\eta_A+\eta_L}{1-\eta_A}$ ,  $\beta_A = \frac{\eta_L}{1-\eta_A}$ . The scale-adjusted variables are defined by  $y := Y/L^{\beta_K}$ ,  $k := K/L^{\beta_K}$ ,  $c := C/L^{\beta_K}$ ,  $a := A/L^{\beta_A}$ ,  $j := J/L^{\beta_A}$  and  $v_a := v/L^{\beta_K-\beta_A}$ . The (unique) stationary solution of this dynamic system corresponds to the (unique) balanced growth path of the economy expressed in original variables.

Equations (3) and (4) are the equations of motion of (scale-adjusted) capital and technology, (5) is the Keynes-Ramsey rule of optimal consumption  $c$ , (6) shows capital market equilibrium with  $v_a$  denoting the (scale-adjusted)

<sup>10</sup>For a detailed derivation of the dynamic system for the general R&D-based non-scale growth model see Steger (2004).

price of blueprints and (7) determines the privately efficient allocation of labour across final-output production and R&D.

### 3.2 Investigation of the transition process

Numerically, the problem is to solve a four dimensional system of differential equations (3) - (6) and simultaneously one static equation (7) that has to be satisfied at all times. The steady state is a saddle point with a two-dimensional stable manifold. Since the steady state can be computed only numerically the algorithm computes the steady state of the system first by applying a Newton algorithm. The choice of  $k(0) = k_0$  and  $a(0) = a_0$  as the initial boundary condition is obvious, since  $k$  and  $a$  are the state variables. Again for choosing the final boundary conditions there is some scope in setting the RHS of two of the four differential equations equal to zero. We take the equations (5) and (6). As an initial guess we choose again all variables to be constant at their steady state values. This always leads to quick convergence supporting the impression that the procedure is relatively robust regarding to the initial guess.

The transition process considered below results from a combination of two simultaneous shocks. Specifically, it is assumed that the overall productivity parameter in the production function for final output  $\alpha_F$  increases from 1.0 to 1.3, while the overall productivity parameter in the production function for new ideas  $\alpha_J$  decreases from 1.0 to 0.9. This shock was chosen to demonstrate that the adjustment can be non-monotonic and therefore the policy functions can not be computed at certain points with conventional methods, which can be seen in the phase diagrams in Figure 1 (vi).<sup>11</sup> Figure 1 gives a summary of the adjustment process. The plots (i) to (iii) show the time path of the jump variables  $c$ ,  $\phi$ ,  $v_a$ , plots (iv) and (v) display the time path of the state variables  $k$  and  $a$ , while plot (vi) contains the

<sup>11</sup>The set of parameters used for the simulations reads  $\sigma_L = 0.6$ ,  $\sigma_K = 0.4$ ,  $\delta = 0.05$ ,  $n = 0.015$ ,  $\eta_A = 0.6$ ,  $\eta_L = 0.5$ ,  $\eta_L^p = 0.6$ ,  $\rho = 0.04$  and  $\gamma = 1$ .

