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Saddle-Point Dynamics in Non-Autonomous Models of Multi-Sector Growth with Variable Returns to Scale

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Abstract

Efficient capital accumulation paths of an expanding multi-sector economy are known to display catenary behavior around a von Neumann saddle-point turnpike if the economy's production possibilities set is a time-free convex cone. We prove that this result generalizes to the case of a time-dependent biconvex production technology with non-constant returns to scale but strong separability and homotheticity conditions imposed. We also demonstrate that these are locally necessary conditions if technical progress is known a priori to be Hicks neutral.

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1 Introduction

It is well understood that the characteristic roots associated with rest points of autonomous variational or Hamiltonian dynamical systems have special properties. In particular, they are known to come in opposite-signed pairs or pairs of reciprocal roots, depending on whether the analysis is carried out in continuous or discrete time. Therefore, if by appropriate assumptions one can rule out roots with zero real parts the stationary equilibrium appears as a symmetric saddle-point (cf. Samuelson [11], Levhari and Liviatan [9]). One such assumption often made in the theory of optimum economic growth of a multi-sector economy is that the economy's technologies set is a convex cone. This generates the familiar catenary motion of efficient capital accumulation paths around a von Neumann ray of fastest proportional expansion of heterogeneous capital stocks, a result that was first conjectured by Dorfman, Samuelson and Solow [4, ch. 12]. It is since termed the DOSSO turnpike theorem or simply the catenary turnpike theorem and is among the most striking results of modern growth theory. (See McKenzie [10] for a review of turnpike theorems in the theory of optimum economic growth.)

The catenary turnpike theorem does not easily generalize to other cases of production technologies which exhibit non-constant returns to scale. Yet non-constant returns in production may be present in many cases of empirical relevance. Consider, e.g., phenomena such as agglomeration economies and diseconomies which play a major role in the process of growth of an urban or regional economy. We will therefore in this paper look into possible extensions of the catenary turnpike theorem with a major focus on production technologies with varying returns to scale. We will also allow for some sort of exogenous technological change over time. We are thus formally concerned with questions relating to the existence and qualitative properties of stationary solutions in non-autonomous variational dynamics.

The paper is organized as follows. We will introduce in Section 2 our basic continuous-time model of a multi-sector pure accumulation economy which seeks to maximize terminal stocks in some future period of time. We assume an instantaneous biconvex production possibilities set which may be time-dependent and can be represented by a differentiable transformation frontier function. The notion of biconvexity of a multiple-input, multiple-output technology is due to Lau [8]. It extends the concept of convexity of a production technology in that it allows for overall non-constant returns but preserves the properties of decreasing marginal substitution rates and increasing marginal rates of transformation between inputs and outputs, respectively. A general characterization of efficient balanced growth solutions to our model is provided in Section 3.

In Section 4 we introduce the first of two theorems: we prove that a turnpike ray of fastest balanced capital stock expansion exists with no assumptions made about returns to scale if we impose upon technology strong separability and homotheticity restrictions. We demonstrate that the linearized system of Euler differential equations can be decomposed under a weak regularity condition into a system of independent equations all of which possess stationary saddle-point solutions. Such decompositions are already known in the literature from the adjustment model of the firm in investment theory (e.g. Scheinkman [12]) as well as from the model of optimum economic growth when the second-order cross partial derivatives of the underlying utility function are symmetric (DasGupta [3]). However, we will be concerned in our paper with non-autonomous dynamical systems and the decomposition can only be done after a suitable transformation of variables.

We then establish in Section 5 our second theorem which states that the technological constraints imposed in Theorem 1 are also in a certain sense locally necessary conditions if technical progress is known a priori to be Hicks neutral. A brief summary and appraisal of results concludes. In particular, we argue that our results are still valid if we allow for the existence of exogenous inputs and outputs different from capital stocks and investment goods once a suitable separability assumption is again maintained. An important implication of this is that nothing has to be known of the decision processes or types of preference orderings from which the time paths of these variables have been generated. In all, however, we feel that our findings are sobering if seen from the applied point of view of a trading economy as the types of technologies involved appear to be rather specific ones.

