# More on Parametric Characterizations of Risk Aversion and Prudence

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**Abstract:** This note provides an alternative proof for the equivalence of decreasing absolute prudence (DAP) in the expected utility framework and in a two-parametric approach where utility is a function of the mean and the standard deviation. In addition, we elucidate that the equivalence of DAP and the concavity of utility as a function of mean and variance, which was shown to hold for normally distributed stochastics in Lajeri and Nielsen [Economic Theory **15** (2000), 469-476], cannot be generalized.

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# 1 Introduction

In their research on alternative representations of individual preferences for risky prospects, several authors explore the relation between standard expected utility approach which employs a von-Neumann-Morgenstern index and the so-called parametric approach where utility depends on the first and second moments (i.e., mean and standard deviation or variance) of the random variable's distribution (see, e.g., Meyer [5]; Sinn [6, 7]; Bar-Shira and Finkelshtain [1]). In a recent contribution to this area, Lajeri and Nielsen [4] provide parametric characterizations of risk aversion and prudence, two concepts which play an important role in the comparative statics of decision making under uncertainty. The linkages between the expected utility approach and the parametric one can be traced back to the following definitions:

$$U(\mu, \sigma) := \mathbf{E}u(y) \equiv \int_a^b u(\mu + \sqrt{\sigma^2}x)\mathrm{d}F(x) =: W(\mu, v) \tag{1}$$

with $v \equiv \sigma^2$. In (1), $\mathbf{E}u(y)$ is a von-Neumann-Morgenstern utility index for random consumption $y$ which itself is a linear function of a standardized random variable $x$ with support $[a, b] \subseteq \mathbb{R}$ ($a < b$) and distribution $F$. By $\mu$, $\sigma$, and $v$ we denote, respectively the mean, the standard deviation and the variance of $y$.

Using different techniques of proof, Meyer [5] (Property 7) and Lajeri and Nielsen [4] (Theorem 1) show that for any two utility functions $u_1$, $u_2$ with corresponding mean/standard deviation-representations $U_1, U_2$, preferences $u_1$ are more risk averse than $u_2$ in the Arrow-Pratt sense if and only if the slope of the indifference curve

$$S(\mu, \sigma) = -\frac{U_\sigma(\mu, \sigma)}{U_\mu(\mu, \sigma)} \tag{2}$$

is always larger for $U_1$ than for $U_2$.[1]

The focus of this note is on Theorem 2 in Lajeri and Nielsen [4]. This result states that, if $x$ is normally distributed, the function $W$ defined in (1) is concave if and only if the function $u$ exhibits decreasing absolute prudence (DAP), i.e. if and only if $-u'''(y)/u''(y)$ is decreasing in $y$ (see Kimball [3] for this definition). Here we wish to make two additions to this theorem: First we provide an alternative proof for the equivalence of the standard notion of DAP (phrased in terms of $u$) and its counterpart in terms of $U$. Our proof follows the procedure inaugurated in Meyer [5] and thus extends this approach to measures of prudence. Second, we wish to demonstrate that the assumption of the underlying stochastics being normally distributed is in fact essential for the equivalence between the concavity of $W$ and DAP of $u$. As we shall argue, in general there is *no* connection between these two concepts. Hence, Theorem 2 in Lajeri and Nielsen [4] cannot be extended to non-gaussian distributions.

[1] Meyer [5] further establishes similar equivalences for the monotonicity of both absolute and of relative risk aversion. These results were recently generalized by Bar-Shira and Finkelshtain [1].

## 2 Decreasing Absolute Prudence of $U$: An alternative proof

As a step towards their Theorem 2, Lajeri and Nielsen [4] show that DAP of $u$ is equivalent to the following:

$$T(\mu,\sigma) := -\frac{U_{\sigma\mu}}{U_{\mu\mu}} \quad \text{is decreasing in} \quad \mu. \tag{3}$$

Phrased geometrically, (3) says that the slope of the indifference curves of the "utility function" $-U_\mu(\mu,\sigma)$ is decreasing in $\mu$.

Kimball [3] has shown that (decreasing) absolute prudence of $u(x)$ corresponds to (decreasing) absolute risk aversion of $-u'(x)$. Translating this into the $(\mu,\sigma)$-language suggests that (decreasing) prudence of $U$ corresponds to (decreasing) risk aversion[2] of $-U_\mu$. Hence, by analogy to (2) one can obtain (3).

Here we wish to present a (different) proof which might further clarify the equivalence between the two notions of prudence for $u$ and $U$. This proof transfers the procedure by which Meyer [5] establishes the equivalences between decreasing absolute risk aversion in terms of $u$ and $U$ to the concept of decreasing absolute prudence.