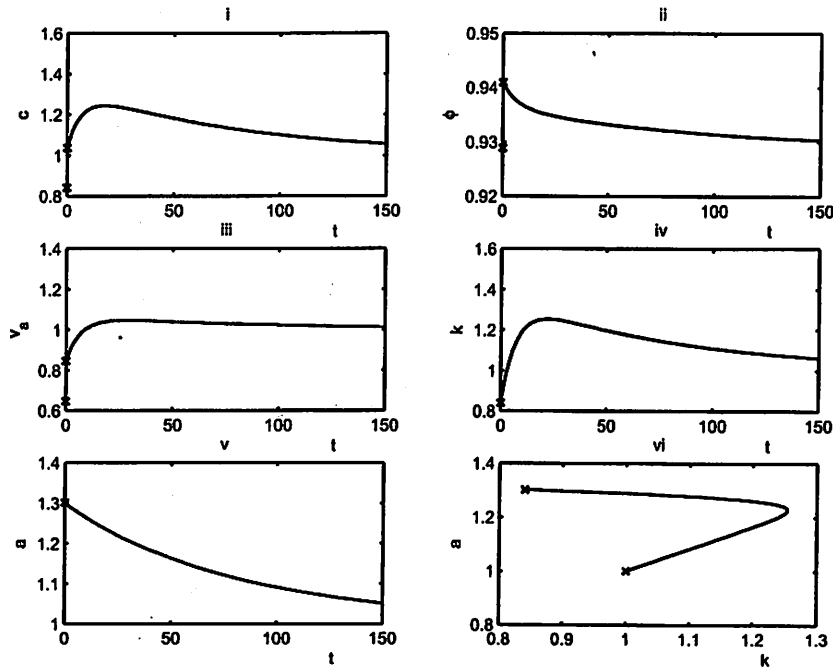


Figure 1: Summary of the transition of the Jones (1995) model

projection of the adjustment trajectory into the  $(k,a)$ -plane. Several aspects are worth being noticed: (1) The transition process shows a pronounced non-monotonicity for  $c$  and  $k$ . This overshooting pattern in scale-adjusted variables implies that the instantaneous growth rate of the respective original variable is initially above the long-run growth rate, then undershoots and finally converges to the long-run value. (2) The (average) speed of convergence appears to be fairly low with half-lives of more than 50 years. This observation underlines the importance of the analysis of transitional dynamics. (3) The intersectoral allocation variable  $\phi$  first jumps up [indicated by the crosses in plot(ii)] and then converges to the initial long-run value. This feature mirrors the basic non-scale character of the underlying model.

## 4 Summary and conclusion

In this paper, we propose the relaxation procedure as a powerful and efficient procedure to investigate the transition process of dynamic macroeconomic models. At a very general level, this procedure has two main advantages: First, it is simpler than most other procedures. Second, and more importantly, the relaxation procedure can easily deal with complex dynamic systems for which standard procedures are generally inadequate. More specifically, the procedure can readily handle dynamic systems which are characterized by multi-dimensional stable manifolds (with the potential of non-monotonic adjustment patterns) and strongly differing (stable) eigenvalues. It is important to notice that such dynamic systems are not at all special cases but arise quite frequently from a large number of widely employed growth models.

As an example, the relaxation procedure has been used to investigate the transition process of the well-known Jones (1995), which represents one of the basic workhorses in modern growth theory. Usual procedures turn out to be inadequate for the analysis of this model. This is probably the reason for the fact that there are only few studies which take the adjustment process of this or related models into account.<sup>12</sup>

## 5 Appendix

In this section we go through some details of the algorithm. Consider a system of  $\tilde{N}$  differential equations on an open set in  $\mathbb{R}^N$ , with  $\tilde{N} \leq N$ . Let  $\tilde{x}$  be the vector of those components of the full vector  $x \in \mathbb{R}^N$  affected by  $f$ .

$$\frac{d\tilde{x}}{dt} = f(t, x), \quad f: \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^{\tilde{N}}$$

---

<sup>12</sup>To the best of our knowledge, Papageorgiou and Pérez-Sebastián (2003) is the only study which, using the projection method of Judd (1992), simulates the adjustment process of an (extended) non-scale R&D-based growth model.



If  $\tilde{N}$  is strictly smaller than  $N$  the differential equations are to be supplemented by  $N - \tilde{N}$  equations  $x$  has to satisfy at any time.

$$0 = g(t, x), \quad g : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^{N-\tilde{N}}$$

Boundary conditions are supposed to be given in form of  $n_1$  initial conditions and  $n_2$  final conditions. For the solution to be well determined we need  $n_1 + n_2$  to equalize  $\tilde{N}$ . Finally, it is convenient to denote the codimension  $N - \tilde{N}$  of the manifold given by  $g = 0$  with  $n_3$ . Summing up we have

$$\begin{array}{ll} n_1 & \text{initial conditions} \\ n_2 & \text{final conditions} \\ n_3 & \text{running equations} \end{array} \quad \text{with } n_1 + n_2 + n_3 = \tilde{N} + n_3 = N$$

For convenience, we rescale the time range  $\mathbb{R}_+$  by introducing a new time parameter  $\tau$  running from 0 to 1

$$\tau = \nu t / (1 + \nu t)$$

In terms of  $\tau$  we get an equivalent differential-algebraic system

$$\begin{aligned} \frac{d\tilde{x}}{d\tau} &= \xi(\tau, x) = f\left(\frac{\tau}{\nu(1-\tau)}, x\right) / \nu(1-\tau)^2 \\ 0 &= \phi(\tau, x) = g\left(\frac{\tau}{\nu(1-\tau)}, x\right) \end{aligned} \quad (8)$$

