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Efficient Stationary Capital Accumulation Structures of a Biconvex Production Technology

Reiner Wolff *

Department of Economics
University of Siegen, FRG

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Abstract

A classical theorem by H. Poincaré asserts that the characteristic roots at a rest point of a continuous-time autonomous variational or Hamiltonian dynamical system come in opposite-signed pairs. This result confirms the catenary motion of efficient capital accumulation paths around a saddle-point turnpike in finite-horizon models of efficient economic growth. Poincaré's theorem will be extended to the non-autonomous case of a time-dependent biconvex production technology which can be represented by a separable transformation frontier function. This extension may prove useful to integrate into optimum growth theory areas of economic analysis such as externalities and public goods.

I. Introduction

A classical theorem by H. Poincaré states that the characteristic roots of a continuous-time autonomous variational or Hamiltonian dynamical system come in opposite-signed pairs of equal absolute value if these roots are evaluated near a rest point. Thus, if τ is a root to the characteristic polynomial then $-\tau$ is also a root (see G.D. Birkhoff [1]). Therefore, if by imposing further restrictions one can rule out roots with zero real parts, the stationary equilibrium is a symmetric saddle-point (cf. D. Levhari and N. Liviatan [6]). In economic dynamic analysis such restrictions are frequently imposed upon technology which is usually assumed a convex cone. We since observe the familiar catenary motion of efficient accumulation paths around a saddle-point turnpike in many finite-horizon models of optimum economic growth. This result was first conjectured by R. Dorfman, P.A. Samuelson and R.M. Solow [3, ch. 12] and is now known as the Dorfman-Samuelson-Solow turnpike theorem or simply as the catenary turnpike theorem.

This paper is concerned with the qualitative dynamics of a non-autonomous optimum growth model into which enters 'time' as an explicit variable of the problem at hand. The model assumes a biconvex instantaneous production technology which may be time-dependent and represented by a differentiable transformation frontier function. The notion of biconvexity of a multiple-input, multiple-output technology is due to L.J. Lau [5]. It can be thought of as a generalization of the concept of convexity in that it allows overall non-constant returns but, at the same time, preserves the properties of decreasing marginal rates of substitution and increasing marginal transformation rates amongst inputs and outputs, respectively. Furthermore, if the production possibilities set is independent in its input and output partitions (see L.J. Lau [5, Section 2.3]) then biconvexity implies (additive) separability of the frontier function. The main purpose of this paper is to establish in Sections II-IV a local turnpike result which occurs if such a separability assumption is made but no further qualifications as to the returns to scale are introduced. We will thereby prepare the ground to cover phenomena such as production externalities as well as public goods which play a key role, e.g., in the creation of agglomeration economies and diseconomies typical of an urban or regional growth process. This will generalize earlier results obtained by R. Wolff [7,8]. The paper's primary concern is with the final-state version of the turnpike theorem in which case the maximization of terminal stocks serves as the intertemporal efficiency criterion. However, our results will also go through in models of intertemporal utility maximization. We conclude with a brief discussion of two global turnpikes in Section V of this paper.

First of all, a few comments on our use of notation are in order. Throughout the paper elements of R^n, $n > 1$, will be referred to as column-vectors or simply called 'vectors'. They will be denoted by lowercase letters and set in a bold typeface for ease of reading. Component i of vector $\mathbf{x}$ will be written as x_i. Correspondingly, emboldened uppercase letters like $\mathbf{A}$ shall represent matrices with elements a_{ij}. Furthermore, let $S \subset R^n$ and $T \subset R$ be two sets and consider a differentiable function $f: S \to T$. Then for each element $\mathbf{x} \in S$ and image $f(\mathbf{x})$ we use f_{x_i} as short-hand notation for $\partial f(\mathbf{x})/\partial x_i$. Much in the same way, $f_{x_i x_j}$ is short-hand notation for $\partial^2 f(\mathbf{x})/\partial x_j \partial x_i$ while $f_{\mathbf{x}}$ stands for the gradient

$\nabla f(\mathbf{x})$. Differentiation with respect to 'time' will be indicated by a dot '$\cdot$' and a prime '$'$' signifies transposition. Finally, we always write as $\mathbf{o}$ and $\mathbf{I}$ the null vector and identity matrix of appropriate length and dimension, respectively.

