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On the Existence of Structural Saddle-Points in Variational Closed Models of Capital Formation

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Abstract

The so-called turnpike theorem can be considered one of the most striking results of modern growth theory. It confirms the catenary behavior of efficient capital accumulation paths around a ray of fastest proportional capital stock expansion in finite-horizon models of efficient economic growth. This paper employs a gradient argument to prove in a continuous-time neoclassical framework with some restrictions imposed upon technical progress that for there to exist a turnpike expansion ray the production technology set must possess along the ray a biconvex transformation frontier function representation in the sense of L.J. Lau.

I. Introduction

The so-called turnpike theorem can be regarded one of the most striking results of modern growth theory. It confirms the catenary behavior of efficient capital accumulation paths around a ray of fastest proportional capital stock expansion in finite-horizon models of efficient economic growth. Standard proofs of the theorem assume that the underlying production technology set is a convex cone (see L.W. McKenzie [4]). More recent research by this author (see R. Wolff [7]) indicates that the theorem also applies in the case of a non-convex production technology set if this set can be assumed a biconvex set in the sense of L.J. Lau [3] with its input and output partitions independent of each other. The present paper tries to shed light on the subsequent, yet open question if there exist production technologies different from the ones just mentioned and still compatible with the catenary property of efficient capital formation paths.

This question will be posed within the common framework of a continuous-time neoclassical optimum growth model to be introduced in Section II. The model assumes as given an instantaneous production technology set which may be time-dependent and represented by a smooth transformation frontier function with some further restrictions imposed upon the change of technology over time. We will then establish and prove in Section III the theorem that for there to exist a turnpike expansion ray the transformation frontier must have *along this ray* a representation of the form *globally* assumed in R. Wolff [7].

Our use of notation will be as follows. Throughout the paper elements of R^n, $n > 1$, will be referred to as column-vectors or simply called 'vectors'. They will be denoted by lowercase letters and set in a bold typeface for ease of reading. Component i of vector $\mathbf{x}$ will be written as x_i. Furthermore, let $S \subset R^n$ and $T \subset R$ be two sets and consider a differentiable function $f: S \to T$. Then for each element $\mathbf{x} \in S$ and image $f(\mathbf{x})$ we use f_{x_i} as short-hand notation for $\partial f(\mathbf{x})/\partial x_i$. Much in the same way, f_x stands for the gradient $\nabla f(\mathbf{x})$. Differentiation with respect to 'time' will be indicated by a dot '$\cdot$' and a prime '$'$' signifies transposition.

II. Statement of the Problem

We will be concerned with the standard optimum investment problem of maximizing under technological constraints a positive linear combination of a vector of capital stocks $\mathbf{k}$ in some future period t_1:

$$(1) \qquad \max_{k(t)} \mathbf{a}'\mathbf{k}(t_1)$$

$$\textit{subject to} \quad T(\mathbf{k},\dot{\mathbf{k}},t) = 0\,, \; t \in [t_0, t_1]\,, \; \textit{and} \quad \mathbf{k}(t_0) = \mathbf{k}_0\,.$$

$T(\cdot)$ is assumed a transformation frontier function with domain $R_+^n \times R^n \times R_+$, where $n > 1$. It thus completely describes all efficient one-period input-output combinations of a firm or of an economy. It shall be increasing in $\mathbf{k}$, decreasing in $\dot{\mathbf{k}}$ and twice-continuously differentiable as well as strictly quasi-concave in both $\mathbf{k}$ and $\dot{\mathbf{k}}$. Differentiability is also assumed with regard to t. Finally, Inada regularity conditions shall hold for $\mathbf{k}$.

Note that (1) is a classical problem of Mayer in the calculus of variations. We call it closed in the sense that all input variables other than capital stocks $\mathbf{k}$ and all output variables other than investments $\dot{\mathbf{k}}$ are assumed to be given from some external decision process and supposed to enter $T(\cdot)$ via t. Most of all, however, will t reflect autonomous technical progress.