Define a mesh of  $M$  points in (transformed) time  $\tau$  by  $T = \{\tau_1, \dots, \tau_M\}$ . Along the mesh, the dependent variable  $x$  falls into a list of vectors. To avoid confusion we denote it by  $y = \{y_1, \dots, y_M\}$  where  $y_k$  is the value of  $x$  at  $\tau_k$ . We use the midpoint of each interval  $(\tau_k, \tau_{k+1})$  for the discretization of the differential equation

$$\tilde{y}_{k+1} - \tilde{y}_k = (\tau_{k+1} - \tau_k) \xi(\bar{\tau}_k, \bar{y}_k) \quad \text{for } k = 1, \dots, M-1 \quad (9)$$

where  $\bar{\tau}_k = (\tau_k + \tau_{k+1})/2$  and  $\bar{y}_k = (y_k + y_{k+1})/2$ . An element of this sequence of difference equations yields an  $\tilde{N}$ -dimensional error function  $H : ([0, \dots, 1] \times \mathbb{R}^N)^2 \rightarrow \mathbb{R}^{\tilde{N}}$

$$H(\tau_k, y_k, \tau_{k+1}, y_{k+1}) = \tilde{y}_{k+1} - \tilde{y}_k - (\tau_{k+1} - \tau_k) \xi(\bar{\tau}_k, \bar{y}_k)$$

Note that the matrix of partial derivatives of  $H$  with respect to  $y_k$  and  $y_{k+1}$  differ only in their derivatives of  $\tilde{y}_{k+1}$  and  $\tilde{y}_k$ , respectively, and this is plus or minus the identity matrix of dimension  $\tilde{N}$ .

Let  $B$  denote the initial conditions

$$B : \mathbb{R}^N \rightarrow \mathbb{R}^{n_1},$$

$F$  denote the final conditions

$$F : \mathbb{R}^N \rightarrow \mathbb{R}^{n_2}$$

and let  $C$  denote the running conditions

$$C : [0, \dots, 1] \times \mathbb{R}^N \rightarrow \mathbb{R}^{n_3}$$

All together this defines a system of equations in  $y = (y_1, \dots, y_M) \in \mathbb{R}^{N \cdot M}$  given a mesh  $\tau = (\tau_1, \dots, \tau_M) \in \mathbb{R}^M$ , and we are looking for a root of this system.

For the description of the algorithm it is convenient to list the equations according to the unknown vectors  $y_k$  involved. We start with the initial conditions which only involve  $y_1$  and end with the equations which only involve  $y_M$ . Ordered this way the system can be seen as a system of  $M + 1$  vector equations  $E_0(y), \dots, E_M(y)$ . The first subsystem  $E_0(y)$  depends only on  $y_1$  and consists of  $n_1$  initial conditions. The intermediate subsystems  $E_k(y)$  for  $k = 1, \dots, M - 1$  depend on  $y_k$  and  $y_{k+1}$  and are of dimension  $N$ . Each of these subsystems begins with  $n_3$  running conditions and is completed by  $n_1 + n_2$  difference equations. The last subsystem  $E_M(y)$  depends on  $y_M$  and consists of  $n_3$  interior conditions together with  $n_2$  final conditions. It has dimension  $n_2 + n_3$ .



- step 0:* Diagonalize the first  $n_1$  columns of  $S^{0,R}$ .  
*step  $k$ ,  $k=1, \dots, M-1$ :* Eliminate the first  $N - n_1$  columns of  $S^{k,L}$ ; diagonalize the remainder of  $S^{k,L}$  together with the first  $N - n_1$  columns of  $S^{k,R}$ .  
*step  $M$ :* Eliminate the first  $N - n_1$  columns of  $S^{M,L}$ ; Diagonalize the remainder of  $S^{M,L}$ .  
*step  $M+k$ ,  $k=1, \dots, M$ :* Solve for  $\Delta y_{M+1-k}$ .

The Newton algorithm refines the current guess of  $y$  by adding  $\Delta y$  or a fraction of this vector to  $y$ . The algorithm stops if the error  $E$  is sufficiently small according to an appropriate norm.

## References

- BARRO, R. J., AND X. SALA-I-MARTIN (2004): *Economic Growth*. MIT-Press, Cambridge, Massachusetts.
- BENHABIB, J., AND R. PERLI (1994): "Uniqueness and Indeterminacy: On the Dynamics of Endogenous Growth," *Journal of Economic Theory*, 63, 113–142.
- BRUNNER, M., AND H. STRULIK (2002): "Solution of Perfect Foresight Saddlepoint Problems: A Simple Method and Applications," *Journal of Economic Dynamics and Control*, 26, 737–753.
- CASS, D. (1965): "Optimum Growth in an Aggregate Model of Capital Accumulation," *Review of Economic Studies*, 32, 233–240.
- EICHER, T. S., AND S. J. TURNOVSKY (1999): "Convergence in a Two-Sector Nonscale Growth Model," *Journal of Economic Growth*, 4, 413–428.
- JONES, C. I. (1995): "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 103(3), 759–784.