We make the following notational conventions. Throughout the paper elements of $\mathbb{R}^n$, $n > 1$, will be referred to as column-vectors or simply called 'vectors'. They will be denoted by lowercase letters and set in a bold typeface for ease of reading. Component i of vector $\mathbf{x}_i$ will be written as x_i. Correspondingly, emboldened uppercase letters like $\mathbf{A}$ shall represent matrices with elements a_{ij}. Furthermore, let $S \in \mathbb{R}^n$ and $T \in \mathbb{R}^n$ be two sets and consider a differentiable function $f : S \rightarrow T$. Then for each element $\mathbf{x} \in S$ and image $f(\mathbf{x})$ we use f_{x_i} as short-hand notation for $\partial f(\mathbf{x})/\partial x_i$. Much in the same way, $f_{x_i x_j}$ is short-hand notation for $\partial^2 f(\mathbf{x})/\partial x_j \partial x_i$ while $f_\mathbf{x}$ stands for the gradient $\nabla f(\mathbf{x})$. Total differentiation with respect to 'time' will be indicated by a dot '$\cdot$' and a superscript 'T' signifies transposition. Finally, we always write as $\mathbf{o}$ the null vector of appropriate length.

2 The Basic Model

To begin with, we will be concerned with a model economy where different outputs are produced from n (> 1) capital stocks $\mathbf{k}^T := (k_1 \ldots k_n)$. All output is used for capital accumulation and consists of quantities of n net investment goods $\dot{\mathbf{k}}^T := (\dot{k}_1 \ldots \dot{k}_n)$.[1] Production technology is assumed to possess in each time period t a frontier function representation of the form

$$T(\mathbf{k}, \dot{\mathbf{k}}, t) = 0 \tag{1}$$

where the 'time' variable t represents exogenous technical progress. In particular, $T(\cdot)$ shall have the following properties:

(P1) It is a continuously differentiable function with domain $\mathbb{R}^n_+ \times \mathbb{R}^n \times \mathbb{R}_+$.

(P2) It has non-vanishing second-order partial and cross-partial derivatives with respect to the components of $\mathbf{k}$. An analogous assumption applies to $\dot{\mathbf{k}}$.

[1] We do not distinguish between gross and net investment for ease of presentation. Gross investment outputs could be modeled by substituting $\dot{\mathbf{k}} + \text{diag}(\mathbf{d})\,\mathbf{k}$ for $\dot{\mathbf{k}}$ where $\mathbf{d}$ is the vector of (constant) depreciation rates.

(P3) It is increasing in $\mathbf{k}$ and decreasing in $\dot{\mathbf{k}}$, respectively.

(P4) It is quasi-concave in both $\mathbf{k}$ and $\dot{\mathbf{k}}$.

(P5) Inada regularity conditions $\lim_{k_i \to 0} T_{k_i} = \infty$ and $\lim_{k_i \to \infty} T_{k_i} = 0$ are satisfied for all k_i.

Properties (P1)-(P5) define a neoclassical type of technology which exhibits the standard attributes of decreasing marginal rates of substitution between any two capital stocks and increasing marginal cost between any pair of investment outputs. Note that nothing is implied by these properties with regard to returns to scale. We are thus implicitly considering a biconvex production possibilities set in the sense of Lau. We will refer to $T(\mathbf{k}, \dot{\mathbf{k}}, t)$ as the *transformation frontier* of our model economy.

The economy shall maximize over a finite and closed time interval $[t_0, t_1]$ some positive linear combination of terminal stocks $\mathbf{k}(t_1)$, given an initial endowment $\mathbf{k}_0$:

$$\max_{\mathbf{k}(t),\, t \in [t_0, t_1]} \mathbf{p}^T \mathbf{k}(t_1) \tag{2}$$
$$\text{subject to} \quad T(\mathbf{k}, \dot{\mathbf{k}}, t) = 0 \quad \text{and} \quad \mathbf{k}(t_0) = \mathbf{k}_0 \,.$$

For example, suppose that $\mathbf{p}$ is a vector of (discounted) stock prices which are expected to prevail in period t_1. We will then be concerned with a classical optimum investment problem: the economy seeks to maximize the expected (present) value of its terminal stocks, thereby using a given technology and given initial endowments.

This is a standard problem of Mayer in the calculus of variations. We assume that there exists an interior solution which can be found in the class of non-negative twice-differentiable functions $\mathbf{k}(t)$ for $t \in [t_0, t_1]$ with $k_i(t_1) > k_i(t_0)$ for at least one k_i. Note that since $T(\cdot)$ is by assumption quasi-concave in $\dot{\mathbf{k}}$ the Legendre definiteness condition for a maximum of $\mathbf{p}^T \mathbf{k}(t_1)$ will be met globally. Uniqueness of the solution is thus assured. (See Kamien and Schwartz [7] as a reference on variational methods in economics.)