II. Statement of the Problem

We will be concerned with the following problem of Mayer in the calculus of variations:

$$\max_{\mathbf{k}(t)} \mathbf{p}'\,\mathbf{k}(t_1) \tag{1}$$

$$\text{subject to} \quad T(\mathbf{k}, \dot{\mathbf{k}}, t) = 0\,, \; t \in [t_0, t_1]\,, \quad \text{and} \quad \mathbf{k}(t_0) = \mathbf{k}_0\,.$$

We take as $\mathbf{p}$ a vector of given non-negative constants with at least one component positive. $T(\cdot)$ is assumed a continuously differentiable function with domain $R^n_+ \times R^n \times R_+$, where $n > 1$. In particular, all second-order derivatives of $T(\cdot)$ with regard to $\mathbf{k}$ and $\dot{\mathbf{k}}$ shall not vanish. Furthermore, we assume that $T(\cdot)$ is increasing in $\mathbf{k}$, decreasing in $\dot{\mathbf{k}}$ and quasi-concave in both $\mathbf{k}$ and $\dot{\mathbf{k}}$. $T(\cdot)$ shall also satisfy Inada regularity conditions $\lim_{k_i \to 0} T_{k_i} = \infty$ and $\lim_{k_i \to \infty} T_{k_i} = 0$ for all k_i.

This setup is motivated by a standard problem in the theory of optimum economic growth. Consider $\mathbf{k}$ a vector of stocks, e.g. machinery of some kind, such that $\dot{\mathbf{k}}$ holds the corresponding amounts of investments into these stocks. Suppose that $\mathbf{p}$ is a vector of stock prices which are expected to prevail in period t_1. Let $T(\cdot)$ completely characterize the set of efficient one-period input-output combinations of a firm or of an economy. $T(\cdot)$ may thus be viewed a transformation frontier function which exhibits the standard neoclassical properties of decreasing marginal rates of substitution between any two capital stocks and increasing marginal transformation rates among any pair of investments. (1) will then represent an economic optimum investment problem: we seek to maximize the expected value of a set of terminal stocks subject to a given environment of investment opportunities and given initial endowments.

In what follows, we will always treat (1) as a classical calculus of variations problem with a free end point. We assume that it has an interior solution in the class of non-negative twice-differentiable functions $\mathbf{k}(t)$ for $t \in [t_0, t_1]$ with $k_i(t_1) > k_i(t_0)$ for at least one k_i. Note that since $T(\cdot)$ is by assumption quasi-concave in $\dot{\mathbf{k}}$ the Legendre definiteness condition for a maximum of $\mathbf{p}'\,\mathbf{k}(t_1)$ will be met globally. Uniqueness of the solution is thus assured (see G. Hadley and M.C. Kemp [4] as a reference on variational methods in economics).

III. Statement of the Theorem

Consider the problem of Mayer (1) which we introduced in Section II. Associated with this problem is the Lagrangian functional

$$\mathbf{p}'\,\mathbf{k}(t_1) + \int_{t_0}^{t_1} \lambda\, T(\mathbf{k}, \dot{\mathbf{k}}, t)\, dt \tag{2}$$

and the time-dependent variational system of Euler differential equations

$$\lambda T_{\mathbf{k}}(\mathbf{k},\dot{\mathbf{k}},t) - \frac{d}{dt}[\lambda T_{\dot{\mathbf{k}}}(\mathbf{k},\dot{\mathbf{k}},t)] = \mathbf{o}\,. \tag{3}$$

Assume that $k_n > 0$. Let $\mathbf{s} := (k_1/k_n, ..., k_{n-1}/k_n)'$ and $\mathbf{z} := (\mathbf{s}', k_n)'$. Next define $F(\mathbf{z},\dot{\mathbf{z}},t) := T(\mathbf{s}k_n, k_n, \dot{\mathbf{s}}k_n + \mathbf{s}\dot{k}_n, \dot{k}_n, t)$ which changes (3) to

$$\lambda F_{\mathbf{z}}(\mathbf{z},\dot{\mathbf{z}},t) - \frac{d}{dt}[\lambda F_{\dot{\mathbf{z}}}(\mathbf{z},\dot{\mathbf{z}},t)] = \mathbf{o}\,. \tag{4}$$