- JUDD, K. L. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58, 410-452.
- (1998): *Numerical Methods in Economics*. MIT Press.
- KOOPMANS, T. C. (1965): "On the Concept of Optimal Economic Growth," In: *The Econometric Approach to Development Planning*, Amsterdam: North Holland.
- LUCAS, R. E. JR.. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22, 3-42.
- MERCENIER, J., AND P. MICHEL (1994): "Discrete-Time Finite Horizon Approximation of Infinite Horizon Optimization Problems with Steady-State Invariance," *Econometrica*, 62(3), 635-656.
- MULLIGAN, C., AND X. SALA-I-MARTIN (1991): "A Note on the Time Elimination Method for Solving Recursive Growth Models," *Technical Working Paper No. 116*.
- (1993): "Transitional Dynamics in Two-Sector Models of Endogenous Growth," *Quarterly Journal of Economics*, 108(3), 737-773.
- PAPAGEORGIOU, C., AND F. PÉREZ-SEBASTIÁN (2002): "Human Capital and Convergence in a Non-Scale R&D Growth Model," *Departmental Working Papers, Department of Economics, Louisiana State University*, 2002-10.
- PRESS, W., B. FLANNERY, S. TEUKOLSKY, AND W. VETTERLING (1989): *Numerical Recipes in Pascal*. Cambridge University Press.
- RAMSEY, F. (1928): "A Mathematical Theory of Saving," *Economic Journal*, 38, 543-559.
- ROMER, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98, S71-S102.

STEGER, T. M. (2003): "Welfare Implications of Non-Scale R&D-based Growth Models," *Scandinavian Journal of Economics*, forthcoming.

TEMPLE, J. (2003): "The Long-Run Implications of Growth Theories," *Journal of Economic Surveys*, 17(3), 497-510.

### Liste der seit 1993 erschienenen Volkswirtschaftlichen Diskussionsbeiträge

Diese Liste, die Zusammenfassungen aller Volkswirtschaftlichen Diskussionsbeiträge und die Volltexte der Beiträge seit 1999 sind online verfügbar unter <http://www.uni-siegen.de/~vwiv/Dateien/diskussionsbeitraege.htm>. Ab dem Beitrag 60-97 können diese Informationen online auch unter der Adresse <http://ideas.repec.org> eingesehen werden. Anfragen nach Diskussionsbeiträgen sind direkt an die Autoren zu richten, in Ausnahmefällen an Prof. Dr. R. Pethig, Universität Siegen, 57068 Siegen.

### List of Economics Discussion Papers released as of 1993

This list, the abstracts of all discussion papers and the full text of the papers since 1999 are available online under <http://www.uni-siegen.de/~vwiv/Dateien/diskussionsbeitraege.htm>. Starting with paper 60-97, this information can also be accessed at <http://ideas.repec.org>. Discussion Papers can be only ordered from the authors directly, in exceptional cases from Prof. Dr. R. Pethig, University of Siegen, D- 57068 Siegen, Germany.