3 A Characterization of Stationary Capital Structures

Optimum time paths $\mathbf{k}(t)$ are long known to satisfy the so-called own-rates of interest relationships of optimum growth theory which are equivalent to the Euler necessary conditions associated with (2):

PROPOSITION 1: *If* $\mathbf{k}(t)$ *is a solution to (2) then*

$$\frac{T_{k_i}}{T_{\dot{k}_i}} = \frac{T_{k_n}}{T_{\dot{k}_n}} + \frac{d}{dt} \log\left(\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}\right) \quad \textit{for all } i \neq n \,. \tag{3}$$

PROOF: See Wan, Jr. [14, pp. 277-278].

These $n-1$ equations along with $2n$ boundary conditions and the technology constraint (1) completely determine the dynamics of $\mathbf{k}(t)$. Now assume that $k_n > 0$ for all $t \in [t_0, t_1]$

and let $\mathbf{s}^T := (k_1/k_n \ldots k_{n-1}/k_n)$. Turnpike theorems are then concerned with balanced growth solutions of the form $\mathbf{k}(t) = k_n(t)\,(\bar{\mathbf{s}}^{*T}, 1)^T$ where $\bar{\mathbf{s}}^*$ is a vector of $n-1$ positive constants which define a ray of efficient proportionate capital stock expansion. Along each such ray the following proposition applies:

PROPOSITION 2: *If $\bar{\mathbf{s}}^*$ is a balanced-growth solution to (2) and $w(t) > 0$ is the uniform rate of growth of all capital stocks in period $t \in [t_0, t_1]$ then*

$$-\frac{T_{k_1}}{T_{\dot{k}_1}} = \ldots = -\frac{T_{k_n}}{T_{\dot{k}_n}} = w + \frac{\dot{w}}{w} - \Theta(t) \tag{4}$$

where $\Theta(t)$ is the relative change in w which can be attributed to technical progress.

PROOF: Note that $\dot{\mathbf{k}} = w\,\mathbf{k}$ along $\bar{\mathbf{s}}^*$ and hence $T(\mathbf{k}, w\,\mathbf{k}, t) = 0$. As $T(\cdot)$ is decreasing in $\dot{\mathbf{k}}$ we may solve for w as a function of $\mathbf{k}$ and t: $w = f(\mathbf{k}, t)$. Using $\mathbf{k} = k_n\,(\bar{\mathbf{s}}^{*T}, 1)^T$ we obtain $w = f(k_n\bar{\mathbf{s}}^*, k_n, t)$. Next observe that for a given expansion ray to be efficient it must never pay, in terms of levels of w, to move off the ray as $\mathbf{k}$ increases over time. In other words, every efficient ray $\bar{\mathbf{s}}^*$ must always point in the direction of the maximum increase or minimum decrease, respectively, of w. Therefore, we conclude from the envelope theorem that along $\bar{\mathbf{s}}^*$:

$$-\frac{T_{k_n}}{T_{\dot{k}_n}} = \frac{\partial \dot{k}_n}{\partial k_n} = \frac{\partial (w k_n)}{\partial k_n} = \sum_{i=1}^{n-1} f_{k_i} \bar{s}_i^* k_n + f_{k_n} k_n + w = \sum_{i=1}^{n} f_{k_i} k_i + w\,. \tag{5}$$

As k_n is really any capital stock the above sequence applies accordingly to all components of $\mathbf{k}$. Finally, totally differentiating $f(\cdot)$ with respect to t yields

$$\dot{w} = \sum_{i=1}^{n} f_{k_i} \dot{k}_i + \frac{\partial f}{\partial t} = w\,(\sum_{i=1}^{n} f_{k_i} k_i + \frac{1}{w}\frac{\partial f}{\partial t})\,. \tag{6}$$

Now define $\Theta(t) := \dfrac{1}{w}\dfrac{\partial f}{\partial t}$ and solve for the summation expression in brackets. Then plug into (5) and our proof of Proposition 2 is completed. □

Proposition 2 implies that the marginal rate of substitution between any two capital stocks coincides along $\bar{\mathbf{s}}^*$ with the marginal rate of transformation between the corresponding pair of investment outputs. (Consider the first $n-1$ equals signs in (4) and rearrange terms.) The latter ratios also follow to be constants as the second terms on the right-hand sides of (3) drop to zero because of (4). This constitutes a special invariant relationship between $\bar{\mathbf{s}}^*$ and the economy's marginal input substitution and output transformation rates. We thus arrive at