Now we may introduce our basic

Theorem: *Suppose that* $\mathbf{k}$ *is additively separable in* $T(\mathbf{k},\dot{\mathbf{k}},t)$ *from* $\dot{\mathbf{k}}$ *and that* $T(\mathbf{k},\dot{\mathbf{k}},t)$ *can be represented by*

$$T(\mathbf{k},\dot{\mathbf{k}},t) = J(G(\mathbf{k}),t) - H(\dot{\mathbf{k}}) \tag{5}$$

with homogeneous functions $G(\mathbf{k})$ *and* $H(\dot{\mathbf{k}})$ *which are quasi-concave and quasi-convex, respectively. Then there exists a real-valued scalar function* $\theta(t)$ *and a vector of positive constants* $\bar{\mathbf{s}}^*$ *such that* $\mathbf{k}(t) = \theta(t)\,(\bar{\mathbf{s}}^{*\prime},1)'$ *is a solution to* (3) *and* $\bar{\mathbf{s}}^*$ *is a saddle-point of* (4).

Our theorem says that if there exists a solution to (4) such that all capital stocks and, therefore, all investment flows change proportionately over time, then the corresponding capital stock and investment ratios $\bar{\mathbf{s}}^*$ will serve as a turnpike for economic growth. This turnpike property of $\bar{\mathbf{s}}^*$ is independent of the levels of input and output variables other than $\mathbf{k}$ and $\dot{\mathbf{k}}$. Consider, e.g., different types of labor inputs and consumption outputs. Such variables may appear as exogeneous functions of time which enter (5) via t. In other words, we do not need to care about the preference orderings from which these functions are generated. The theorem thus still applies if we wish to maximize, e.g., a Ramsey-type utility functional rather than terminal stocks. The turnpike property of $\bar{\mathbf{s}}^*$ does also not depend on autonomous types of technical progress. Furthermore, our theorem does not require $T(\cdot)$ to exhibit constant returns to scale. In passing, note that we may assume $G(\cdot)$ and $H(\cdot)$ homogeneous of degree one without loss of generality.

The first part of the next section contains a brief discussion of some important existence and uniqueness related questions. Then we shall present in a subsequent second part a formal proof of the saddle-point property of $\bar{\mathbf{s}}^*$. We will not, however, address the assumed existence of aggregator functions $G(\cdot)$ and $H(\cdot)$. The interested reader is referred to C. Blackorby and W. Schworm [2] for a comprehensive study of the existence of input and output aggregates in aggregate production functions and for further references.

IV. Proof of the Theorem

Part 1. Existence and Uniqueness

The own-rates of interest equations of optimum growth theory provide the key to proving the existence and uniqueness of a stationary solution $\bar{s}^*$ to (4). These equations link to each other the own-rates of interest $-T_{k_i}/T_{\dot{k}_i}$ of different capital stocks k_i in the process of efficient economic growth. We attach a quick proof for the reader's convenience:

Lemma 1: *If* $\mathbf{k}(t)$ *is a solution to (3) then*

$$\text{(6)} \qquad \frac{T_{k_i}}{T_{\dot{k}_i}} = \frac{T_{k_n}}{T_{\dot{k}_n}} + \frac{d}{dt} ln\left[\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}\right], \quad \textit{for all} \;\; i \neq n.$$

Proof: Consider equations (3) which are equivalent to (4). From the former follows with regard to each arbitrary variable k_i that $\dot{\lambda}/\lambda = (T_{k_i} - \dot{T}_{\dot{k}_i})/T_{\dot{k}_i}$. Hence,

$$\text{(7)} \qquad \frac{T_{k_i}}{T_{\dot{k}_i}} - \frac{\dot{T}_{\dot{k}_i}}{T_{\dot{k}_i}} = \frac{T_{k_n}}{T_{\dot{k}_n}} - \frac{\dot{T}_{\dot{k}_n}}{T_{\dot{k}_n}}, \quad \text{for all} \;\; i \neq n.$$