Standard practice suggests to define a Lagrangian functional associated with (1) and solve the corresponding system of Euler differential equations for optimum time paths $\mathbf{k}(t)$. These equations are long known to be equivalent to the familiar own-rates of interest relationships of optimum growth theory:

Lemma 1: *If $k(t)$ is a solution to (1) then*

$$\frac{T_{k_i}}{T_{\dot{k}_i}} = \frac{T_{k_n}}{T_{\dot{k}_n}} + \frac{d}{dt} ln\left[\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}\right], \quad for\ all\quad i \neq n. \tag{2}$$

Proof: see H.Y. Wan, Jr. ([6], pp. 277-278).

The above $n-1$ equations along with $2n$ boundary conditions and the technology constraint $T(\cdot) = 0$ completely determine the dynamics of $\mathbf{k}(t)$. In particular, the following theorem holds:

Theorem 1: *If the transformation frontier of (1) assumes the form*

$$T(\mathbf{k}, \dot{\mathbf{k}}, t) = J(G(\mathbf{k}), t) - H(\dot{\mathbf{k}}) \tag{3}$$

with first-order homogeneous functions $G(\cdot)$ and $H(\cdot)$ then there exists a real-valued scalar function $\theta(t)$ and a vector of $n-1$ positive constants $\bar{\mathbf{s}}^*$ such that $\mathbf{k}(t) = \theta(t)(\bar{\mathbf{s}}^{*\prime}, 1)'$ is a solution to (2) and $\bar{\mathbf{s}}^*$ constitutes a turnpike expansion ray of (1).

Proof: see R. Wolff [7].

Theorem 1 refers to the case of a production technology set which is globally biconvex and has independent input and output partitions. Biconvexity of the production technology set then involves (additive) separability of the transformation frontier function (see L.J. Lau [3]). Note that our theorem does not imply constant returns to scale. Finally, as $\bar{\mathbf{s}}^*$ is indicative of the composition of the capital stocks $\mathbf{k}(t)$, we label it a 'structural saddle-point' of (1).

III. A Representation Theorem

We may now introduce our basic

Theorem 2: *Suppose a turnpike expansion ray $\bar{\mathbf{s}}^*$ of (1) exists. Also suppose that along this ray all marginal substitution and transformation rates between inputs $\mathbf{k}$ and outputs $\dot{\mathbf{k}}$, respectively, are independent of t. Then the production technology set will possess a local frontier function representation of the form in (3).*

Proof: Our proof consists of two parts. To begin with the first part, define as w the rate of growth uniformly assigned to all stock variables along an arbitrary expansion ray at a given point in time. Hence, $\dot{\mathbf{k}} = w\mathbf{k}$ such that $T(\mathbf{k}, w\mathbf{k}, t) = 0$ must hold. As $T(\cdot)$ is decreasing in $\dot{\mathbf{k}}$ we may solve for w as a function of $\mathbf{k}$ and t: $w = f(\mathbf{k}, t)$. Furthermore, assume that $k_n > 0$ and let $\mathbf{s} := (k_1/k_n, \ldots, k_{n-1}/k_n)'$ in which case $w = f(\mathbf{s}\, k_n, k_n, t)$. Now observe as an intrinsic necessary condition for a given expansion ray to be efficient that it must never pay, in terms of levels of w, to move off the ray as $\mathbf{k}$ increases over time. In other words, every efficient ray $\bar{\mathbf{s}}^*$ must always point in the direction of the maximum increase or minimum decrease, respectively, of w. Therefore, we conclude from the envelope theorem that along $\bar{\mathbf{s}}^*$:

$$(4) \qquad -\frac{T_{k_n}}{T_{\dot{k}_n}} = \frac{\partial \dot{k}_n}{\partial k_n} = \frac{\partial (w k_n)}{\partial k_n} = \sum_{i=1}^{n-1} f_{k_i} \bar{s}_i^* k_n + f_{k_n} k_n + w = \sum_{i=1}^{n} f_{k_i} k_i + w \, .$$