COROLLARY 1: *Consider a ray $\bar{\mathbf{s}}^*$ of efficient balanced capital stock expansion. Then each marginal substitution rate and related marginal cost take identical constant values for all $t \in [t_0, t_1]$:*

$$\frac{T_{k_i}}{T_{k_n}} = \frac{T_{\dot{k}_i}}{T_{\dot{k}_n}} \quad \textit{and} \quad \frac{d}{dt}(\frac{T_{k_i}}{T_{k_n}}) = \frac{d}{dt}(\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}) = 0 \quad \textit{for all } i \neq n\,. \tag{7}$$

Now assume for a moment that the transformation frontier $T(\cdot)$ is both time-free and homogeneous of degree one in $\mathbf{k}$ and $\dot{\mathbf{k}}$. This is the case of a stationary constant-returns technology originally studied by Dorfman, Samuelson and Solow [4, ch. 12]. We then have $\Theta(t) \equiv \dot{w}(t) \equiv 0$ and can be sure that all marginal substitution and transformation rates between inputs and outputs stay constant along $\bar{\mathbf{s}}^*$. Conditions (7) will thus establish as $\bar{\mathbf{s}}^*$ the economy's von Neumann ray of balanced exponential growth at a maximum possible rate. This ray is well-known to possess the saddle-point property and hence to serve as a turnpike for optimum accumulation paths $\mathbf{k}(t)$ which do not start or terminate along $\bar{\mathbf{s}}^*$.

However, in other cases of production technologies each ratio T_{k_i}/T_{k_n} and $T_{\dot{k}_i}/T_{\dot{k}_n}$ ($i = 1, \ldots, n-1$) will normally depend on 'time' t (due to technical progress) as well as on 'size' k_n (due to non-constant returns) and will change over time even with the composition of capital stocks and investment flows held constant. As a result, no ray of proportionate capital stock expansion may be efficient in the sense of (3). In other words: the turnpike theorem does not in general go through for non-stationary technologies with non-constant returns to scale.

Therefore, for there to exist a turnpike ray that allows for accelerated or decelerated balanced growth of our economy we must impose upon technology further a-priori structure. This brings up the question whether we can find functional forms of frontier functions $T(\cdot)$ which are compatible with (7).

4 Separable Production Technologies

We have argued in Section 3 that a turnpike growth ray may not in general exist due to 'time' and 'size' effects on input substitution and output transformation rates. We therefore now require that technology satisfies more specific properties. The first of these refers to technical progress:

(P6) Technical progress is Hicks neutral.

We also assume that the production technologies set is independent in its input and output partitions. In particular:

(P7) $T(\cdot)$ is both (additively) separable and homothetic with respect to $\mathbf{k}$ on one hand and $\dot{\mathbf{k}}$ on the other hand.

(Note that (P7) implies (P6).) These are sufficient assumptions to warrant constant substitution and transformation rates T_{k_i}/T_{k_n} and $T_{\dot{k}_i}/T_{\dot{k}_n}$ ($i = 1, \ldots, n-1$) along each arbitrary expansion ray $\bar{\mathbf{s}}$. We may now establish the following theorem:

THEOREM 1: *If the transformation frontier (1) assumes the form*

$$T(\mathbf{k}, \dot{\mathbf{k}}, t) = J\big(G(\mathbf{k}), t\big) - H(\dot{\mathbf{k}}) \tag{8}$$

with first-order homogeneous functions $G(\cdot)$ and $H(\cdot)$ and a regularity condition satisfied then there exists a unique vector of positive constants $\bar{\mathbf{s}}^*$ which will possess the local turnpike property.

The regularity condition is the following weak requirement. Its significance will become clear soon:

(P7) The Hessian matrices $G_{k_j k_i}$ and $H_{\dot{k}_j \dot{k}_i}$ with $i,j < n$ will be of maximum rank if evaluated along $\bar{\mathbf{s}}^*$.

PROOF: The existence and uniqueness of a vector of constants $\bar{\mathbf{s}}^*$ which is efficient in the sense of (3) is immediate from our discussion of Corollary 1 and from (P1)-(P4). We can also be sure of this vector's components being strictly positive due to (P5). However, we still have to prove the saddle-point property of $\bar{\mathbf{s}}^*$. We will now do so by taking three steps.