Finally, straightforward differentiation yields

$$\text{(8)} \qquad \frac{d}{dt} ln\left[\frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}\right] = \frac{\dot{T}_{\dot{k}_i}}{T_{\dot{k}_i}} - \frac{\dot{T}_{\dot{k}_n}}{T_{\dot{k}_n}}, \quad \text{for all} \;\; i \neq n.$$

Adding $\dot{T}_{\dot{k}_n}/T_{\dot{k}_n}$ to both sides of (7) and using (8) will give (6). This proves Lemma 1. Q.E.D.

Now suppose that all components of $\mathbf{k}$ change proportionately, i.e., suppose that $\mathbf{k} = \theta(t)\,(\bar{s}',1)'$ and $\dot{\mathbf{k}} = \dot{\theta}(t)\,(\bar{s}',1)'$ with $\bar{s}$ constant. Furthermore, recall that $T(\cdot)$ has been assumed separable with respect to $\mathbf{k}$ on one hand and $\dot{\mathbf{k}}$ on the other hand according to (5). Therefore, since $H(\cdot)$ is homogeneous by assumption, the ratios $T_{\dot{k}_i}/T_{\dot{k}_n}$ will depend solely on $\bar{s}$ and will thus stay constant along the ray $\dot{\mathbf{k}} = \dot{\theta}(t)\,(\bar{s}',1)'$. As a result, the second term on the right-hand side of (6) will drop to zero and (6) can be rearranged to

$$\text{(9)} \qquad \frac{T_{k_i}}{T_{k_n}} = \frac{T_{\dot{k}_i}}{T_{\dot{k}_n}}, \quad \text{for all} \;\; i \neq n.$$

By the same line of reasoning, we conclude that the ratios T_{k_i}/T_{k_n} on the left-hand side of (9) will also only depend on $\bar{s}$. Furthermore, because of the regularity conditions imposed upon $T_{\mathbf{k}}$, these ratios can assume any non-negative value. There will thus exist a strictly positive solution $\bar{s}^*$ to (9). Furthermore, because of the quasi-concavity of $G(\cdot)$ and quasi-convexity of $H(\cdot)$, this solution will be unique. Q.E.D.

The following shall elucidate the significance of the given functional form (5) of $T(\cdot)$ for the above results. Consider, e.g., the case of a non-separable transformation frontier

$T(\cdot) = J(G(\mathbf{k}, \dot{\mathbf{k}}), t)$ with homogeneous $G(\cdot)$. There is no loss in generality if we assume $G(\cdot)$ homogeneous of degree one. Using $\mathbf{k}(t) = \theta(t)(\bar{\mathbf{s}}', 1)'$ and $\dot{\mathbf{k}}(t) = \dot{\theta}(t)(\bar{\mathbf{s}}', 1)'$ will then give $G(\cdot) = \theta(t)\, G((\bar{\mathbf{s}}', 1)', w(t)(\bar{\mathbf{s}}', 1)')$ with $w(t) := \dot{\theta}(t)/\theta(t)$. Hence, both sides of (9) will depend on $w(t)$ and will therefore, in general, change as t changes. No stationary solution $\bar{\mathbf{s}}^*$ may then exist.

The economic content of (9) is that the marginal rates of substitution between any two capital stocks coincide with the marginal transformation rates which prevail between the corresponding pair of investments. In addition, since $T_{\dot{k}_i}/T_{\dot{k}_n}$ is constant for all $i \neq n$ along $\dot{\mathbf{k}}(t) = \dot{\theta}(t)(\bar{\mathbf{s}}^{*\prime}, 1)'$, we conclude from (6) that the own-rates of interest are the same for all components of $\mathbf{k}(t)$.