**Result 1** $T_\mu(\mu,\sigma) \le 0 \quad \Longleftrightarrow \quad$ *$u$ satisfies DAP.*

**Proof:** Calculate that

$$T_\mu(\mu,\sigma) \quad = \quad \frac{-U_{\mu\mu}U_{\sigma\mu\mu} + U_{\sigma\mu}U_{\mu\mu\mu}}{U_{\mu\mu}^2}. \tag{4}$$

The sign of $T_\mu$ equals that of its numerator. Verify that:

$$\begin{aligned}
U_{\mu\mu} &= \int_a^b u''(\mu+\sigma x)\mathrm{d}F(x),\\
U_{\sigma\mu} &= \int_a^b x u''(\mu+\sigma x)\mathrm{d}F(x),\\
U_{\sigma\mu\mu} &= \int_a^b x u'''(\mu+\sigma x)\mathrm{d}F(x),\\
U_{\mu\mu\mu} &= \int_a^b u'''(\mu+\sigma x)\mathrm{d}F(x).
\end{aligned}$$

Define $x^*$ such that $\int_a^b x u''(\mu+\sigma x)\mathrm{d}F(x) = x^* \int_a^b u''(\mu+\sigma x)\mathrm{d}F(x)$. Use this to rewrite the numerator of (4) as

$$\begin{aligned}
&\int_a^b u''(\mu+\sigma x)\mathrm{d}F(x) \cdot \left[\int_a^b (x^*-x)\cdot u'''(\mu+\sigma x)\mathrm{d}F(x)\right]\\
&= \int_a^b u''(\mu+\sigma x)\mathrm{d}F(x) \cdot \left[\int_a^b \left(-\frac{u'''(\mu+\sigma x)}{u''(\mu+\sigma x)}\right)\cdot (x-x^*)\cdot u''(\mu+\sigma x)\mathrm{d}F(x)\right].
\end{aligned}$$

[2] Lajeri and Nielsen [4] use the expression variance aversion instead of risk aversion. Since their variance aversion is not in accordance with Löffler's [2] variance aversion, we have decided to replace variance aversion by risk aversion.

Note that by definition $\int_a^b (x - x^*) u'' \mathrm{d}F = 0$. The integrand changes its sign once, from positive to negative. Given that $\int u'' \mathrm{d}F < 0$, the numerator of (4) is thus non-positive if and only if $-u'''/u''$ is non-increasing. ■

# 3 Decreasing Prudence and the Concavity of $W$

In their Theorem 2, Lajeri and Nielsen [4] establish the equivalence between the concavity of $W$ and DAP of $U$ (and thus of $u$) for the case that the underlying random variable is normally distributed. Since this is a rather special (and often not very attractive) assumption, one might wonder whether the equivalence can be extended to other cases, too. Our observations imply that the answer is negative.

To establish results on the concavity of $W$ we need its second-order derivatives. From (1):

$$\begin{aligned}
W_{\mu\mu} &= U_{\mu\mu} = \int_a^b u''(\mu + \sigma x) \mathrm{d}F(x) < 0, && (5a) \\
W_{\mu v} &= \frac{U_{\mu\sigma}}{2\sigma} = \frac{1}{2\sigma} \int_a^b x u''(\mu + \sigma x) \mathrm{d}F(x) > 0, && (5b) \\
W_{vv} &= \frac{1}{4\sigma^2} \left( U_{\sigma\sigma} - \frac{U_\sigma}{\sigma} \right) \\
&= \frac{1}{4\sigma^3} \int_a^b \left[ \sigma x^2 u''(\mu + \sigma x) - x u'(\mu + \sigma x) \right] \mathrm{d}F(x). && (5c)
\end{aligned}$$

The signs of (5a) and (5b) follow from risk aversion and its decreasingness; cf. Meyer [5]. Concavity of $W$ requires $W_{vv} < 0$ and $W_{\mu\mu} W_{vv} - W_{\mu v}^2 > 0$. With respect to the sign of (5c) we get

**Lemma 1** *a) $W_{vv} < 0$ if and only if $U_{\sigma\sigma\sigma} < 0$.*

*b) $U_{\sigma\sigma\sigma} < 0$ if (i) the distribution $F$ of $x$ is symmetric and $u'''' < 0$ or (ii) the distribution $F$ of $x$ is negatively skewed and $u'''' < 0 < u'''$.*

**Proof:** From(5c), $W_{vv} < 0$ if and only if $U_{\sigma\sigma} < \frac{U_\sigma}{\sigma}$ for all $(\mu, \sigma)$. Verify from (1) that:

$$U_\sigma(\mu, 0) = u(\mu) \int_a^b x \mathrm{d}F(x) = 0. \qquad (6)$$

Now assume that $U_\sigma$ is concave in $\sigma$ (i.e., $U_{\sigma\sigma\sigma} < 0$). Using (6), this implies that $U_{\sigma\sigma} < \frac{U_\sigma}{\sigma}$ for all $(\mu, \sigma)$. To establish the other direction assume that $U_{\sigma\sigma} < \frac{U_\sigma}{\sigma}$ for all $\sigma$. Combining the mean value theorem of differential calculus and (6) we get that for all $(\mu,\sigma)$ there exists $\lambda \in (0,1)$ such that

$$U_{\sigma\sigma}(\mu,\lambda\sigma) = \frac{U_\sigma(\mu,\sigma)}{\sigma} > U_{\sigma\sigma}(\mu,\sigma), \tag{7}$$

which can only hold if $U_{\sigma\sigma\sigma} < 0$.