To begin with the first step, define $F(\mathbf{s},\dot{\mathbf{s}},t) := T(k_n\mathbf{s}, k_n, k_n\dot{\mathbf{s}} + \dot{k}_n\mathbf{s}, \dot{k}_n, t)$ for presupposed solution paths $k_n(t)$ and $\dot{k}_n(t)$ and note that associated with (2) is the Lagrangian functional

$$k_n(t)(\sum_{i=1}^{n-1} p_i s_i + p_n) + \int_{t_0}^{t_1} \lambda\, F(\mathbf{s},\dot{\mathbf{s}},t)\, dt \tag{9}$$

and the subsequent system of $n-1$ Euler differential equations:

$$\lambda\, F_{\mathbf{s}}(\mathbf{s},\dot{\mathbf{s}},t) - \frac{d}{dt}[\lambda\, F_{\dot{\mathbf{s}}}(\mathbf{s},\dot{\mathbf{s}},t)] = \mathbf{o}\,. \tag{10}$$

Exploiting (8) one obtains:

$$\lambda J_G G_{\mathbf{s}} + \dot{\lambda} H_{\dot{\mathbf{s}}} + \lambda \dot{H}_{\dot{\mathbf{s}}} = \mathbf{o}\,. \tag{11}$$

We assume that λ, $\dot{\lambda}$ and J_G are known functions of time and proceed by expanding (11) into a Taylor series approximation around $\bar{\mathbf{s}}^*$. (We neglect second-order and higher-order terms.) Because of $\dot{\bar{\mathbf{s}}}^* \equiv \mathbf{o}$, this gives after division by k_n ($\neq 0$):

$$\lambda(t)\, J_G(t)\, \mathbf{A}\mathbf{y} + \dot{\lambda}(t)\, \mathbf{B}\dot{\mathbf{y}} + \lambda(t)\, \mathbf{B}\ddot{\mathbf{y}} = \mathbf{o}\,, \tag{12}$$

where $\mathbf{y}$ stands for the deviations $\mathbf{s} - \bar{\mathbf{s}}^*$ and $\mathbf{A}$ and $\mathbf{B}$ denote constant matrices which are built of the first $n-1$ rows and columns of the Hessians of $G(\cdot)$ and $H(\cdot)$, respectively, multiplied by k_n and evaluated along $\bar{\mathbf{s}}^*$. We are thus facing a non-autonomous linear system of differential equations with respect to $\mathbf{y}$. Note that both $\mathbf{A}$ and $\mathbf{B}$ are regular due to (P7). This rules out borderline cases where the qualitative properties of $\bar{\mathbf{s}}^*$ depend on terms which involve higher powers of $\mathbf{s}$.

In order to study these properties any further consider in a second step a real-valued and regular $(n-1, n-1)$-matrix $\mathbf{X}$ and the transformation

$$\mathbf{y}(t) = \mathbf{X}\mathbf{z}(x) \quad \text{where} \quad x := \int \lambda(t)^{-1}\, dt\,, \tag{13}$$

which, after rearranging terms, changes (12) into

$$\mathbf{BXz}'' + \tau(x)\mathbf{AXz} = \mathbf{o}\,, \tag{14}$$

with $\tau := \lambda^2 J_G$. Note that in (14) we have used the following intermediate result (a proof is provided in the appendix):

LEMMA 1: $\lambda(t) > 0$ *for all* $t \in [t_0, t_1]$.

Hence, there is a one-to-one correspondence between x and t, so τ can be expressed as a function of x. Also note that as $\mathbf{B}$ is regular its inverse exists. Therefore, premultiplying (14) by $\mathbf{X}^{-1}\mathbf{B}^{-1}$ yields

$$\mathbf{z}'' + \tau(x)\mathbf{X}^{-1}\mathbf{CXz} = \mathbf{o}\,, \tag{15}$$

introducing $\mathbf{C} := \mathbf{B}^{-1}\mathbf{A}$. A characterization of optimum $\mathbf{z}(x)$ and $\mathbf{s}(t)$ can now be derived from (15) and the following properties of $\mathbf{C}$ (see the Appendix for proofs):

LEMMA 2: *All eigenvalues of* $\mathbf{C}$ *are real-valued and negative.*

LEMMA 3: $\mathbf{C}$ *has a complete set of* $n-1$ *independent (right) eigenvectors.*