Part 2. Stability

We will now prove that $\bar{\mathbf{s}}^*$ is a saddle-point of the variational dynamical system (4). Our proof will be a proof of the local saddle-point property of $\bar{\mathbf{s}}^*$ to begin with. Later on in Section V, we will also present some global results which are available in special cases of (5). First of all, however, we shall briefly comment on how we will seek to arrive at the desired conclusion.

One might as a first try of a proof of the local saddle-point property of $\bar{\mathbf{s}}^*$ expand equations (4) into a linear Taylor series approximation around $\bar{\mathbf{s}}^*$ and then check the roots of the associated characteristic polynomial. The problem with this approach is that the resulting linear equations will, in most cases of (5), possess time-dependent coefficients which makes our stability analysis rather intricate.[1] We will therefore pursue a different strategy which involves a detour. Our proof resorts to the fact that associated with $\bar{\mathbf{s}}^*$ is a unique set of stationary ratios $T^*_{k_i}/T^*_{k_n}$ and $T^*_{\dot{k}_i}/T^*_{\dot{k}_n}$ according to (9). Now let $u_i := T_{k_i}/T_{k_n} =: a_i(\mathbf{s})$ for all $i \neq n$. Furthermore, let $\dot{\mathbf{k}}_{n-1} := (\dot{k}_1, \ldots, \dot{k}_{n-1})'$ such that $v_i := T_{\dot{k}_i}/T_{\dot{k}_n} =: b_i(\dot{\mathbf{k}}_{n-1}/\dot{k}_n)$ for all $i \neq n$. Also define $\bar{\mathbf{u}}^* := \mathbf{a}(\bar{\mathbf{s}}^*)$ and $\bar{\mathbf{v}}^* := \mathbf{b}(\bar{\mathbf{s}}^*)$. It is sufficient then for a proof of the local saddle-point property of $\bar{\mathbf{s}}^*$ to prove the local saddle-point property of $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$ instead. As we proceed, we will show how this can be accomplished.

We start with the algebra of the dynamics of $\mathbf{u}$ and $\mathbf{v}$. First of all, recall that if $\mathbf{k}(t)$ is a solution to (3) and, therefore, $\mathbf{s}(t)$ is a solution to (4) then (6) will hold. Thus, since $d\{ln[\cdot]\}/dt = [\cdot]^{-1}d[\cdot]/dt$, rearranging terms and substituting u_i and v_i for T_{k_i}/T_{k_n} and $T_{\dot{k}_i}/T_{\dot{k}_n}$ will give

$$\dot{v}_i = f(t)(u_i - v_i), \quad \text{for all } i \neq n, \tag{10}$$

[1] It is sometimes possible to transform the linearized model into an autonomous system. Two such examples are given in R. Wolff [7,8]. However, R. Wolff [8] assumes that the own-rates of interest of the capital stocks are approximately equal to w(t) which is valid only in the neighborhood of a certain point along the ray of efficient proportionate capital stock expansion.

with $f(t) := T_{k_n}(\mathbf{k}(t), \dot{\mathbf{k}}(t), t) \,/\, T_{\dot{k}_n}(\mathbf{k}(t), \dot{\mathbf{k}}(t), t)$. Furthermore, we conclude from $u_i = a_i(\mathbf{s})$ and $s_j := k_j/k_n$ that

$$\text{(11)} \qquad \dot{u}_i = \sum_{j=1}^{n-1} \frac{\partial a_i(\mathbf{s})}{\partial s_j} \dot{s}_j = \sum_{j=1}^{n-1} \frac{\partial a_i(\mathbf{s})}{\partial s_j} \frac{\dot{k}_j \, k_n - k_j \, \dot{k}_n}{k_n^2}$$

$$= w_n(t) \sum_{j=1}^{n-1} \frac{\partial a_i(\mathbf{s})}{\partial s_j} \left(\frac{\dot{k}_j}{\dot{k}_n} - s_j \right), \quad \text{for all } i \neq n,$$