- 38-93 Reiner Wolff, Saddle-Point Dynamics in Non-Autonomous Models of Multi-Sector Growth with Variable Returns to Scale
- 39-93 Reiner Wolff, Strategien der Investitionspolitik in einer Region: Der Fall des Wachstums mit konstanter Sektorstruktur
- 40-93 Axel A. Weber, Monetary Policy in Europe: Towards a European Central Bank and One European Currency
- 41-93 Axel A. Weber, Exchange Rates, Target Zones and International Trade: The Importance of the Policy Making Framework
- 42-93 Klaus Schöler und Matthias Schlemper, Oligopolistisches Marktverhalten der Banken
- 43-93 Andreas Pfingsten and Reiner Wolff, Specific Input in Competitive Equilibria with Decreasing Returns to Scale
- 44-93 Andreas Pfingsten and Reiner Wolff, Adverse Rybczynski Effects Generated from Scale Diseconomies
- 45-93 Rüdiger Pethig, TV-Monopoly, Advertising and Program Quality
- 46-93 Axel A. Weber, Testing Long-Run Neutrality: Empirical Evidence for G7-Countries with Special Emphasis on Germany
- 47-94 Rüdiger Pethig, Efficient Management of Water Quality
- 48-94 Klaus Fiedler, Naturwissenschaftliche Grundlagen natürlicher Selbstreinigungsprozesse in Wasserressourcen
- 49-94 Rüdiger Pethig, Noncooperative National Environmental Policies and International Capital Mobility
- 50-94 Klaus Fiedler, The Conditions for Ecological Sustainable Development in the Context of a Double-Limited Selfpurification Model of an Aggregate Water Recourse
- 51-95 Gerhard Brinkmann, Die Verwendung des Euler-Theorems zum Beweis des Adding-up-Theorems impliziert einen Widerspruch
- 52-95 Gerhard Brinkmann, Über öffentliche Güter und über Güter, um deren Gebrauch man nicht rivalisieren kann
- 53-95 Marlies Klemisch-Ahlert, International Environmental Negotiations with Compensation or Redistribution
- 54-95 Walter Buhr and Josef Wagner, Line Integrals In Applied Welfare Economics: A Summary Of Basic Theorems
- 55-95 Rüdiger Pethig, Information als Wirtschaftsgut
- 56-95 Marlies Klemisch-Ahlert, An Experimental Study on Bargaining Behavior in Economic and Ethical Environments
- 57-96 Rüdiger Pethig, Ecological Tax Reform and Efficiency of Taxation: A Public Good Perspective
- 58-96 Daniel Weinbrenner, Zur Realisierung einer doppelten Dividende einer ökologischen Steuerreform
- 59-96 Andreas Wagener, Corporate Finance, Capital Market Equilibrium, and International Tax Competition with Capital Income Taxes
- 60-97 Daniel Weinbrenner, A Comment on the Impact of the Initial Tax Mix on the Dividends of an Environmental Tax Reform
- 61-97 Rüdiger Pethig, Emission Tax Revenues in a Growing Economy
- 62-97 Andreas Wagener, Pay-as-you-go Pension Systems as Incomplete Social Contracts
- 63-97 Andreas Wagener, Strategic Business Taxation when Finance and Portfolio Decisions are Endogenous
- 64-97 Thomas Steger, Productive Consumption and Growth in Developing Countries
- 65-98 Marco Runkel, Alternative Allokationsmechanismen für ein Rundfunkprogramm bei endogener Programmqualität
- 66-98 Jürgen Ehgen, A Comparison of Solution Methods for Real Business Cycle Models
- 67-98 Peter Seethaler, Zum Einfluß von Devisentermingeschäften auf das Marktgleichgewicht bei asymmetrischer Information
- 68-98 Thomas Christiaans, A Note on Public Goods: Non-Excludability Implies Joint Consumability
- 69-98 Michael Gall, Stylized Facts and International Business Cycles - The German Case
- 70-98 Thomas Eichner, The state as social insurer: labour supply and investments in human capital