Therefore, select in our third step as $\mathbf{X}$ the matrix of (right) eigenvectors of $\mathbf{C}$ in which case $\mathbf{X}^{-1}\mathbf{CX}$ is known to be a diagonal matrix with the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ of $\mathbf{C}$ as its diagonal entries. As a result, (15) disintegrates into $n-1$ independent second-order differential equations:

$$z_i'' + \tau(x)\,\mu_i z_i = 0 \quad \text{for all } i \neq n\,. \tag{16}$$

These are equivalent to the following systems of first-order equations:

$$\begin{aligned} y_i' + \tau(x)\,\mu_i z_i &= 0 \\ y_i - z_i' &= 0 \end{aligned} \quad \text{for all } i \neq n\,. \tag{17}$$

Each such system has a stationary solution $\{\bar{y}_i^*, \bar{z}_i^*\} = \{0, 0\}$ which is independent of $\tau(x)$. Now recall that $\mu_i < 0$ for all $i \neq n$ and also note that $\tau(x) > 0$. Then an inspection of the phase trajectories in $\{y_i, z_i\}$-space will reveal that the origin is a saddle-point of (17). Hence, we conclude that the null vector $\mathbf{o}$ (of length $n-1$) is a 'generalized' saddle-point of (12) due to (13). This in turn means that $\bar{\mathbf{s}}^*$ is a 'generalized' local saddle-point of (10). Our proof of Theorem 1 is thereby finished. □

5 A Representation Theorem

Our first theorem states that a frontier function specification of the form in (8) is a *sufficient* pre-condition for the turnpike result established in the previous section. The question thus comes up whether or not there exist other functional forms of the transformation frontier which generate local saddle-point behavior of optimum capital accumulation paths. In particular, how can we characterize such functional forms? Note that since (8) appears to stand for a rather exceptional case of a production technology of a trading economy much of the empirical significance of the catenary turnpike theorem will depend on the answers to these questions. Therefore, we will now look into related *necessary* conditions in terms of the kind of structure to be requested of a frontier function $T(\cdot)$ in the presence of 'time' and 'size' effects of economic growth.

However, we make two strong qualifications. First of all, since we have been concerned in Theorem 1 with local behavior of efficient growth paths, we will now also focus on local conditions only, i.e. necessary conditions along the ray $\bar{\mathbf{s}}^*$. Secondly, we make the a-priori assumption that in the neighborhood of each such ray technical progress is neutral with regard to input substitution and output transformation. We then obtain the following representation theorem:

THEOREM 2: *Suppose a turnpike expansion ray $\bar{\mathbf{s}}^*$ exists. Also suppose that along this ray all marginal substitution and transformation rates between inputs $\mathbf{k}$ and outputs $\dot{\mathbf{k}}$, respectively, are independent of t. Then the production technology set will possess a local frontier function representation of the form in (8).*

PROOF: To begin with, define again as w the rate of growth uniformly assigned to all stock variables along $\bar{\mathbf{s}}^*$ at a given point in time and recall from Section 3 that $w = f(\mathbf{k}, t)$. Now let $\tilde{T}(k_n, \dot{\mathbf{k}}, t) := T(k_n \bar{\mathbf{s}}^*, k_n, \dot{\mathbf{k}}, t) = 0$. Since $\tilde{T}(\cdot)$ is increasing in k_n because of (P3) we can solve for k_n in terms of $\dot{\mathbf{k}}$ and t: $k_n = \tilde{F}(\dot{\mathbf{k}}, t)$. Now recall that all output transformation rates are by assumption independent of t. Therefore, and considering (7), it follows from the quasi-concavity of $T(\cdot)$ with respect to $\dot{\mathbf{k}}$ that $\tilde{F}(\cdot)$ must be a homothetic and quasi-convex function of $\dot{\mathbf{k}}$. (See Färe [5] and [6, pp. 49-61]; also see Shephard [13].) Hence, $k_n = F(H(\dot{\mathbf{k}}), t) = F(w\, H(\mathbf{k}), t)$ where $H(\cdot)$ has been assumed a first-order homogeneous function without loss of generality. Consequently, multiplying both sides of $w = f(\mathbf{k}, t)$ by $H(\mathbf{k})$ yields

$$w\, H(\mathbf{k}) = H(\dot{\mathbf{k}}) = f(\mathbf{k}, t)\, H(\mathbf{k}) =: \tilde{J}(\mathbf{k}, t) \tag{18}$$