where $w_n(t) := \dot{k}_n(t)/k_n(t)$. Now observe that $G(\cdot)$ and $F(\cdot)$ are by assumption quasi-concave and quasi-convex, respectively. Hence, the Jacobian determinants of $\mathbf{a}(\cdot)$ and $\mathbf{b}(\cdot)$ will be globally non-zero. Therefore, by the implicit function theorem, there will always exist inverse functions $\mathbf{s} = \mathbf{a}^{-1}(\mathbf{u}) =: \tilde{\mathbf{a}}(\mathbf{u})$ and $\dot{\mathbf{k}}_{n-1}/\dot{k}_n = \mathbf{b}^{-1}(\mathbf{v}) =: \tilde{\mathbf{b}}(\mathbf{v})$. We thus find that

$$\text{(12)} \qquad \dot{u}_i = w_n(t) \sum_{j=1}^{n-1} \frac{\partial a_i(\mathbf{s})}{\partial s_j}\bigg|_{\tilde{\mathbf{a}}(\mathbf{u})} \left(\tilde{b}_j(\mathbf{v}) - \tilde{a}_j(\mathbf{u}) \right), \quad \text{for all } i \neq n.$$

Equations (10) along with (12) make a system of $2(n-1)$ first-order differential equations with respect to yet unknown functions $\mathbf{u}(t)$ and $\mathbf{v}(t)$. A stationary solution to (10) and (12) is a solution $(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = (\mathbf{q}, \mathbf{q})$ with $\tilde{\mathbf{a}}(\mathbf{q}) = \tilde{\mathbf{b}}(\mathbf{q})$. This latter condition is equivalent to (9), so we can be sure that $\mathbf{q}$ will exist and will also be unique. For the same reason, $\mathbf{q} = \bar{\mathbf{u}}^* = \bar{\mathbf{v}}^*$.

The next step is to expand (10) and (12) into a Taylor series around $\bar{\mathbf{u}}^*$ and $\bar{\mathbf{v}}^*$. Therefore, recall that $\dot{\mathbf{u}} = \dot{\mathbf{v}} = \mathbf{o}$ and $\tilde{\mathbf{b}}(\mathbf{v}) = \tilde{\mathbf{a}}(\mathbf{u})$ as well as $w_n(t) = w(t)$ along $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$. Also observe that the product of the Jacobian matrices of $\mathbf{a}(\mathbf{s})$ and $\tilde{\mathbf{a}}(\mathbf{u})$ will come out as the identity matrix, since $\tilde{\mathbf{a}}(\mathbf{u})$ is the inverse of $\mathbf{a}(\mathbf{s})$. We will thus end up with the following linear dynamical system if we neglect all second-order and higher-order terms:

$$\text{(13)} \qquad \dot{\mathbf{y}} = f(t)\,(\mathbf{x} - \mathbf{y}), \quad \dot{\mathbf{x}} = w(t)\,(\mathbf{A}\tilde{\mathbf{B}}\,\mathbf{y} - \mathbf{x}),$$

where $\mathbf{x} := \mathbf{u} - \bar{\mathbf{u}}^*$ and $\mathbf{y} := \mathbf{v} - \bar{\mathbf{v}}^*$ while $\mathbf{A}$ and $\tilde{\mathbf{B}}$ are symmetric $(n-1, n-1)$-matrices with elements

$$\text{(14)} \qquad a_{ij} = \frac{\partial a_i(\mathbf{s})}{\partial s_j}\bigg|_{\tilde{\mathbf{a}}(\mathbf{u})}, \quad \tilde{b}_{ij} = \frac{\partial \tilde{b}_i(\mathbf{v})}{\partial v_j},$$

taken at $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$. Note that associated with the stationary solution $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$ to (10) and (12) is a stationary solution $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*) = (\mathbf{o}, \mathbf{o})$ to (13).