To see (b) use the definition of $U$ in (1) to calculate via integration by parts:

$$\begin{aligned} U_{\sigma\sigma\sigma}(\mu,\sigma) &= \int_a^b x^3 u'''(\mu+\sigma x)\mathrm{d}F(x) \\ &= \left[u'''(\mu+\sigma x)\int_a^x z^3 \mathrm{d}F(z)\right]_a^b - \sigma\int_a^b u''''(\mu+\sigma x)\left(\int_a^x z^3\mathrm{d}F(z)\right)\mathrm{d}x. \end{aligned} \tag{8}$$

If $F$ is symmetric, all odd central moments of $x$ are zero. Hence, the first expression in (8) is zero while the inner integral in the second expression is negative. Thus, $U_{\sigma\sigma\sigma} < 0$ if $u'''' < 0$.

If $F$ is negatively skewed (i.e., if $\int_a^b x^3 \mathrm{d}F(x) < 0$), the first term in (8) will be negative whenever $u''' > 0$. The second term will be negative (including the negative sign) when $u'''' < 0$. ■

Lemma 1 relates $W_{vv}$ to the third and fourth derivatives of $u$. In particular, if $x$ is symmetrically distributed (e.g., normally distributed), then $u'''' < 0$ – which is implied by DAP – is sufficient for $W_{vv} < 0$.

To gain some further insights into the concavity of $W$ let us restrict attention to the case $\sigma \to 0$. From (5a) and applying de l'Hospital's rule to (5b) and (5c) we infer

$$\begin{aligned} \lim_{\sigma\to 0} W_{\mu\mu} &= u''(\mu), \\ \lim_{\sigma\to 0} W_{\mu v} &= \lim_{\sigma\to 0}\frac{U_{\mu\sigma\sigma}}{2} = \frac{1}{2}u'''(\mu), \\ \lim_{\sigma\to 0} W_{vv} &= \lim_{\sigma\to 0}\left(\frac{1}{8}U_{\sigma\sigma\sigma\sigma} - \frac{1}{24}U_{\sigma\sigma\sigma\sigma}\right) = \frac{1}{12}u''''(\mu)\int_a^b x^4\mathrm{d}F(x). \end{aligned}$$

(Recall that $x$ is a standardized random variable). Combining these results yields:

$$\lim_{\sigma\to 0}\left(W_{\mu\mu}W_{vv} - W_{\mu v}^2\right) = \frac{1}{12}u''(\mu)u''''(\mu)\int_a^b x^4\mathrm{d}F(x) - \frac{1}{4}u'''(\mu)^2, \tag{9}$$

which should be positive for all $\mu$ when $W$ is supposed to be concave. The function $u$ being DAP, however, is formally represented by

$$u''(x)u''''(x) > u'''(x)^2 \tag{10}$$

for all $x$. Comparing (9) and (10) we thus get

**Lemma 2** *Decreasing absolute prudence of $u$ can only be sufficient for $W$ being concave if the distribution of $x$ has a non-negative kurtosis.*

The *kurtosis* of the distribution of $x$ is

$$\frac{\mathbf{E}((x-\mathbf{E}x)^4)}{Var^2(x)} - 3 = \int_a^b x^4 \mathrm{d}F(x) - 3,$$

since $x$ is standardized. The kurtosis is a classical measure of nongaussianity of a probability distribution: It is zero for the normal distribution and nonzero for most (but not quite all) nongaussian distributions. Roughly speaking, it is negative if the density function is "flatter" and positive if the density function is more "spiky" than the normal density. If $x$ is normally distributed – as Lajeri and Nielsen [4] assume – then (9) and (10) coincide. As (10) clearly necessitates $u''''(x) < 0$ and as the normal distribution is symmetric, DAP also implies $W_{vv} < 0$ from Result 1. This confirms Theorem 2 in Lajeri and Nielsen [4]. However, this theorem cannot be generalized for distributions other than the normal distribution:

**Result 2** *DAP of $u$ is in general neither sufficient nor necessary for $W$ to be concave.*

In view of Results 1 and 2 our conclusion is that DAP is not a behavioral assumption that can or should be associated with the concavity of two-parametric utility functions $W$ of mean and variance.

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