and thereby separates $T(\cdot)$ into a homothetic and quasi-convex output branch $H(\cdot)$ and a remaining input branch $\tilde{J}(\cdot)$. At this point note that the gradients $H_{\dot{\mathbf{k}}}$ and $\tilde{J}_{\mathbf{k}}$ always indicate the same expansion ray because of (7). Also note that we have assumed all input substitution rates not to be affected by t. Finally, remind that $T(\cdot)$ is a quasi-concave function of $\mathbf{k}$. As a result, $\tilde{J}(\cdot)$ must be both homothetic and quasi-concave in $\mathbf{k}$: $\tilde{J}(\mathbf{k}, t) = J(G(\mathbf{k}), t)$. We thus obtain from (18) a frontier function representation of the form

$$T(\mathbf{k}, \dot{\mathbf{k}}, t) = J(G(\mathbf{k}), t) - H(\dot{\mathbf{k}}) \tag{19}$$

which is quasi-concave in both $\mathbf{k}$ and $\dot{\mathbf{k}}$ and with $H(\cdot)$ and $G(\cdot)$ homogeneous of degree one. This ends our proof of Theorem 2. □

6 Conclusions

We have so far been concerned with the final-state version of the turnpike theorem where the economy seeks to maximize terminal stocks and where all output is used for capital accumulation, physical capital being the only input to production. Two basic results have emerged. First, we proved that the theorem generalizes to the case of a time-dependent biconvex production technology with non-constant returns to scale but strong separability and homotheticity restrictions imposed. Second, we showed that these restrictions are also locally necessary conditions if technical progress is Hicks neutral.

Now assume that there are inputs and outputs different from capital stocks and investment goods, e.g. labor inputs and consumption outputs. Next suppose that society has chosen to select known growth patterns for each of these variables. We may then simply re-interpret our transformation frontier function (1). In particular, we may attach a new meaning to the time variable t: it does not any longer only stand for exogenous technical progress but now also represents the time paths of all inputs and outputs other than $\mathbf{k}$ and $\dot{\mathbf{k}}$. Therefore, all of our results remain valid provided that appropriate separability assumptions are again made such that (8) still applies. Note that we even do not have to care about the decision processes or preference orderings from which the presupposed growth patterns have been generated. Therefore, these patterns may well be 'optimum patterns' in that they maximize some intertemporal utility functional subject to the technology constraint.

This once more again demonstrates the strength of separability assumptions in economics. It is for such reasons that separability is a maintained hypothesis in a great deal of empirical research. On the other hand, however, we have not addressed any of the complex questions relating to the assumed existence of aggregator functions like $G(\cdot)$ and $H(\cdot)$. (The interested reader is referred to Blackorby and Schworm [2] for a comprehensive discussion of the existence of input and output aggregates in aggregate production functions and for further references.) In all, it seems that the types of technologies which are compatible with (8) are rather specific ones. Therefore, we feel that our findings appear to be sobering if seen from the applied point of view of a trading economy.

Appendix: Proofs of Lemmas 1-3

PROOF OF LEMMA 1: Let $\tilde{F}(\mathbf{s}, \dot{\mathbf{s}}, k_n, \dot{k}_n, t) := T(k_n\mathbf{s}, k_n, k_n\dot{\mathbf{s}} + \dot{k}_n\mathbf{s}, \dot{k}_n, t)$ in which case $p_n = \lambda(t_1)\tilde{F}_{\dot{k}_n}(\cdot, t_1)$ constitutes a transversality condition with respect to the optimum path $k_n(t)$. Hence:

$$p_n = \lambda(t_1)\,(\sum_{i=1}^{n-1} T_{k_i} s_i + T_{k_n} + \sum_{i=1}^{n-1} T_{\dot{k}_i} \dot{s}_i)\,,$$

where all derivatives have to be taken at $t = t_1$. Now assume for a moment that in (2) $\mathbf{k}_0$ and $\mathbf{p}$ are such that the optimum time path of $\mathbf{s}$ happens to coincide with $\bar{\mathbf{s}}^*$ in all of the interval $[t_0, t_1]$. Then the following chain of reasoning applies. To begin with, recall that $\bar{\mathbf{s}}^* > \mathbf{o}$, $\dot{\bar{\mathbf{s}}}^* \equiv \mathbf{o}$ and that $T_{k_i} > 0$ for all i due to (P3). Therefore, since p_n is positive by assumption, we conclude that $\lambda(t_1)$ is also a positive number. Next consider (11) and observe that $J_G, G_{s_i}, H_{\dot{s}_i} > 0$ by (P3). Also remind that $H(\cdot)$ is a first-order homogeneous function which has constant derivatives along each arbitray expansion ray. Therefore, $\dot{H}_{\dot{s}_i} = H_{\dot{k}_i}\dot{k}_n > 0$ where the latter inequality follows again from (P3) and from $\dot{k}_n > 0$. We thus find that $\dot{\lambda}(t_1) < 0$. Hence, $\lambda(t)$ can be traced backwards in time to stay positive for all $t \in [t_0, t_1]$. At this point, observe that the optimum time path of λ changes continuously as we change $\mathbf{k}_0$ or $\mathbf{p}$. Hence, $\lambda(t)$ will always come out positive if evaluated in some neighborhood of $\bar{\mathbf{s}}^*$ which proves Lemma 1. □