Since in (13) both $f(t)$ and $w(t)$ may in cases be rather complicated functions of the time variable t we can only look for qualitative results. As it turns out, such results can already be obtained from the time-free dynamical system

$$\text{(15)} \qquad \dot{\mathbf{y}} = \mathbf{y} - \mathbf{x}, \quad \dot{\mathbf{x}} = \mathbf{A}\tilde{\mathbf{B}}\,\mathbf{y} - \mathbf{x},$$

to which $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*) = (\mathbf{o}, \mathbf{o})$ is also a stationary solution. Our interest in (15) is due to

Lemma 2: *If $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*) = (\mathbf{o}, \mathbf{o})$ is a saddle-point of the time-free system (15), then it is also a saddle-point of (13).*

Proof: Consider the families in $(\mathbf{x}, \mathbf{y})$–space of phase trajectories implicitly defined by (13) and (15), respectively. Observe that $f(t) < 0$ and $w(t) > 0$. Hence, the set $S = \{(\mathbf{x}, \mathbf{y}) \,|\, \dot{\mathbf{x}} = \mathbf{o} \text{ or } \dot{\mathbf{y}} = \mathbf{o}\}$ is the same for both systems. It also follows with regard to all other points $(\mathbf{x}, \mathbf{y})$ not in S that the signs of $\dot{\mathbf{y}}$ and $\dot{\mathbf{x}}$ do not change if we switch from (13) to (15). The topological structure of the phase trajectories of (13) is thus preserved. Q.E.D.

Associated with (15) is the characteristic polynomial

$$|\mathbf{A}\tilde{\mathbf{B}} - (1 - \eta^2)\mathbf{I}| = 0. \tag{16}$$

We thus have

Lemma 3: *If η is a root to (16) then $-\eta$ is also a root to (16). Both η and $-\eta$ come as non-zero real numbers.*

Proof: The first result is immediate. The second requires further thought. To begin with, note that $\mathbf{A}$ is a negative definite matrix while both $\mathbf{B}$ and its inverse $\tilde{\mathbf{B}}$ are positive definite matrices as we have assumed $G(\cdot)$ and $H(\cdot)$ quasi-concave and quasi-convex functions, respectively. By a standard theorem of linear algebra there exists a regular matrix $\mathbf{R}$ such that $\tilde{\mathbf{B}} = \mathbf{R}\mathbf{R}'$. Another theorem says that for any regular $(n-1, n-1)$-matrix $\mathbf{C}$ the eigenvalues of $\mathbf{A}\tilde{\mathbf{B}}$ and $\mathbf{C}^{-1}\mathbf{A}\tilde{\mathbf{B}}\mathbf{C}$ are the same (see any textbook on linear algebra for a proof of these theorems). Now choose $\mathbf{C} = \mathbf{R}'^{-1}$. Then

$$\begin{aligned} \mathbf{C}^{-1}\mathbf{A}\tilde{\mathbf{B}}\mathbf{C} &= \mathbf{R}'\mathbf{A}\tilde{\mathbf{B}}\mathbf{R}'^{-1} \\ &= \mathbf{R}'\mathbf{A}\mathbf{R}\mathbf{R}'\mathbf{R}'^{-1} \\ &= \mathbf{R}'\mathbf{A}\mathbf{R}. \end{aligned} \tag{17}$$

Next consider an arbitrary vector $\mathbf{c} \in R^{n-1}$ with at least one component different from zero. Let $\mathbf{r} := \mathbf{R}\mathbf{c}$. Note that $\mathbf{r} \neq \mathbf{o}$ as $\mathbf{R}$ is regular. Hence, since $\mathbf{A}$ is negative definite,

$$\mathbf{c}'\mathbf{R}'\mathbf{A}\mathbf{R}\mathbf{c} = \mathbf{r}'\mathbf{A}\mathbf{r} < 0. \tag{18}$$

We thus conclude that $\mathbf{R}'\mathbf{A}\mathbf{R} = \mathbf{C}^{-1}\mathbf{A}\tilde{\mathbf{B}}\mathbf{C}$ is also a negative definite matrix so its eigenvalues will be real-valued and negative. As a result, all eigenvalues of $\mathbf{A}\tilde{\mathbf{B}}$ are real-valued and negative, too. Now note that if μ is an eigenvalue of $\mathbf{A}\tilde{\mathbf{B}}$ then $1 - \eta^2 = \mu < 0$. Therefore, both η and $-\eta$ are non-zero reals. This completes the proof of our second result. Q.E.D.