- 71-98 **Thomas Steger**, Aggregate Economic Growth with Subsistence Consumption
- 72-98 **Andreas Wagener**, Implementing Equal Living Conditions in a Federation
- 73-99 **Thomas Eichner and Rüdiger Pethig**, Product Design and Markets for Recycling, Waste Treatment and Disposal
- 74-99 **Peter Seethaler**, Zum Einfluß des Hedging auf das Kreditvergabeverhalten der Banken
- 75-99 **Thomas Christiaans**, Regional Competition for the Location of New Facilities
- 76-99 **Thomas Eichner and Rüdiger Pethig**, Product Design and Efficient Management of Recycling and Waste Treatment
- 77-99 **Rüdiger Pethig**, On the Future of Environmental Economics
- 78-99 **Marco Runkel**, Product Durability, Solid Waste Management, and Market Structure
- 79-99 **Hagen Bobzin**, Dualities in the Functional Representations of a Production Technology
- 80-99 **Hagen Bobzin**, Behandlung von Totzeitsystemen in der Ökonomik
- 81-99 **Marco Runkel**, First-Best and Second-Best Regulation of Solid Waste under Imperfect Competition in a Durable Good Industry
- 82-99 **Marco Runkel**, A Note on 'Emissions Taxation in Durable Goods Oligopoly'
- 83-99 **Thomas Eichner and Rüdiger Pethig**, Recycling, Producer Responsibility and Centralized Waste Management
- 84-00 **Thomas Eichner und Rüdiger Pethig**, Das Gebührenkonzept der Duales System Deutschland AG (DSD) auf dem ökonomischen Prüfstand
- 85-00 **Thomas Eichner und Rüdiger Pethig**, Gebührenstrategien in einem disaggregierten Modell der Abfallwirtschaft
- 86-00 **Rüdiger Pethig and Sao-Wen Cheng**, Cultural Goods Consumption and Cultural Capital
- 87-00 **Michael Gail**, Optimal Monetary Policy in an Optimizing Stochastic Dynamic Model with Sticky Prices
- 88-00 **Thomas Eichner and Marco Runkel**, Efficient and Sustainable Management of Product Durability and Recyclability
- 89-00 **Walter Buhr and Thomas Christiaans**, Economic Decisions by Approved Principles: Rules of Thumb as Behavioral Guidelines
- 90-00 **Walter Buhr**, A Macroeconomic Growth Model of Competing Regions
- 91-00 **Hagen Bobzin**, Computer Simulation of Reallocating Resources among Growing Regions
- 92-00 **Sao-Wen Cheng and Andreas Wagener**, Altruism and Donations
- 93-01 **Jürgen Ehlgén**, Geldpolitische Strategien. Die Deutsche Bundesbank und die Europäische Zentralbank im Vergleich
- 94-01 **Thomas Christiaans**, Economic Growth, the Mathematical Pendulum, and a Golden Rule of Thumb
- 95-01 **Thomas Christiaans**, Economic Growth, a Golden Rule of Thumb, and Learning by Doing
- 96-01 **Michael Gail**, Persistency and Money Demand Distortions in a Stochastic DGE Model with Sticky Prices
- 97-01 **Rüdiger Pethig**, Agriculture, pesticides and the ecosystem
- 98-01 **Hagen Bobzin**, Das duale Programm der Erlösmaximierung in der Außenhandelstheorie
- 99-01 **Thomas Eichner and Andreas Wagener**, More on Parametric Characterizations of Risk Aversion and Prudence
- 100-01 **Rüdiger Pethig**, Massenmedien, Werbung und Märkte. Eine wirtschaftstheoretische Analyse
- 101-02 **Karl-Josef Koch**, Beyond Balanced Growth: On the Analysis of Growth Trajectories
- 102-02 **Rüdiger Pethig**, How to Internalize Pollution Externalities Through 'Excess Burdening' Taxes
- 103-02 **Michael Gail**, Persistency and Money Demand Distortions in a Stochastic DGE Model with Sticky Prices and Capital
- 104-02 **Hagen Bobzin**, Fundamentals of Production Theory in International Trade A Modern Approach Based on Theory of Duality
- 105-03 **Rüdiger Pethig**, The 'materials balance approach' to pollution: its origin, implications and acceptance
- 106-03 **Rüdiger Pethig and Andreas Wagener**, Profit Tax Competition and Formula Apportionment
- 107-03 **Walter Buhr**, What is infrastructure?
- 108-03 **Thomas Eichner**, Imperfect Competition in the Recycling Industry
- 109-03 **Thomas Eichner and Rüdiger Pethig**, The impact of scarcity and abundance in food chains on species population dynamics
- 110-03 **Thomas Eichner and Rüdiger Pethig**, A Microfoundation of Predator-Prey Dynamics



- 111-03 Michael Gail, Habit Persistence in Consumption in a Sticky Price Model of the Business Cycle
- 112-03 Thomas Christiaans, Aging in a Neoclassical Theory of Labor Demand
- 113-03 Thomas Christiaans, Non-Scale Growth, Endogenous Comparative Advantages, and Industrialization
- 114-04 Michael Gail, Sticky Wages in a Stochastic DGE Model of the Business Cycle
- 115-04 Thomas Eichner and Rüdiger Pethig, Efficient nonanthropocentric nature protection
- 116-04 Thomas Eichner and Rüdiger Pethig, Economic land use, ecosystem services and microfounded species dynamics
- 117-04 Thomas Eichner and Rüdiger Pethig, An analytical foundation of the ratio-dependent predator-prey model
- 118-04 Thomas Christiaans, Population Dynamics in a Microfounded Predator-Prey Model
- 119-05 Sao-Wen Cheng, Cultural Goods Production, Cultural Capital Formation and the Provision of Cultural Services
- 120-05 Jens Siebel, Budget deficit, size of the public sector and majority voting