PROOF OF LEMMA 2: The proof can be obtained from an extension of a theorem by Arrow [1, p. 200] on the eigenvalues of the product of a positive quasi-definite matrix and a symmetric matrix (see DasGupta [3, p. 315]). We provide a direct proof which exploits the properties of **A** and **B**. In particular, we show that the eigenvalues of **C** are the same as those of **A**. Therefore, note that **A** is a negative definite matrix while both **B** and its inverse $\mathbf{B}^{-1}$ are positive definite matrices because of (P4), (8) and (P7). By a standard theorem of linear algebra there exists a regular matrix **R** such that $\mathbf{B}^{-1} = \mathbf{R}\mathbf{R}^T$. Another theorem says that for any regular $(n-1, n-1)$-matrix **X** the product $\mathbf{X}^{-1}\mathbf{B}^{-1}\mathbf{A}\mathbf{X}$ is a similarity transformation of $\mathbf{B}^{-1}\mathbf{A}$ which leaves its eigenvalues unchanged. (See any textbook on linear algebra for a proof of these theorems.) Now choose $\mathbf{X} = \mathbf{R}$. Then

$$\begin{aligned}\mathbf{X}^{-1}\mathbf{B}^{-1}\mathbf{A}\mathbf{X} &= \mathbf{X}^{-1}\mathbf{R}\mathbf{R}^T\mathbf{A}\mathbf{X} \\ &= \mathbf{R}^{-1}\mathbf{R}\mathbf{R}^T\mathbf{A}\mathbf{R} \\ &= \mathbf{R}^T\mathbf{A}\mathbf{R}\,.\end{aligned}$$

Next consider an arbitrary vector $\mathbf{x} \in I\!R^{n-1}$ with at least one component different from zero and let $\mathbf{r} := \mathbf{R}\mathbf{x}$. Note that $\mathbf{r} \neq \mathbf{o}$ as **R** is regular. Hence, since **A** is negative definite,

$$\mathbf{x}^T\mathbf{R}^T\mathbf{A}\mathbf{R}\mathbf{x} = \mathbf{r}^T\mathbf{A}\mathbf{r} < 0\,.$$

We thus conlude that $\mathbf{R}^T\mathbf{A}\mathbf{R} = \mathbf{X}^{-1}\mathbf{B}^{-1}\mathbf{A}\mathbf{X} = \mathbf{X}^{-1}\mathbf{C}\mathbf{X}$ is also a negative definite matrix, so its eigenvalues are real-valued and negative. As a result, all eigenvalues of **C** are real-valued and negative, too. (However, note that **C** is not, in general, a negative-definite matrix as the product of two symmetric matrices like $\mathbf{B}^{-1}$ and **A** need not be symmetric.) This proves Lemma 2. □

PROOF OF LEMMA 3: Suppose that μ is an eigenvalue of $\mathbf{X}^{-1}\mathbf{C}\mathbf{X}$ and that **r** is the associated eigenvector. Hence, $\mathbf{X}^{-1}\mathbf{C}\mathbf{X}\mathbf{r} = \mu\mathbf{r}$. Premultiply with **X** to obtain $\mathbf{C}\mathbf{X}\mathbf{r} = \mu\mathbf{X}\mathbf{r}$. Therefore, **Xr** is the corresponding eigenvector of **C**. Now let $\mathbf{X} := \mathbf{R}$ in which case $\mathbf{X}^{-1}\mathbf{C}\mathbf{X}$ has $n-1$ independent real-valued eigenvectors as this matrix is negative definite (see proof of Lemma 2). Therefore, since the transformation matrix **X** (= **R**) is real-valued and regular, **C** will possess $n-1$ real-valued eigenvectors which are linearly independent. This completes the proof of Lemma 3. □

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