Since k_n is really any capital stock the above sequence applies accordingly to all components of $\mathbf{k}$. We thus obtain from (4) a system of $n-1$ necessary conditions for there to exist a ray of efficient proportional capital stock formation. They require that the own-rates of interest be the same for all capital stocks:

$$(5) \qquad \frac{T_{k_i}}{T_{\dot{k}_i}} = \frac{T_{k_n}}{T_{\dot{k}_n}} \quad for\ all\ t\ and\ all\ \ i \neq n \, .$$

Equivalently, we may also say that all corresponding marginal substitution and transformation rates must coincide:

$$(6) \qquad \frac{T_{k_i}}{T_{k_n}} = \frac{T_{\dot{k}_i}}{T_{\dot{k}_n}} \quad for\ all\ t\ and\ all\ \ i \neq n \, .$$

Plugging (5) into (2) reveals that intertemporal efficiency is achieved only if the second term on the right-hand side of (2) drops to zero for all t and i. Therefore, and because of (6), we find that

$$(7) \qquad \frac{d}{dt}\left(\frac{T_{k_i}}{T_{k_n}}\right) = \frac{d}{dt}\left(\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}\right) = 0 \quad for\ all\ t\ and\ all\ \ i \neq n \, .$$

These equations establish an invariant relationship between $\mathbf{s}^*$ and the ratios in (6) and end the first part of our proof.

We will now discuss in the final part of the proof what is implied by (5)-(7) for the functional form of the transformation frontier. First of all, let $\tilde{T}(k_n, \dot{\mathbf{k}}, t) := T(\mathbf{s}^* k_n, k_n, \dot{\mathbf{k}}, t) = 0$. Since $\tilde{T}(\cdot)$ is increasing in k_n we can solve for k_n in terms of $\dot{\mathbf{k}}$ and t: $k_n = \tilde{F}(\dot{\mathbf{k}}, t)$. Now recall that all output transformation rates are by assumption independent of t. Therefore, and considering (7), it follows from the quasi-concavity of $T(\cdot)$ with respect to $\dot{\mathbf{k}}$ that $\tilde{F}(\cdot)$ must be a homothetic and quasi-convex function of $\mathbf{k}$, at least if evaluated along $\mathbf{s}^*$ (see R. Färe [1] and [2], pp. 49-61; also see Shephard [5]). Hence, $k_n = F(H(\dot{\mathbf{k}}), t) = F(wH(\mathbf{k}), t)$. Consequently, multiplying both sides of $w = f(\mathbf{k}, t)$ by $H(\mathbf{k})$ yields

$$w\, H(\mathbf{k}) = H(\dot{\mathbf{k}}) = f(\mathbf{k}, t)\, H(\mathbf{k}) =: \tilde{J}(\mathbf{k}, t) \tag{8}$$

and thereby separates $T(\cdot)$ into a homothetic and quasi-convex output branch $H(\cdot)$ and a remaining input branch $\tilde{J}(\cdot)$. At this point note that the gradients H_k and $\tilde{J}_k$ always indicate the same direction because of (6). Also note that we have assumed all input substitution rates not to be affected by t. Finally, remind that $T(\cdot)$ is a quasi-concave function of $\mathbf{k}$. As a result, $\tilde{J}(\cdot)$ must be both homothetic and quasi-concave in $\mathbf{k}$ along $\mathbf{s}^*$: $\tilde{J}(\mathbf{k}, t) = J(G(\mathbf{k}), t)$. We thus obtain from (8) a frontier function representation of the form

$$T(\mathbf{k}, \dot{\mathbf{k}}, t) := J(G(\mathbf{k}), t) - H(\dot{\mathbf{k}}) \tag{9}$$

which is quasi-concave in both $\mathbf{k}$ and $\dot{\mathbf{k}}$ and with $H(\cdot)$ and $G(\cdot)$ homogeneous of degree one. This completes our proof of Theorem 2. Q.E.D.

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