We have so far shown that the roots to (16) are both real and non-zero and appear in opposite-signed pairs of equal absolute value. The origin $(\mathbf{o}, \mathbf{o})$ is thus a symmetric saddle-point of (15) and hence of (13) because of Lemma 2. From this we conclude that $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$ constitutes a local saddle-point of (10) and (12). Furthermore, since there exists a one-to-one correspondence between $(\bar{\mathbf{u}}^*, \bar{\mathbf{v}}^*)$ and $\bar{\mathbf{s}}^*$, it follows that $\bar{\mathbf{s}}^*$ is a local saddle-point of (4). Our proof of the local turnpike property of $\bar{\mathbf{s}}^*$ is thereby finished. Q.E.D.

V. Conclusions

We will now briefly present in our final section two global turnpike results which relate to special cases of (5). To begin with, take as $\mathbf{k}$ a vector of two capital stocks, i.e., let $n = 2$ in which case there is only one capital stock ratio k_1/k_2 and only one ratio of investment outputs $\dot{k}_1/\dot{k}_2$. We may therefore suppress all subscripts attached to variables s, u and v. Equations (10) and (12) can then be written concisely as

$$\text{(19)} \qquad \dot{v} = f(t)\,(u - v), \quad \dot{u} = w_2(t)\,\frac{da(s)}{ds}\bigg|_{\tilde{a}(u)} \left(\tilde{b}(v) - \tilde{a}(u)\right),$$

with scalar functions $a(s)$, $\tilde{a}(u)$ and $\tilde{b}(v)$ which are decreasing in s and u and increasing in v, respectively. Hence, $da(s)/ds < 0$. Recall likewise that $f(t) < 0$. Now suppose that $w_2(t) > 0$ for all $t \in [t_0, t_1]$. Then the global saddle-point property of $(\bar{u}^*, \bar{v}^*)$ and thus the turnpike-property of $\bar{s}^*$ will be ensured. We leave it to the reader to provide a proof of this result, e.g. by means of a graphical analysis of the phase trajectories in (u, v)–space.

Next let $n > 2$ and assume that both $G(\cdot)$ and $H(\cdot)$ in (5) are ACMS-type functions. This latter assumption implies that the marginal rate of substitution between any two capital stocks k_i and k_n and the marginal transformation rate between any two investment outputs $\dot{k}_i$ and $\dot{k}_n$ will depend solely on the respective ratios k_i/k_n and $\dot{k}_i/\dot{k}_n$. As a consequence, all off-diagonal elements of the Jacobian matrices of $\mathbf{a}(\mathbf{s})$, $\tilde{\mathbf{a}}(\mathbf{u})$ and $\tilde{\mathbf{b}}(\mathbf{v})$ will identically drop to zero and equations (10) and (12) disintegrate into $n-1$ separate systems of the form (19). The above global turnpike result applies accordingly.

References

1. G.D. Birkhoff, *Dynamical Systems,* 1st rev. ed., Providence/R.I. 1966.
2. C. Blackorby and W. Schworm, "The Existence of Input and Output Aggregates in Aggregate Production Functions," *Econometrica* **56** (1988), 613-643.
3. R. Dorfman, P.A. Samuelson, and R.M. Solow, *Linear Programming and Economic Analysis,* New York 1958.
4. G. Hadley and M.C. Kemp, *Variational Methods in Economics,* 2nd ptg., Amsterdam 1973.
5. L.J. Lau, "A Characterization of the Normalized Restricted Profit Function," *Journal of Economic Theory* **12** (1976), 131-163. Reprinted in D. Cass and K. Shell (eds.), *The Hamiltonian Approach to Dynamic Economics,* New York 1976.

6. D. Levhari and N. Liviatan, "On Stability in the Saddle-point Sense," *Journal of Economic Theory* **4** (1972), 88-93.
7. R. Wolff, "A Neoclassical Turnpike Theorem on Maximum Consumption Growth of Heterogeneous Capital Stocks in the Case of Increasing Returns to Scale," *Methods of Operations Research* **48** (1984), 227-258.
8. R. Wolff, "Efficient Growth of an Agglomerating Regional Economy," *Regional Science and Urban Economics* **15** (1985), 555-572.

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